Approved by Curriculum Development Centre, Sanothimi; Bhaktapur for the Years 2078 to 2080 B.S.

Revised New Edition 2078 B.S.

BASIC MATHEMATICS

GRADE XII

DHUP RATNA BAJRACHARYA
RAM MAN SHRESHTHA
MOHAN BIR SINGH
YOG RATNA STHAPIT
BHANU CHANDRA BAJRACHARYA

नेपालको नक्सा
(राजनीतिक तथा प्रशासनिक)
सुदूरपश्चिम प्रदेश
कर्णाली प्रदेश
गण्डकी प्रदेश
बागमती प्रदेश
प्रदेश नं. १
प्रदेश नं. २

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सानोठिमी, भक्तपुर
(........................ शाखा)
मान्यता, समकक्षता तथा मूल्याङ्कन

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विषय : पाठ्यसामग्री स्वीकृति सम्बन्धमा ।

श्री सुकुन्दा पुस्तक भवन,

काठमाण्डौं ।

प्रस्तुत विषयमा त्यस प्रकाशनबाट मूल्याङ्कन र स्वीकृतिका लागि तोकिएको अवधिभित्र पेस हुन आएका तपसिलबमोजिमका पाठ्यसामग्री आवश्यक निर्णयार्थ पाठ्यसामग्री व्यवस्थापन तथा मूल्याङ्कन समितिमा पेस हुँदा विद्यालय शिक्षाको राष्ट्रिय पाठ्यक्रम प्रारूप २०७६, माध्यमिक शिक्षा पाठ्यक्रम (कक्षा ११ र १२), पाठ्यसामग्री विकाससम्बन्धी विद्यमान प्रावधान, ऐन, कानुन, निर्देशिका, कार्यविधि, प्रकाशन शैलीका प्रावधान, पठ्यकम विकास केन्द्रले विभिन्न समयमा जारी गरेका र पाठ्यसामग्री सुधार/परिमार्जन/पुनर्लेखनका लागि दिइएका सुझाव र निर्देशनको परिपालना गरी स्वीकृति दिन सिफारिस भएअनुसार यस कार्यालयको मिति २०७८।०४।२६ गतेको निर्णयानुसार तपसिलमा उल्लिखित निर्देशनको पूर्ण परिपालना गरी शैक्षिक वर्ष २०७८, २०७९ र २०८० गरी तीन शैक्षिक वर्षका लागि गुणस्तरीय एवम् त्रुटिरहित पाठ्यसामग्री विकास गरी प्रकाशन गर्न स्वीकृति प्रदान गरिएको छ । विद्यमान संवैधानिक व्यवस्था, ऐन, कानुन, निर्देशिका, कार्यविधि, पाठ्यकम विकास केन्द्रले विभिन्न समयमा जारी गरेका निर्देशनलगायतका प्रावधानहरूको पूर्ण परिपालना नगरी गुणस्तरहीन पाठ्यसामग्रीको विकास, प्रकाशन र विकीवितरण गरेको पाइएमा, पाठ्यकम परिवर्तन भएमा वा यस केन्द्रबाट अन्य निर्णय भएमा यो स्वीकृति जुनसुकै बेला रद्ढ हुने छ ।

तपसिल :

(क)पाठ्यसामग्रीको नाम :

१	Basic Mathematics	माध्यमिक तह/ कक्षा १२

(ख) निर्देशन :

१.आवरण पृष्ठको अघिल्लो (Front) भागको बायाँ (Verso) पृष्ठमा नेपालको आधिकारिक नक्सा र आवरण पृष्ठको पछिल्लो (Back) भागको दायाँ (Recto) पृष्ठमा कोभिड १९ सङ्कमण रोकथामसम्बन्धी सूचना यस केन्द्रको वेबसाइटबाट डाउनलोड गरी समावेश गर्ने । विषयवस्तुको प्रकृति र आवश्यकताका आधारमा पाठ्यसामग्री भित्रका विषयवस्तुमा समावेश गरिने नक्सा आधिकारिक र प्रामाणिक हुनुपर्ने

२.स्वीकृति पत्र स्क्यान गरी पाठ्यसामग्रीको शीर्षक पृष्ठभन्दा पछि दायाँ (Recto) पृष्ठमा समावेश गर्ने । पाठ्यसामग्रीको प्रत्येक पृष्ठको पुच्छर (Footer) मा पाठ्यकम विकास केन्द्रबाट स्वीकृत भन्ने व्यहोरा उल्लेख गरी प्रकाशन गरेको पाठ्यसामग्रीको तीन प्रति यस केन्द्रमा पेस गरेपछि मात्र विकीवितरण गर्ने । शिक्षा, विज्ञान तथा प्रविधि मन्त्रालयको निर्णयअनुसारको मूल्य कायम गर्ने र मूल्य सर्वाधिकार पृष्ठमा राख्नुपर्ने । प्रतिलिपि अधिकार (Copy right) को सम्बन्धमा लेखक र प्रकाशक स्वयम् जिम्मेवार हुने

३.पाठ्यसामग्री विकाससम्बन्धी विद्यमान प्रावधान तथा पाठ्यसामग्री सुधार र परिमार्जनका लागि यस अघि दिइएका सुझाव र निर्देशनको पूर्ण परिपालना गर्ने । माध्यमिक शिक्षा पाठ्यक्रम (कक्षा ११ र १२) २०७६ का तहगत सक्षमता, कक्षागत सिकाइ उपलब्धि हासिल गर्न सहयोग पुग्नेगरी विषयवस्तुको क्षेत्र र क्रम, प्रयोगात्मक तथा परियोजना कार्यअन्तर्गतका सम्भाव्य क्रियाकलापका उदाहरण, क्षेत्र वा एकाइगत कार्यघण्टा, सिकाइ सहजीकरण विधि र प्रक्रिया, विद्यार्थी मूल्याङ्न विधि तथा प्रक्रिया आदि समेतलाई ध्यानमा राखी पाठ्यसामग्रीलाई पूर्णरुप दिने

४.विद्यार्थीलाई थप भार पर्ने गरी पाठ्यकममा समावेश नगरिएका विषयवस्तु, अभ्यास तथा सिकाइ कियाकलाप पाठ्यसामग्रीमा समावेश नगर्ने

५.राष्ट्र, राष्ट्रिय एकता, सार्वभौमिकता, भौगोलिक अखण्डता, स्वाधीनता, राष्ट्रिय हित, पहिचान, सम्मान र समृद्धिमा आँच आउने तथा विभिन्न जातजाति, भाषा, धर्म, संस्कृति, सामाजिक सहिष्णुता, सद्भाव, सांस्कृतिक मूल्यमान्यता, रहनसहन आदिमा प्रतिकूल प्रभाव पार्ने कुनै पनि विषयवस्तु, उदाहरण, चित्र, अभ्यास, सिकाइ क्रियाकलाप समावेश नगर्ने

६.जातजाति, भाषा, धर्म, संस्कृति, वर्ण, क्षेत्र, लैङ्गिकता, अपाङ्गता, पेसा, व्यवसाय, सामाजिक सांस्कृतिक अवस्थाका आधारमा भावनात्मक रूपमा चोट पुऱ्याउने, आक्षेप लाग्ने, होच्याउने र विभेदीकरण गर्ने किसिमका विषयवस्तु, उदाहरण, चित्र, अभ्यास, सिकाइ क्रियाकलाप समावेश नगर्ने

७.प्रत्येक एकाइ/पाठमा अभ्यासका लागि प्रश्नहरु समावेश गर्दा सरलदेखि जटिल हुनेगरी समावेश गर्ने । विभिन्न एकाइ/पाठमा अभ्यासका लागि समावेश गरिएका प्रश्नहरुको उत्तर समावेश नगर्ने । प्रश्नोत्तरको शैलीमा समावेश गरिएका अभ्यास हटाउने

मीनबहादुर थापा
पाठ्यक्रम अधिकृत

पुनश्च : यो स्वीकृति शैक्षिक वर्ष २०७८, २०७९ र २०८० का लागि प्रदान गरिएकाले सोहीबमोजिम प्रकाशन, बिकी वितरण र प्रयोग गर्नु गराउनहुन सम्बन्धित सरोकारवाला सबैमा अनुरोध छ ।

Preface

Everything changes in the universe with the passage of time. The change occurs for the better performance and better result. Similarly, the change has occurred in the Mathematics Curriculum as well. The present book entitled "Basic Mathematics" grade 12 is the new form of our books "Higher Secondary Level Basic Mathematics for grade XI and XII". The new and updated curriculum in Mathematics needs a revision of our books "Higher Secondary Level Basic Mathematics Vol. I and Vol. II" for grade XI and XII in order to meet the necessity of the latest changes. So, we revised our previous books and presented under the title "Basic Mathematics" grade 12.

In this edition, we have as usual preserved many features and characteristics of the previous edition which are highly liked and appreciated by the teachers and taught while incorporating many changes in both content and extent. New chapters on use of quadratic functions, differential equations and its application such as cobweb model, keynesian macro economic model have been added. These topics have been carefully written so that they may easily be understood. The questions at the end of the exercises under the heading A, B, C, … are either for the practical work or for the project work.

We must not forget to thank Mr. Ananda Krishna Shrestha of Sukunda Pustak Bhawan for the determination and dedication he has shown on publishing the book in time. We also thank Mr. Kiran Shakya for the appreciable work done by him in computer typing and figure designing.

Any suggestion for the improvement of the book will be highly appreciated and thankfully acknowledged.

Poush, 2077 **Authors**

Secondary Education Curriculum
2076
Maths

Grade: 12
Credit hrs: 5

Subject code: Mat. 402 (Grade 12)
Working hrs: 160

1. Introduction

Mathematics is an indispensable in many fields. It --is essential in the field of engineering, medicine, natural sciences, finance and other social sciences. The branch of mathematics concerned with application of mathematical knowledge to other fields and inspires new mathematical discoveries. The new discoveries in mathematics led to the development of entirely new mathematical disciplines. School mathematics is necessary as the backbone for higher study in different disciplines. Mathematics curriculum at secondary level is the extension of mathematics curriculum offered in lower grades (1 to 10).

This course of Mathematics is designed for grade 11 and 12 students as an optional subject as per the curriculum structure prescribed by the National Curriculum Framework, 2075. This course will be delivered using both the conceptual and theoretical inputs through demonstration and presentation, discussion, and group works as well as practical and project works in the real world context. Calculation strategies and problem solving skills will be an integral part of the delivery.

This course includes different contents like; Algebra, Trigonometry, Analytic Geometry, Vectors, Statistics and Probability, Calculus, Computational Methods and Mechanics or Mathematics for Economics and Finance.

Student's content knowledge in different sectors of mathematics with higher understanding is possible only with appropriate pedagogical skills of their teachers. So, classroom teaching must be based on student-centered approaches like project work, problem solving etc.

2. Level-wise Competencies

On completion of this course, students will have the following competencies:

1. apply numerical methods to solve algebraic equation and calculate definite integrals and use simplex method to solve linear programming problems (LPP).
2. use principles of elementary logic to find the validity of statement.
3. make connections and present the relationships between abstract algebraic structures with familiar number systems such as the integers and real numbers.
4. use basic properties of elementary functions and their inverse including linear, quadratic, reciprocal, polynomial, rational, absolute value, exponential, logarithm, sine, cosine and tangent functions.
5. identify and derive equations or graphs for lines, circles, parabolas, ellipses, and hyperbolas,
6. use relative motion, Newton's laws of motion in solving related problems.

7. articulate personal values of statistics and probability in everyday life.
8. apply derivatives to determine the nature of the function and determine the maxima and minima of a function and normal increasing and decreasing function into context of daily life.
9. explain anti-derivatives as an inverse process of derivative and use them in various situations.
10. use vectors and mechanics in day to day life.
11. develop proficiency in application of mathematics in economics and finance.

3. Scope and Sequence of Contents

S.N.	Content area	Contents	Working hrs
1	**Algebra**	1.1 **Permutation and combination:** Basic principle of counting, Permutation of (a) set of objects all different (b) set of objects not all different (c) circular arrangement (d) repeated use of the same objects, Combination of things all different, Properties of combination 1.2 **Binomial Theorem:** Binomial theorem for a positive integral index, general term, Binomial coefficient, Binomial theorem for any index (without proof), application to approximation, Euler's number, Expansion of e^x, a^x and $\log(1+x)$ (without proof) 1.3 **Elementary Group Theory:** Binary operation, Binary operation on sets of integers and their properties, Definition of a group, finite and infinite groups. Uniqueness of identity, Uniqueness of inverse, Cancelation law, Abelian group. 1.4 **Complex numbers:** De Moivre's theorem and its application in finding the roots of a complex number, properties of cube roots of unity. Euler's formula. 1.5 **Quadratic equation:** Nature and roots of a quadratic equation, Relation between roots and coefficient. Formation of a quadratic equation, Symmetric roots, one or both roots common. 1.6 **Sequence and series:** Sum of finite natural numbers, sum of squares of first n-natural numbers, Sum of cubes of first n- natural numbers, principle of mathematical induction. 1.7 **Matrix based system of linear equation:** Solution of a system of linear equations by Cramer's rule and matrix method (row- equivalent and inverse) up to three variables.	31

2	**Trigonometry**	**2.1** Inverse circular functions. **2.2** Trigonometric equations and general values	8
3	**Analytic Geometry**	**3.1** **Conic section:** Standard equations of Ellipse and hyperbola. **3.2** **Coordinates in space:** direction cosines and ratios of a line, general equation of a plane, equation of a plane in intercept and normal form, plane through 3 given points, plane through the intersection of two given planes, parallel and perpendicular planes, angle between two planes, distance of a point from a plane.	13
4	**Vectors**	**4.1** **Product of Vectors:** vector product of two vectors, geometrical interpretation of vector product, properties of vector product, application of vector product in geometry and trigonometry.	7
5	**Statistics & Probability**	**5.1** **Correlation and Regression:** Correlation, nature of correlation, correlation coefficient by Karl Pearson's method, interpretation of correlation coefficient, properties of correlation coefficient (without proof), rank correlation by Spearman, regression equation, regression line of y on x and x on y. **5.2** **Probability:** Dependent cases, conditional probability (without proof), binomial distribution, mean and standard deviation of binomial distribution (without proof)	9
6	**Calculus**	**6.1** **Derivatives:** derivative of inverse trigonometric, exponential and logarithmic function by definition, relationship between continuity and differentiability, rules for differentiating hyperbolic function and inverse hyperbolic function, L'Hospital's rule (0/0, ∞/∞), differentials, tangent and normal, geometrical interpretation and application of Rolle's theorem and mean value theorem. **6.2** **Anti-derivatives:** anti-derivatives of standard integrals, integrals reducible to standard forms, integrals of rational function. **6.3** **Differential equations:** differential equation and its order, degree, differential equations of first order and first degree, differential equations with separable variables, homogenous, linear and exact differential equations.	31

7	**Computational Methods**	7.1 **Linear programming problems (LPP):** simplex method (maximization problems only) 7.2 **System of linear equations:** Gauss Elimination method	10
8	**Mechanics Or Mathematics for Economics and Finance**	8.1 **Statics:** Resultant of like and unlike parallel forces. 8.2 **Dynamics:** Newton's laws of motion and projectile. 8.3 **Mathematics for economics and finance:** Consumer and Producer Surplus, Quadratic functions in Economics, Input-Output analysis, Dynamics of market price, Difference equations, The Cobweb model, Lagged Keynesian macroeconomic model.	11
		Total	120

4. Practical and project activities

The students are required to do different practical activities in different content areas and the teachers should plan in the same way. Total of 34 working hours is allocated for practical and project activities in each of the grade 12. The following table shows estimated working hours for practical activities in different content areas of grade 12.

S.No.	Content area/domain	Working hrs in each of the grade 12
1.	Algebra	11
2.	Trigonometry	2
3.	Analytic geometry	5
4.	Vectors	3
5.	Statistics & Probability	3
6.	Calculus	11
7.	Computational methods	2
8.	Mechanics or Mathematics for Economics and Finance	3
	Total	40

Here are some sample (examples) of practical and project activities.

Sample project works/mathematical activities for grade 12

1. Represent the binomial theorem of power 1, 2, and 3 separately by using concrete materials and generalize it with n dimension relating with Pascal's triangle.

2. Take four sets R, Q, Z, N and the binary operations +, –, ×. Test which binary operation forms group or not with R, Q, Z, N.
3. Prepare a model to explore the principal value of the function $\sin^{-1}x$ using a unit circle and present in the classroom.
4. Draw the graph of $\sin^{-1}x$, using the graph of sin x and demonstrate the concept of mirror reflection (about the line y = x).
5. Fix a point on the middle of the ceiling of your classroom. Find the distance between that point and four corners of the floor.
6. Construct an ellipse using a rectangle.
7. Express the area of triangle and parallelogram in terms of vector.
8. Verify geometrically that: → × (→ + →→) = → × → + → × →→
9. Collect the grades obtained by 10 students of grade 11 in their final examination of English and Mathematics. Find the correlation coefficient between the grades of two subjects and analyze the result.
10. Find two regression equations by taking two set of data from your textbook. Find the point where the two regression equations intersect. Analyze the result and prepare a report.
11. Find, how many peoples will be there after 5 years in your districts by using the concept of differentiation.
12. Verify that the integration is the reverse process of differentiation with examples and curves.
13. Correlate the trapezoidal rule and Simpson rule of numerical integration with suitable example.
14. Identify different applications of Newton's law of motion and related cases in our daily life.
15. Construct and present Cobweb model and lagged Keynesian macroeconomic model.

5. Student Assessment

Evaluation is an integral part of learning process. Both formative and summative evaluation system will be used to evaluate the learning of the students. Students should be evaluated to assess the learning achievements of the students. There are two basic purposes of evaluating students in Mathematics: first, to provide regular feedback to the students and bringing improvement in student learning-the formative purpose; and second, to identify student's learning levels for decision making.

a. Internal Examination/Assessment

i. **Project Work:** Each Student should do one project work from each of eight content areas and has to give a 15 minute presentation for each project work in classroom. These seven project works will be documented in a file and will be submitted at the time of external examination. Out of eight projects, any one should be presented at the time of external examination by each student.

ii. **Mathematical activity:** Mathematical activities mean various activities in which students willingly and purposefully work on Mathematics. Mathematical activities can include various activities like (i) Hands-on activities (ii) Experimental activities (iii) physical activities. Each student should do one activity from each of eight content area (altogether seven activities). These activities will be documented in a file and will be submitted at the time of external examination. Out of eight activities, any one should be presented at the time of external examination by each student.

iii. **Demonstration of Competency in classroom activity:** During teaching learning process in classroom, students demonstrate 10 competencies through activities. The evaluation of students' performance should be recorded by subject teacher on the following basis.

- Through mathematical activities and presentation of project works.
- Identifying basic and fundamental knowledge and skills.
- Fostering students' ability to think and express with good perspectives and logically on matters of everyday life.
- Finding pleasure in mathematical activities and appreciate the value of mathematical approaches.
- Fostering and attitude to willingly make use of mathematics in their lives as well as in their learning.

iv. **Marks from trimester examinations:** Marks from each trimester examination will be converted into full marks 3 and calculated total marks of two trimester in each grade.

The weightage for internal assessment are as follows:

Classroom participation	Project work / Mathematical activity (at least 10 work/activities from the above mentioned project work/ mathematical activities should be evaluated)	Demonstration of competency in classroom activity	Marks from terminal exams	Total
3	10	6	6	25

b. External Examination/Evaluation

External evaluation of the students will be based on the written examination at the end of each grade. It carries 75 percent of the total weightage. The types and number questions will be as per the test specification chart developed by the Curriculum Development Centre.

Contents

1. Algebra

2. Trigonometry

3. Analytical Geometry

4. Vectors

5. Statistics and Probability

6. Calculus

7. Computational Methods

8. Mechanics or
Mathematics for Economics and Finance

Practical Work/Activity/Project Work is given at the end of the exercise or chapter under the heading A, B, C, ...

Chapter 1
Permutation & Combination

Introduction

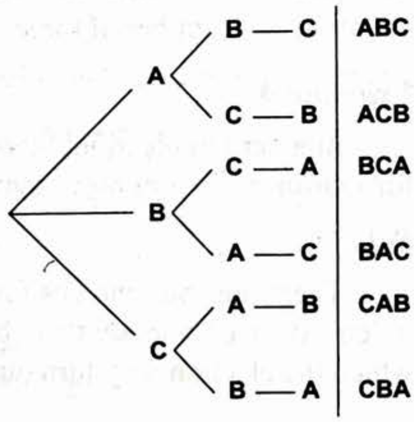
We very often come across problems in which we have to compute the number of ways a set of objects can be arranged under some conditions. As examples concerning these problems, we can have, in particular in the probability theory. To tackle them we have to study different counting methods. The study of counting methods comes under the field of combinatorial algebra. From this field, we have selected only the topics, permutation and combination which will prove helpful in our future study.

The Basic Principle of Counting

This principle can be best understood with some examples. Let us consider three letters A, B, C and see in how many ways they can be arranged in a row.

ABC BCA CAB

ACB BAC CBA



Thus we see that they can be arranged in six ways. This can be explained in a simple way. When we choose a first letter we can choose any one of the three. So there are three choices. For each choice of the first letter there are only two letters left from which we have to choose the second letter. So these first two letters can be chosen in 3.2 = 6 ways. For the third choice there is no alternative but to choose the only letter left. Thus we have 3.2.1 = 6 possibilities. This is illustrated more clearly in a tree diagram shown aside.

This consideration leads us to the following principle known as the **basic principle of counting**. "If one thing can be done independently in n_1 different ways and if a second thing can be done in n_2 different ways and if a third thing can be done in n_3 ways and so on (for any finite number of things), then the total number of ways in which all the things can be done in the stated order is $n_1 \, n_2 \, n_3 \, \ldots\ldots$".

Worked Out Examples

Example 1

8 buses run between Kathmandu and Biratnagar. In how many ways can a man travel from Kathmandu to Biratnagar and return by a different bus?

Solution:

A man can travel from Kathmandu to Biratnagar by 8 different ways. As he has to return by different bus, so he can return by 7 different ways. Now by the basic principle of counting, the total number of ways $= 8 \times 7 = 56$

Example 2

Find how many numbers of two different digits can be formed from the integers 1, 2, 3 and 4.

Solution :

There are 4 choices for an integer in the tens digit and 3 choices in the units digit. So there are altogether 4.3 = 12 two-digit numbers that can be formed from four given integers.

Example 3

How many odd numbers of three digits can be formed from the integers 1, 2, 3, 4 and 5?

Solution:

For the three digits odd numbers, the digit in the unit's place must be 1, 3 or 5. So, the unit's place can be filled up in 3 ways. After filling up the unit's place, 4 integers are left. The ten's place can be filled by 4 ways and hundred's place by 3 ways. Then by the basic principle of counting, the number of three digit odd numbers $= 3 \times 4 \times 3 = 36$

Example 4

In a certain election, there are four candidates for president, six for secretary and only two for treasurer. Find in how many ways the election may turn out.

Solution :

There are four choices for president, six for secretary and two for treasurer. As they are all independent of one another, by the basic principle of counting there are 4.6.2 = 48 ways in which the election may turn out.

Example 5

How many three digit numbers less than 400 can be formed from the digits 1, 2, 3, 4, 5, 6 if no digit is repeated? How many of these are divisible by 5?

Solution:

The numbers formed must be of three digits and less than 400, so the digit in the hundred's place must be 1, 2 or 3. Thus, there are 3 ways of filling up the hundred's place. After filling the hundred's place, 5 digits are left. So, the ten's place can be filled up in 5 ways and the unit's place by 4 ways

$\therefore$ no. of three digit numbers less than $400 = 3 \times 5 \times 4 = 60$

Again for the numbers divisible by 5, we fix the digit 5 in the unit's place. So, there is only one choice for filling up the unit's place. There are 3 ways of filling up the hundred's place and 4 ways of filling up the ten's place.

$\therefore$ no. of three digits numbers less than 400 and divisible by 5 $= 1 \times 3 \times 4 = 12$

EXERCISE

1. A football stadium has four entrance gates and nine exists. In how many different ways can a man enter and leave the stadium?
2. There are six doors in a hostel. In how many ways can a student enter the hostel and leave by a different door?
3. In how many ways can a man send three of his children to seven different colleges of a certain town?
4. Suppose there are five main roads between the cities A and B. In how many ways can a man go from a city to the other and return by a different road?
5. There are five main roads between the cities A and B and 4 between B and C. In how many ways can a person drive from A to C and return without driving on the same road twice?
6. How many numbers of at least three different digits can be formed from the integers 1, 2, 3, 4, 5, 6?
7. How many numbers of three digits less than 500 can be formed from the integers 1, 2, 3, 4, 5, 6?
8. Of the numbers formed by using all the figures 1, 2, 3, 4, 5 only once, how many are even?
9. How many numbers between 4000 and 5000 can be formed with the digits 2, 3, 4, 5, 6, 7?
10. How many numbers of three digits can be formed from the integers 2, 3, 4, 5, 6? How many of them will be divisible by 5?

Answers

1. 36	2. 30	3. 210	4. 20	5. 240
6. 1920	7. 80	8. 48	9. 60	10. 60, 12

Factorial notations

In many cases, we come across the products of the type 1.2.3.4 or 1.2.3.4.5 etc. These products are denoted by a notation known as the factorial notation.

The product 1.2.3.4 is written as 4! or $\lfloor\underline{4}$ and is known as factorial 4.

The product 1.2.3.4.5 written as 5! or $\lfloor\underline{5}$ and is known as factorial 5.

Thus if $n \in N$, then the continued product from 1 to n is known as factorial n and is written as n! or $\lfloor\underline{n}$.

Thus $\quad n! = 1.2.3.4.5 \ldots\ldots (n-1).n$

So, $1! = 1, \quad 2! = 1.2, \quad 3! = 1.2.3 \quad$ and $\quad 4! = 1.2.3.4$

Also, we observed that

$$5! = 1.2.3.4.5 = 5(1.2.3.4) = 5 \times 4!$$
$$12! = 12 \times (12-1)! = 12 \times 11!$$
$$15! = 15 \times (15-1)! = 15 \times 14!$$

Also, $n! = n \times (n-1)! = n.(n-1) \times (n-2)!$

It is to be noted that 0! = 1 though by definition 0! has no meaning.

Example

Find : $\dfrac{8!-6!}{3!}$

Solution:

$$\frac{8!-6!}{3!} = \frac{8 \times 7 \times 6! - 6!}{3 \times 2 \times 1} = \frac{6! \times (56-1)}{6} = \frac{6 \times 5! \times 55}{6}$$
$$= 5 \times 4 \times 3 \times 2 \times 1 \times 55 = 6600$$

Permutations

a) Set of objects all different

Permutation of a set of objects means an arrangement of objects in some order. Consider the numbers 456, 654. Both of them consist of the same digits 4, 5, 6. But they are arranged in different order. So they are different permutations of the digits 4, 5, 6. So we can form many different permutation from a given set of objects taken all at a time or taken particular number of objects at a time. The number of permutations that can be formed taken r at a time out of n given objects is given by the following theorem. We shall denote this number of permutations by $P(n, r)$ or by ${}^nP_r$.

Theorem 1

The total number of permutations of a set of n objects taken r at a time is given by

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1) \quad (n \geq r).$$

Proof :

The number of permutations of a set of n objects taken r at a time is equivalent to the number of ways in which r positions can be filled up by those n objects. Now there are n choices to fill up the first position. When the first position has been filled up, there will be left only $n-1$ objects to fill up the second position, i.e., there are $n-1$ choices to fill up the second position.

Similarly there are $n-2$ choices to fill up the third position and so on. Ultimately to fill up the r^{th} position, there are $n-(r-1) = n-r+1$ choices. Then by the basic principle of counting the number of ways in which r positions can be filled up is

$$n(n-1)(n-2) \ldots (n-r+1)$$

Therefore,

$$P(n, r) = n(n-1)(n-2) \ldots (n-r+1)$$
$$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 3.2.1}{(n-r)\ldots 3.2.1.} = \frac{n!}{(n-r)!}.$$

Corollary. The permutations of a set of n things taken n (or all) at a time given by

$$P(n, n) = n(n-1)(n-2) \ldots 1 = n!$$

Take $r = n$. The proof immediately follows.

Example 1

Find P(15, 2)

Solution:

$$P(15, 2) = \frac{15!}{(15-2)!} = \frac{15!}{13!} = \frac{15 \times 14 \times 13!}{13!}$$
$$= 210$$

Example 2

Find the number of permutations of the three letters A, B, C taken all at a time.

Solution:

The number of permutations of 3 letters taken all at a time is $P(3, 3) = 3! = 6$.

Example 3

How many license plates consisting of 3 different digits can be made out of given integers 3, 4, 5, 6, 7 ?

Solution :

This is just like arranging 3 objects out of 5 objects. So, we have

$$P(5, 3) = \frac{5!}{(5-3)!} = \frac{5.4.3.2.1}{2.1} = 5.4.3 = 60$$

b) Permutation of objects not all different

To find the permutation of n objects taken all at a time when p of the objects are of first kind, q of them are of second kind, r of them are of the third kind and the rest all are different.

Let the total number of permutations of n objects taken all at a time be x. Out of these n objects, p of the objects of the first kind be replaced by p new objects different from one another and also different from each of the remaining objects. If these p different objects be arranged among themselves, keeping the positions of all other objects same, they will give $p!$ permutations corresponding to each x permutations. Hence there will be $x \times p!$ permutations in all. In the same way if q of the objects of the second kind be replaced by q new objects different from one another and from the remaining objects, in each of $x \times p!$ permutations, the total permutations is $x \times p! \times q!$. Similarly, if r of the objects be replaced by r different objects then we will have $x \times p! \times q! \times r!$ as the total number of permutations when all objects are different.

But the total permutations of n different objects taken all at a time is $n!$

$$\therefore \quad x \times p! \times q! \times r! = n! \quad \Rightarrow \quad x = \frac{n!}{p!\,q!\,r!}$$

$$\therefore \text{ the total number of permutations} = \frac{n!}{p!\,q!\,r!}$$

Example

In how many ways can the letters of the word "MEMBER" be arranged?

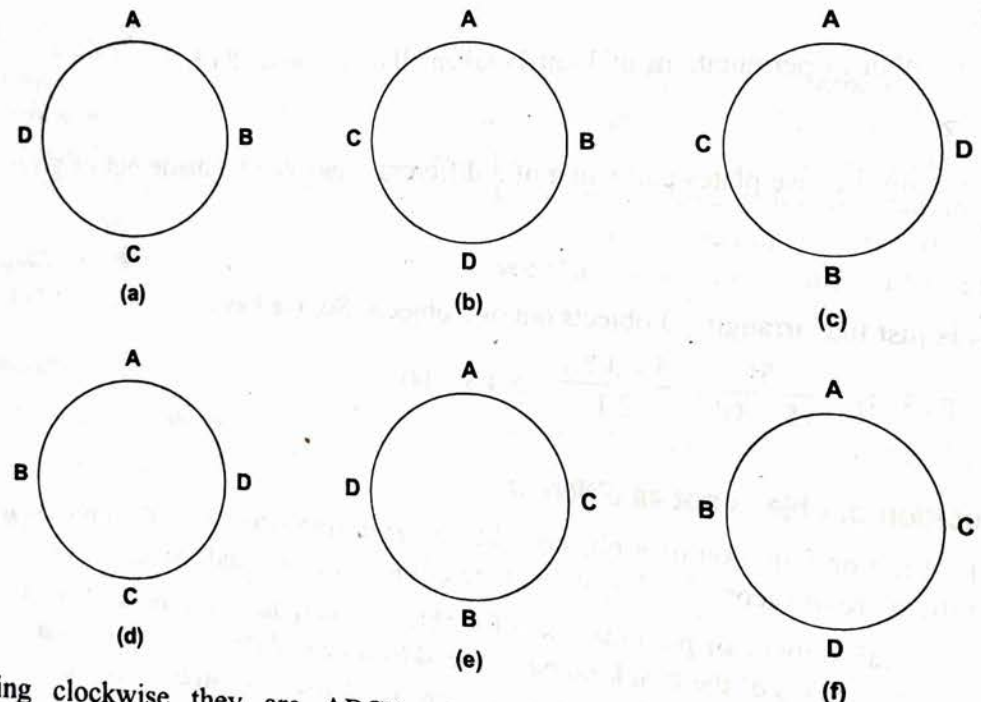
Solution:

The word "MEMBER" consists of 6 letters, 2 of which are M's, 2 are E's and the rest are different. So, writing n = 6, p = 2 and q = 2 the required number of permutations

$$= \frac{n!}{p!\,q!} = \frac{6!}{2!\,2!} = 180$$

c) Circular Permutation

If the four letters A, B, C, D be arranged in a circle, we can easily see that the following six different arrangements are possible and no more.



Going clockwise they are ABCD, ABDC, ADBC, ADCB, ACBD, ACDB. The arrangements BCDA, CDAB, DABC are same as the arrangement ABCD.

If they were arranged in a line they will be all different. Hence for every circular arrangement there are four arrangements in a line. This shows that if P be the number of circular arrangements, 4P is the number of arrangements in a line. Hence 4P = 4! or P = 3! = 6

We may arrive at the same conclusion by the following reasoning.

We have to fill in 4 blank spaces in a circle with 4 different objects A, B, C, D. Let any one of them, say A, be placed in one of the spaces so there are three things left to be arranged in three vacant spaces which can be done in 3! ways.

This can be generalised to the following theorem.

Theorem 3.

The total number of permutations of a set of n objects arranged in a circle is $(n-1)!$.

Proof:

Let $a_1, a_2 \ldots, a_n$ be n objects arranged in a circle. Then the arrangements $a_1\, a_2\, a_3 \ldots a_n$, $a_2\, a_3\, a_4 \ldots a_n\, a_1$, $a_3\, a_4 \ldots a_n\, a_1\, a_2, \ldots$ are not distinguishable. But if they are arranged in a straight line, to each arrangement in a circle, there are n arrangements in a line. Hence if P is the number of arrangements in a circle, we have

$$nP = n! \quad \text{or} \quad P = \frac{n!}{n} = (n-1)!$$

Note :

If the clockwise and anticlockwise arrangements are not to be taken as different as in the case of making a necklace of beads or arranging people in a circle in such a way that they do not have the same neighbours in any two arrangements then the number of arrangements will be $\frac{1}{2}(n-1)!$. In the illustration above the arrangements (a) and (b) are same, so are (c) and (d) as also (e) and (f), if we consider the clockwise and anticlockwise arrangements to be same. Hence the number of arrangements is $\frac{1}{2} \times 3! = 3$.

However, if the arrangements be with respect to the seats (i.e. think of the seats to be numbered) then the one that was fixed first can be done so in n ways as he can be given any one of the n different seats. In this case the number of arrangements will be $n \times (n-1)! = n!$.

Example

In how many ways can the numbers on a clock face be arranged ?

Solution:

In a clock face there are 12 numbers. So they can be arranged in $(12-1)! = 11!$ ways

d) Permutation of repeated things

To find the permutation of n things taken r at a time when each thing may occur any number of times.

Suppose there are r places and there are n objects. The first place can be filled up by any of the n objects. So there are n choices for filling up the first place. After filling up the first place, the second place can also be filled up by n objects as the object occupying the first place may also occupy the second place. Thus the first two places can be filled up in $n \times n = n^2$ ways. Similarly the third place can be filled in n ways and the first three places can be filled up in $n \times n \times n = n^3$ ways.

Proceeding in the same way, the r places can be filled by n objects in

$$n \times n \times n \times n \ldots\ldots \times r \text{ times} = n^r \text{ ways}$$

Example

Find the number of ways in which three letters can be arranged in a row with repetition.

Solution:

The first letter can be arranged in 3 ways. As the repetition is allowed, the second letter can be arranged in 3 ways. Similarly, the third letter can be arranged in 3 ways. So, the total of ways of arranging the three letters is $3 \times 3 \times 3 = 27$.

Worked Out Examples

Example 1

Find the value of P(n, 2) if 16 P(n, 3) = 13 P(n + 1, 3)

Solution:

$$16\,P(n, 3) = 13\,P(n+1, 3)$$

$$\Rightarrow\ 16\frac{n!}{(n-3)!} = 13\frac{(n+1)!}{(n-2)!}$$

$$\Rightarrow\ 16\frac{n!}{(n-3)!} = \frac{13.(n+1) \times n!}{(n-2) \times (n-3)!}$$

$$\Rightarrow\ 16(n-2) = 13(n+1)$$

$$\Rightarrow\ 16n - 13n = 13 + 32$$

$$\Rightarrow\ 3n = 45$$

$$\therefore\ n = 15$$

Now, $P(n, 2) = P(15, 2)$

$$= \frac{15!}{13!} = \frac{15 \times 14 \times 13!}{13!} = 210$$

Example 2

In how many ways can 5 boys and 3 girls be arranged for a group photograph if the girls are to sit on the chairs in a row and the boys are to stand in a row behind them.

Solution:

Here the boys and the girls are to be arranged separately. Firstly we arrange 5 boys in a row.

5 boys in a row can be arranged in P(5, 5) ways

i.e. $\frac{5!}{0!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1} = 120$ ways

Now, we arrange 3 girls

3 girls in a row can be arranged in P(3, 3) ways

i.e. $\frac{3!}{0!} = \frac{3 \times 2 \times 1}{1} = 6$ ways

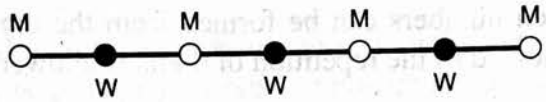
$\therefore$ total number of arrangements = 120 × 6 = 720 ways

Example 3

Find the number of ways in which 4 men and 3 women can be seated in a row having seven seats so that the men and the women must alternate.

Solution:

Since 7 persons are to be arranged in a row with 7 seats so that the men and the women are in alternate, so men are to be arranged in odd seats and women in even seats.



There are 4 seats for 4 men, so they can be arranged in P(4, 4) ways

i.e. $\frac{4!}{0!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24$ ways

Again there are 3 seats for 3 women so they can be arranged in 3! ways

i.e. 3 × 2 × 1 = 6 ways

$\therefore$ total numbers of arrangements = 24 × 6 = 144 ways

Example 4

In how many ways can the letters of the word 'CALCULUS' be arranged so that the two C's do not come together?

Solution:

There are 8 letters in the word 'CALCULUS'

Also there are 2C's and 2U's and 2L's and the rest are different.

The total number of arrangements without restriction $= \frac{8!}{2!\,2!\,2!} = 5040$

Firstly, we find the arrangements each of which contain 2 C's together. For this, we consider 2 C's as one. Now, there are 7 letters in which there are 2 U's and 2 L's.

The number of arrangements in which 2C's are always together $= \frac{7!}{2!\,2!} = 1260$

Now, the required no. of arrangements in which 2 C's do not come together

= 5040 – 1260 = 3780

Example 5

How many different numbers of six digits can be formed with the digits 3, 1, 7, 0, 9, 5? How many of them will have 0 in the unit's place?

Solution:

The number of 6 digits numbers formed from given digits $= P(6, 6) = 6! = 720$

Some of these numbers have 0 in the first place. For the number of 6-digits, each of which has 0 in the first place we keep, one digit 0 fixed, so there are only 5 digits left.

The number of 6-digit numbers with 0 in the first place $= P(5, 5) = 5! = 120$ ways

$\therefore$ the required no. of 6-digit significant numbers $= 720 - 120 = 600$

For the numbers with 0 in the unit's place, we fix 0.

So, the required number of 6-digit numbers with 0 in the unit's place

$$= P(5, 5) = 5! = 120$$

Example 6

How many 4 digit even numbers can be formed from the digits 4, 5, 6, 7, 8, 9 if a) the repetition of digits is not allowed b) the repetition of digits is allowed?

Solution:

a) Since the numbers to be formed are even, so the digit in the unit's place must be 4 or 6 or 8 i.e. there are 3 choices for unit's place to fill up. After filling the unit's place, the number of digits left = 6 – 1 = 5 and the number of places still to be filled up = 3. These 3 places can be filled by P(5, 3) ways i.e $\frac{5!}{(5-3)!} = 5 \times 4 \times 3$ ways = 60 ways

$\therefore$ total no. of 4 digit numbers $= 3 \times 60 = 180$

b) The unit's place has 3 choices. For the rest of the places i.e. ten's place, hundred's place and thousand's place, each can be filled by 6 ways

$\therefore$ total no. of 4 digit numbers $= 3 \times 6 \times 6 \times 6 = 648$

Example 7

How many different 4 digit numbers can be formed from the digits 2, 3, 4, 5, 6? How many of them are divisible by 5? How many of these numbers are not divisible by 5?

Solution:

n = given no. of digits = 5

r = No. of digits to be taken = 4

P(n, r) = ?

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\Rightarrow P(5, 4) = \frac{5!}{(5-4)!} = \frac{5!}{1!}$$

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$\therefore$ total no. of 4 digit numbers = 120

To form 4-digit numbers divisible by 5, 5 must be always in the unit's place. So, there is only one choice for the unit's place. After that, remaining 3 places are to be filled up by remaining 4 digits which can be done in P(4, 3) ways

i.e. $\frac{4!}{1!} = 4 \times 3 \times 2 \times 1$ ways = 24 ways

$\therefore$ total number of 4 digit numbers divisible by 5 = 24

Also total number of 4 ditit numbers not divisible by 5 = 120 – 24 = 96

Example 8

Eight members of a committee sit at a round table. In how many ways can they be seated if the president and secretary choose to sit together?

Solution:

Since the president and the secretary choose to sit together, so we consider them to be one. So, there are 7 members to be arranged. These 7 members in the circular table can be arranged in (7 – 1)! i.e. 6! = 720 ways

But the positions of the president and the secretary can be interchanged in 2 ways

$\therefore$ total number of arrangements = 2 × 720 = 1440 ways.

Example 9

In how many ways can 6 different beads be strung on a necklace?

Solution:

The 6 different beads in a necklace can be strung in (6 – 1)! ways

i.e. 5! = 120 ways.

Since the clockwise and the anticlockwise arrangements are the same, so the required no. of arrangement = $\frac{1}{2} \times 120 = 60$ ways.

Example 10

In how many ways can 3 letters be posted in 4 letter boxes?

Solution:

The first letter can be posted in 4 ways.

Similarly, each of second and third can be posted in 4 ways

$\therefore$ total no. of ways = 4 × 4 × 4 = 64

Example 11

In how many ways can the letters of the word 'LOGIC' be arranged so that

a) the vowels may occupy odd position?

b) no two vowels are together?

c) the relative positions of the vowels and consonants are not changed?

Solution:

There are five letter in the word 'LOGIC'.

a) There are 2 vowels and 3 positions for them. So, 2 vowels can be arranged in

$P(3, 2) = 3!$ ways

Also, the remaining 3 letters (i.e. L, G, C) can be arranged in $P(3, 3) = 3!$ ways

$\therefore$ total no. of arrangements $= 3! \times 3! = 36$ ways.

b) All letters are different, so they can be arranged in $P(5, 5) = 5!$ ways

Considering 2 vowels as one, there are 4 letters L, (OI), G, C. They can be arranged in 4! ways.

But the two vowels can be interchanged their positions in 2 ways.

$\therefore$ no. of arrangements when two vowels always come together $= 4! \times 2$

$\therefore$ no. of arrangements when no two vowels are together $= 5! - 2 \times 4! = 72$ ways

c) The three consonants can be arranged in 3! ways and the two vowels can be arranged in 2! ways

$\therefore$ total no. of arrangements $= 3! \times 2! = 12$ ways.

EXERCISE

1. Find the number of permutations of five different objects taken three at a time.
2. If three persons enter a bus in which there are ten vacant seats, find in how many ways they can sit.
3. a) How many plates of vehicles consisting of 4 different digits can be made out of the integers 4, 5, 6, 7, 8, 9? How many of these numbers are divisible by 2?
   b) How many numbers of 4 different digits can be formed from the digits 2, 3, 4, 5, 6, 7? How many of these numbers are i) divisible by 5 ii) not divisible by 5.
   c) How many 5 digit odd numbers can be formed using the digits 3, 4, 5, 6, 7, 8 and 9. If i) repetition of digits is not allowed ii) repetition of digits is allowed?
4. In how many ways can four boys and three girls be seated in a row containing seven seats
   a) if they may sit anywhere
   b) if the boys and girls must alternate
   c) if all three girls are together?
   d) if girls are to occupy odd seats
5. In how many ways can eight people be seated in a row of eight seats so that two particular persons are a) always together b) never together?
6. Six different books are arranged on a shelf. Find the number of different ways in which the two particular books are a) always together b) not together.
7. In how many ways can four red beads, five white beads and three blue beads be arranged in a row?

8. In how many ways can the letters of the following words be arranged?
   a) ELEMENT　　b) NOTATION
   c) MATHEMATICS　　d) MISSISSIPPI
9. How many numbers of 6 digits can be formed with the digits 2, 3, 2, 0, 3, 3?
10. In how many ways can 4 Art students and 4 Science students be arranged in a circular table if a) they may sit any where b) they sit alternately.
11. In how many ways can eight people be seated in a round table if two people insist in sitting next to each other?
12. In how many ways can seven different coloured beads be made into a bracelet?
13. a) In how many ways can 4 letters be posted in six letter boxes?
    b) How many even numbers of 3 digits can be formed when repetition of digits is allowed?
    c) In how many ways can 3 prizes be distributed among 4 students so that each student may receive any number of prizes?
14. In how many ways can the letters of the word "MONDAY" be arranged? How many of these arrangements do not begin with M? How many begin with M and do not end with Y?
15. Show that the number of ways in which the letters of the word
    a) "COLLEGE" can be arranged so that the two E's always come together is 360.
    b) "ARRANGE" can be arranged so that no two R's come together is 900.
16. In how many ways can the letters of the word 'COMPUTER' be arranged so that
    a) all the vowels are always together?
    b) the vowels may occupy only odd positions?
    c) the relative positions of vowels and consonants are not changed?
17. Find the number of arrangements of the letters of the word "Laptop" so that
    a) the vowels may never be separated;
    b) all consonants may not be together;
    c) they always begin with L and end with T
    d) they do not begin with L but always end with T.
18. How many different words can be formed with all the letters of the word "Internet" if
    a) each word is to begin with vowel?
    b) each word is to end with consonant?

Answers

1. 60　　2. 720　　3. a) 360, 180　　b) 360, i) 60　　ii) 300　　c) i) 1440　　ii) 9604

4. a) 5040　　b) 144　　c) 720　　d) 576　　5. a) 10080　　b) 30240

6. a) 240　　b) 480　　7. $\frac{12!}{3!\,4!\,5!}$　　8. a) 840　　b) $\frac{8!}{2!\,2!\,2!}$

c) $\frac{11!}{2!\,2!\,2!}$ d) $\frac{11!}{4!\,4!\,2!}$ 9. 50 10. a) 5040 b) 144 11. 1440

12. 360 13. a) 1296 b) 450 c) 64 14. 720, 600, 96

16. a) 4320 b) 2880 c) 720 17. a) 120 b) 288 c) 12 d) 48

18. a) 1890 b) 3150

Combinations

Combination of objects means just their collection without any regard to order or arrangement. The absence of order in the combination of objects makes it different from the permutations of the objects. There is only one combination of n objects; but for the same n objects the number of permutations is $n!$. Similarly, the number of combinations of n objects taken r at a time is less than the number of permutations of n objects taken r at a time. The combination of n things taken r at a time is denoted by ${}^nC_r$ or $C(n, r)$.

Theorem 5.

The total number of combinations of n objects taken r at a time, $C(n, r)$, is given by the expression $C(n, r) = \frac{n!}{(n-r)!\,r!}$

Proof.

Consider any one of the $C(n, r)$ combinations. This combination contains r objects, these r objects can be arranged among themselves in $r!$ different ways. So for each combination there are $r!$ permutations. Consequently for the $C(n, r)$ possible combinations, there are $C(n, r).r!$ different permutations. Since these are all possible permutations of n objects taken r at a time, we have,

$$C(n, r).\,r! = P(n, r) = \frac{n!}{(n-r)!}$$

$$C(n, r) = \frac{n!}{(n-r)!\,r!}$$

This proves the theorem.

Cor.1 Complementary combination:

The number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $n - r$ at a time. Symbolically,

$$C(n, r) = C(n, n-r).$$

Using theorem 5, we have

$$C(n, n-r) = \frac{n!}{(n - n + r)!\,(n-r)!}$$

$$= \frac{n!}{r!(n-r)!} = C(n, r)$$

$$\therefore \quad C(n, r) = C(n, n-r)$$

Cor.2: If $C(n, r) = C(n, r')$, then $r + r' = n$

Since, $C(n, r) = C(n, r')$

or, $C(n, r) = C(n, n - r')$

So, $r = n - r'$

$r + r' = n$

Also, $C(n, r) = C(n, r')$

$\Rightarrow r = r'$

$\therefore$ if $C(n, r) = C(n, r')$ then either $r = r'$ or $r + r' = n$.

Cor. 3 $C(n, r) + C(n, r - 1) = C(n + 1, r)$

Using theorem 5, we have

$$C(n, r) = \frac{n!}{(n-r)!\, r!} \quad \text{and} \quad C(n, r-1) = \frac{n!}{(n-r+1)!\,(r-1)!}$$

Now, $C(n, r) + C(n, r - 1)$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\,\frac{n+1}{r\,(n-r+1)}$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!} = C(n+1, r)$$

$\therefore C(n, r) + C(n, r - 1) = C(n+1, r)$

Worked Out Examples

Example 1

Calculate the value of C(12, 9).

Solution :

We have $C(12, 9) = C(12, 12 - 9) = C(12, 3)$

$$= \frac{12!}{9!\,3!} = \frac{10.11.12}{1.2.3} = 220$$

Example 2

a) If C(18, r) = C(18, r + 2), find r and C(r, 3).

b) C(21, 2r + 1) = C(21, 3r – 5), find the value of 4.

Solution:

a) $C(18, r) = C(18, r + 2)$

$\Rightarrow C(18, r) = C(18, 18 - r - 2)$

$\Rightarrow C(18, r) = C(18, 16 - r)$

$\Rightarrow \quad r = 16 - r$

$\Rightarrow \quad 2r = 16$

$\therefore \quad r = 8$

Now, $\quad C(r, 3) = C(8, 3) = \dfrac{8!}{5!\,3!} = \dfrac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} = 56$

b) $\quad C(21, 2r + 1) = C(21, 3r - 5)$

$\Rightarrow \quad 2r + 1 = 3r - 5$

$\Rightarrow \quad 2r - 3r = -5 - 1$

$\Rightarrow \quad -r = -6$

$\therefore \quad r = 6$

Again, $C(21, 2r + 1) = C(21, 3r - 5)$

$\Rightarrow \quad 2r + 1 + 3r - 5 = 21 \qquad (\because r + r' = n)$

$\Rightarrow \quad 5r - 4 = 21$

$\Rightarrow \quad 5r = 25$

$\therefore \quad r = 5$

$\therefore r = 5$ or 6

Example 3

In an examination paper containing 10 questions a candidate has to answer 7 questions only; in how many ways can he choose the questions ? If two questions are made compulsory, in how many ways can he choose 7 questions in all?

Solution:

The candidate can choose 7 questions out of 10 in C (10, 7) ways, which gives

$$\frac{10\,!}{(10 - 7)\,!\,7!} = \frac{8.9.10.}{1.2.3} = 120$$

If two questions are compulsory, then the candidate has to select 5 questions from the remaining 8 questions. This selection can be made in C(8, 5) ways

i.e. $\quad \dfrac{8!}{(8-5)!\,5!} = \dfrac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 1 \times 5!} = 56$ ways

Example 4

A committee is to be chosen from 12 men and 8 women and is to consist of 3 men and 2 women. How many committee can be formed ?

Solution:

3 men can be chosen from 12 men in

$$C\,(12, 3) = \frac{12\,!}{(12 - 3)\,!\,3!} = 220.$$

2 women can be chosen from 8 women in

$$C(8, 2) = \frac{81}{(8-2)!\,2!} = 28 \text{ ways.}$$

Therefore, total number of committees = 220 × 28 = 6160.

Example 5

5 men in a group of 12 are graduates. In how many committees of 6 members be made so that each committee may consist of 3 graduates?

Solution:

There are 12 men of which 5 are graduates so 7 of them are non-graduates. Now selection of 3 graduates from 5 graduates and rest 3 from remaining 7 non-graduates are to be made.

3 graduates from 5 graduates can be selected in C(5, 3) ways.

Again 3 non-graduates from 7 can be selected in C(7, 3) ways

∴ total no. of committees = C(5, 3) × C(7, 3)

$$= \frac{5!}{2!\,3!} \times \frac{7!}{4!\,3!}$$

$$= \frac{5 \times 4 \times 3!}{2 \times 3!} \times \frac{7 \times 6 \times 5 \times 4!}{4! \times 3 \times 2 \times 1}$$

$$= 10 \times 35 = 350$$

Example 6

From 6 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include at least 3 ladies?

Solution:

The selection of the members in the committee can be made as follows:

Ladies (4)	Gentlemen (6)	Selection
3	2	C(4, 3) × C(6, 2)
4	1	C(4, 4) × C(6, 1)

∴ the required no. of committees

$$= C(4, 3) \times C(6, 2) + C(4, 4) \times C(6, 1)$$

$$= \frac{4!}{1!\,3!} \times \frac{6!}{4!\,2!} + \frac{4!}{0!\,4!} \times \frac{6!}{5!\,1!}$$

$$= 4 \times \frac{6 \times 5}{2} + 1 \times 6 = 66$$

Example 7

In an examination, a candidate has to pass in each of the 4 subjects. In how many ways can he fail?

Solution:

A candidate fails in an examination if he can not pass either in 1 or 2 or 3 or 4 subjects.

∴ total no. of ways by which he fails

$$= C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4)$$

$$= \frac{4!}{3!\,1!} + \frac{4!}{2!\,2!} + \frac{4!}{1!\,3!} + \frac{4!}{0!\,4!}$$

$$= 4 + 6 + 4 + 1 = 15$$

EXERCISE

1. A boy puts his hand into a bag which contains 10 differently coloured marbles and brings out 3. How many different results are possible?
2. Find the number of ways in which a student can select 5 courses out of 8 courses. If 3 courses are compulsory, in how many ways can the selections be made?
3. From 10 persons, in how many ways can a selection of 4 be made
   i) when one particular person is always included?
   ii) when two particular persons are always excluded?
4. A bag contains 8 white balls and 5 blue balls. In how many ways can 5 white balls and 3 blue balls be drawn?
5. How many committees can be formed from a set of 7 boys and 5 girls if each committee contains 4 boys and 3 girls?
6. From a group of 11 men and 8 women, how many committees consisting of 3 men and 2 women are possible?
7. From 4 mathematicians, 6 statisticians and 5 economists, how many committees of 6 members can be formed so as to include 2 members from each category?
8. A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them may be relatives?
9. There are ten electric bulbs in the stock of a shop out of which there are three defectives. In how many ways can a selection of 6 bulbs be made so that 4 of them may be good bulbs?
10. From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady?
11. A candidate is required to answer 6 out of 10 questions which are divided into 2 groups each containing 5 questions and he is not permitted to attempt more than 4 from any group. In how many different ways can he make up his choice?
12. A man has 5 friends. In how many ways can he invite one or more of them to a dinner?
13. a) If C(20, r + 5) = C(20, 2r − 7), find C(15, r).

b) If $C(n, 10) + C(n, 9) = C(20, 10)$ find n and $C(n, 17)$

c) Solve for n the equation $C(n + 2, 4) = 6C(n, 2)$

d) If $P(n, r) = 336$ and $C(n, r) = 56$, find n and r.

e) If ${}^nC_{r-1} = 45$, ${}^nC_r = 120$ and ${}^nC_{r+1} = 210$, find n and r.

14. An examination paper consisting of 10 questions, is divided into two groups A and B. Group A contains 6 questions. In how many ways can an examinee attempt 7 questions
    a) selecting 4 from group A and 3 from group B?
    b) selecting atleast two questions from each group?

15. Six men in a group of 8 are skilled. Find the number of ways by which 5 men can be selected such that
    a) atleast 3 of them may be the skilled men.
    b) atleast one of them may be the unskilled man.

16. In a group of 10 students, 6 are boys. In how many ways can 4 students be selected for mathematical competition so as to include
    a) exactly two boys
    b) atleast two boys
    c) at most two girls.

A. Can you show that $0! = 1$ though it has no meaning from the definition.

B. Perform the following experiment.

Given 4 objects arranged then taking 1, 2, 3, 4 at a time and then generalize the result for n objects taking r at a time and show that ${}^np_4 = P(n, r) = \frac{n!}{(n-r)!}$

C. Perform the following experiment

Given 4 objects, select them taking 1, 2, 3, 4 at a time and then generalize the result for n objects taken r at a time and show that

i) ${}^nC_r = C(n, r) = \frac{n!}{(n-r)!\, r!}$ ii) ${}^np_r = r! = C(n, r)$

Answers

1. 120	2. 56, 10	3. i) 84	ii) 70	4. 560	5. 350	6. 4620
7. 900	8. 336	9. 105	10. 246	11. 20012. 31	13. a) 455	b) 19, 171
c) 7	d) 8, 3	e) 10, 3	14. a) 60	b) 116	15. a) 56	b) 50
16. a) 90	b) 185	c) 185				

Chapter 2
Binomial Theorem, Exponential and Logarithmic Series

Introduction

By a binomial expression, we mean an expression consisting of two terms. For example, $x + a$, $x + y$, $x + b$ are all binomial expressions. It is not difficult to expand the power of a binomial expression, when the power is a very small positive integer such as 2, 3, 4. But it is difficult to expand, when the power is big. So, we need a formula for the expansion of binomial expression with any index or power. That formula is called Binomial Theorem. Here we shall be just content with a proof, when the index of the binomial expression is any positive integer. The Binomial theorem was first discovered by Issac Newton.

Binomial Theorem

For any positive integer n,

$$(a + x)^n = C(n, 0)\, a^n + C(n, 1)\, a^{n-1}x + C(n, 2)\, a^{n-2}x^2 + \ldots\ldots + C(n, r)a^{n-r}x^r + \ldots\ldots + C(n, n)\, x^n.$$

Proof :

We have, $(a + x)^n = (a + x)\,(a + x)\,(a + x) \ldots\ldots$ to n factors.

In the process of multiplication of n factors in the right hand side, we shall choose either a or x from each factor and get n letters. If we choose a from each factor we get a^n. If we choose a from each of $n - 1$ factors and x from the remaining one, we get $a^{n-1}x$. In this way we get a term $a^{n-r}x^r$ by choosing a from $n - r$ factors and x from the other r factors. Moreover, the number of times $a^{n-r}x^r$ appears in the expression is equal to the number of combination of n taken r at a time which is $C(n, r)$. Now if we allow r to vary from 0 to n we get the required expansion. Now let us note some of the properties of the expression, when n is any positive integer.

(i) The number of terms in the expansion is $n + 1$.

(ii) In each term, the sum of the exponents is n.

(iii) The expansion starts with the first term a^n and ends with the last term x^n. When we pass from one term to the next, we find that the exponent of a decreases by one and that of x increases by one.

(iv) The coefficients of the terms equidistant from the beginning and the end are always equal.

Alternative Method

By actual multiplication, we have

$$(a+x)^2 = a^2 + 2ax + x^2$$
$$= C(2, 0)\,a^2 + C(2, 1)ax + C(2, 2)x^2$$
$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$
$$= C(3, 0)a^3 + C(3, 1)a^2x + C(3, 2)ax^2 + C(3, 3)x^3.$$

and so on.

Thus, the theorem holds good when $n = 2$ and $n = 3$.

Let us assume that the theorem is true when n has some particular value, say m, so that

$$(a+x)^m = C(m, 0)a^m + C(m, 1)a^{m-1}x + C(m, 2)a^{m-2}x^2 + \dots\dots + C(m, r)a^{m-r}x^r + \dots\dots + C(m, m)x^m.$$

Multiplying both sides by $(a + x)$, we have

$$(a+x)^{m+1} = (a+x)\,\{C(m, 0)a^m + C(m, 1)a^{m-1}x + C(m, 2)a^{m-2}x^2 + \dots\dots + C(m, r)a^{m-r}x^r + \dots\dots + C(m, m)x^m\}$$

$$= C(m, 0)a^{m+1} + C(m, 1)a^m x + \dots + C(m, m)ax^m \dots + C(m, 0)a^m x + \dots + C(m, m-1)ax^m + \dots + C(m, m)\,x^{m+1}$$

$$= a^{m+1} + \{C(m, 1) + 1\}\,a^m x + \{C(m, 2) + C(m, 1)\}\,a^{m-1}x^2 + \{C(m, 3) + C(m, 2)\}a^{m-2}x^3 + \dots\dots + 1\cdot x^{m+1}.$$

$$= C(m+1, 0)a^{m+1} + C(m+1, 1)a^m x + C(m+1, 2)a^{m-1}x^2 + \dots\dots + C(m+1, m+1)x^{m+1}.$$

Therefore, if the theorem is true for $n = m$, it is true for $n = m + 1$ also.

But we have seen that the theorem is true for $n = 3$, therefore it must be true for $n = 4$ and hence for $n = 5$ and so on. That is, it is true for all positive integral values of n.

Cor : When $a = 1$, the above binomial expansion takes the following form:

$$(1+x)^n = 1 + C(n, 1)x + C(n, 2)x^2 + \dots\dots + C(n, r)x^r + \dots + x^n$$
$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n\,(n-1)\,(n-2)\,\dots\dots\,(n-r+1)}{r!}x^r + \dots + x^n.$$

Particular forms :

We have

$$(a+x)^n = C(n, 0)a^n + C(n, 1)a^{n-1}x + \dots\dots + C(n, n)x^n$$

Now putting $-x$ for x,

$$(a-x)^n = C(n, 0)a^n - C(n, 1)a^{n-1}x + C(n, 2)a^{n-2}x^2 - C(n, 3)a^{n-3}x^3 + \dots\dots + (-1)^n\,C(n, n)x^n.$$

Also we have

$$(1+x)^n = 1 + nx + \frac{n\,(n-1)}{2!}x^2 + \dots\dots + x^n$$

Putting $x = -x$, we have

$$(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \dots \dots + (-1)^n x^n.$$

General Term

The $(r + 1)$th term in the expansion of $(a + x)^n$ is usually called its *general term*, because any required term may be obtained from it, by giving a suitable value to r. The $(r + 1)$th term is denoted by t_{r+1}.

In the expansion of $(a + x)^n$

t_1 = 1st term $= C(n, 0)\, a^n$

t_2 = 2nd term $= C(n, 1)\, a^{n-1}x$

t_3 = 3rd term $= C(n, 2)\, a^{n-2}x^2$

t_4 = 4th term $= C(n, 3)\, a^{n-3}x^3$

t_{r+1} = $(r + 1)$th term $= C(n, r)\, a^{n-r}x^r$.

Using the general term, we may state the binomial theorem in the following abbreviated form

$$(a + x)^n = \sum_{r=0}^{n} C(n, r)a^{n-r}x^r$$

Cor. The general term in the expansion of $(a - x)^n$ is $(-1)^r . C(n, r)a^{n-r}x^r$.

Middle Term

Now let us find the middle term or terms in the expression of $(a + x)^n$. We have to consider the cases when n is an *even* number and when it is an *odd* number.

(i) When *n* is even

When n is even, we write $n = 2m$, where $m = 1, 2, 3, \dots$. The number of terms after expansion is $2m + 1$, which is odd. So, it has only one middle term, namely $(m + 1)$th term. So,

$$t_{m+1} = C(2m, m)\, a^m x^m = \frac{2m!}{(m!)^2} a^m x^m$$

$\therefore$ middle term $t_{m+1} = t_{\frac{1}{2}n+1} = C\left(n, \frac{1}{2}n\right) a^{n-n/2}x^{n/2}$

$$= \frac{n!}{\left\{\left(\frac{1}{2}n\right)!\right\}^2} a^{n/2}x^{n/2}$$

(ii) When *n* is odd

When n is odd, we write $n = 2m - 1$, where $m = 1, 2, 3, \dots$. The number of terms after expansion is $2m$, which is even. So, there will be two middle terms, namely mth and $(m + 1)$th term. So,

$$t_m = C(2m-1, m-1)\, a^m x^{m-1} = \frac{(2m-1)!}{m!\,(m-1)!} a^m x^{m-1}$$

and $$t_{m+1} = C(2m-1, m)\, a^{m-1} x^m = \frac{(2m-1)!}{m!\,(m-1)!} a^{m-1} x^m$$

$\therefore$ middle terms are

$$t_m = t_{\frac{n+1}{2}} = t_{\frac{n-1}{2}+1} = C\left(n, \frac{n-1}{2}\right) a^{n-(n-1)/2} x^{(n-1)/2}$$

$$= \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} a^{(n+1)/2} x^{(n-1)/2}$$

and $$t_{m+1} = t_{\frac{n+1}{2}+1} = C\left(n, \frac{n+1}{2}\right) a^{n-(n+1)/2} x^{(n+1)/2}$$

$$= \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} a^{(n-1)/2} x^{(n+1)/2}$$

Binomial Coefficients

The coefficients $C(n, 0), C(n, 1), \ldots, C(n, n)$ in the expansion of $(a + x)^n$ are known as *binomial coefficients*. We have the following properties of binomial coefficients :

(i) In the expansion of $(1 + x)^n$, the sum of the binomial coefficients is 2^n.

We have $(1 + x)^n = C(n, 0) + C(n, 1)x + C(n, 2)x^2 + \ldots + C(n, n)x^n$

Putting $x = 1$, we have

$$2^n = C(n, 0) + C(n, 1) + C(n, 2) + C(n, 3) + \ldots + C(n, n)$$

= sum of all the binomial coefficients.

Note 1 : The binomial coefficients $C(n, 0), C(n, 1), C(n, 2), \ldots, C(n, n)$ are also written as ${}^nC_0$, ${}^nC_1, {}^nC_2, \ldots \ldots {}^nC_n$. For simplicity, we denote ${}^nC_0$, ${}^nC_1$ etc. by C_0, C_1, etc. thus, $C_r = {}^nC_r$.

Note 2 : The sum of the binomial coefficients

$$= C_0 + C_1 + C_2 + C_3 + \ldots\ldots\ldots\ldots + C_n = 2^n$$

$$\therefore \quad C_1 + C_2 + C_3 + \ldots\ldots\ldots\ldots + C_n = 2^n - 1$$

i.e. the total number of combinations of n different things is $2^n - 1$.

(ii) In the expansion $(1 + x)^n$, the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1}.

We have $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots\ldots\ldots + C_nx^n$

Putting $x = -1$

$$(1-1)^n = C_0 - C_1 + C_2 - C_3 + C_4 - \ldots\ldots\ldots\ldots\ldots$$

$$0 = (C_0 + C_2 + C_4 + \ldots\ldots) - (C_1 + C_3 + C_5 + \ldots\ \ldots\ldots)$$

$\therefore C_0 + C_2 + C_4 + \ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots\ldots$

But $C_0 + C_1 + C_2 + C_3 + C_4 + \ldots\ldots\ldots = 2^n$

$\therefore \quad C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}.$

Worked Out Examples

Example 1

Expand $(2a + b)^5$ by the binomial theorem.

Solution :

With $2a$ as the first term and b as the second term in the binomial expression to be expanded, we get

$$(2a + b)^5 = (2a)^5 + 5(2a)^4 \cdot b + \frac{5\cdot 4}{2!}(2a)^3 \cdot b^2 + \frac{5\cdot 4\cdot 3}{3!}(2a)^2 \cdot b^3 + 5\cdot 2a\cdot b^4 + b^5$$

$$= 32a^5 + 80a^4b + 80a^3b^2 + 40a^2b^3 + 10ab^4 + b^5.$$

Example 2

Find the 7^{th} term in the expansion of $\left(2x + \frac{1}{y}\right)^{10}$

Solution:

$$7^{th} \text{ term} = t_7 = t_{6+1} = C(10, 6)\,(2x)^{10-6}.\left(\frac{1}{y}\right)^6$$

$$= \frac{10.9.8.7}{4.3.2.1} . 2^4 . x^4 . \frac{1^6}{y^6} = \frac{3360x^4}{y^6}$$

Example 3

Find the general term in the expansion of $\left(x^2 + \frac{a^2}{x}\right)^5$. Then find the coefficient of x.

Solution :

Let t_{r+1} be the general term in the expansion of $\left(x^2 + \frac{a^2}{x}\right)^5$.

Then, $t_{r+1} = C(5, r)\,(x^2)^{5-r}.\left(\frac{a^2}{x}\right)^r$

$$= C(5, r)\, x^{10-2r} . \frac{a^{2r}}{x^r}$$

$$= C(5, r)\, x^{10-3r} \cdot a^{2r}$$

To get the coefficient of x,

$$10 - 3r = 1$$

or, $9 = 3r$

$\therefore \quad r = 3$

$\therefore \quad t_4 = 4^{th}$ term contains x.

$\therefore$ coefficient of $x = C(5, 3)\, a^6 = 10a^6$

Example 4

Find the middle term in the expansion of $(2a + 3x)^{30}$.

Solution :

Here the number of terms in the expansion of $(2a + 3x)^{30}$ is 30 + 1 i.e. 31. So, there is only one middle term.

$$\therefore \text{ middle term } = t_{(30/2)+1} = t_{15+1}$$
$$= C(30, 15)\, (2a)^{30-15}.\, (3x)^{15}$$
$$= C(30, 15)\, (2a)^{15}.\, (3x)^{15}$$
$$= \frac{30\,!}{(15!)^2} (2a)^{15}.\, (3x)^{15}.$$

Example 5

Find the middle term in the expansion of $\left(1 + \frac{x}{2}\right)^{15}$.

Solution :

Here the number of terms in the expansion is 15 + 1 = 16 which is even. So, there are two middle terms. The middle terms are $t_{(15+1)/2}$ and $t_{(15+1)/2+1}$ i.e. t_8 and t_9.

$$t_8 = t_{7+1} = C(15, 7)\, (1)^{15-7}.\left(\frac{x}{2}\right)^7 = \frac{15!}{8!\,7!} \frac{x^7}{2^7}$$

and $$t_9 = t_{8+1} = C(15, 8)\, (1)^{15-8} \left(\frac{x}{2}\right)^8 = \frac{15!}{7!\,8!} \frac{x^8}{2^8}.$$

Example 6

Find the term independent of x in the expansion of $\left(x - \frac{1}{3x^2}\right)^{12}$

Solution:

Let t_{r+1} be the term independent of x.

Then, $$t_{r+1} = C(12, r)\, (x)^{12-r}.\left(-\frac{1}{3x^2}\right)^r$$
$$= (-1)^r \frac{C(12, r)}{3^r}.\, x^{12-r} \frac{1}{x^{2r}}$$
$$= \frac{(-1)^r.C(12, r)}{3^r}.\, x^{12-3r}$$

For the term independent of x,

$$12 - 3r = 0$$

$$\therefore \quad r = 4$$

$\therefore$ t_5 is the term independent of x and $t_5 = \dfrac{(-1)^4 \, C(12, 4)}{3^4} = \dfrac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1 \times 81} = \dfrac{55}{9}$

Example 7

In the expansion of $(1 + x)^{43}$, coefficient of (2r + 1)th term and (r + 2)th term are equal. Find r.

Solution:

$t_{2r+1} = C(43, 2r)\, 1.x^{2r}$

$\therefore$ coefficient of $t_{2r+1} = C(43, 2r)$

Again, $t_{r+2} = C(43, r+1) \,.\, 1 \,.\, x^{r+1}$

$\therefore$ coefficient of $t_{r+2} = C(43, r+1)$

By given, $C(43, 2r) = C(43, r+1)$

$\Rightarrow \quad 2r = r + 1$

$\Rightarrow \quad r = 1$

Also, $2r + r + 1 = 43$

$\Rightarrow \quad 3r = 42$

$\therefore \quad r = 14$

$\therefore \quad r = 1$ or 14

Example 8

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots + C_n x^n$, prove that

(a) $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots + nC_n = n\,2^{n-1}$

(b) $C_0C_2 + C_1C_3 + C_2C_4 + \ldots + C_{n-2}C_n = \dfrac{2n\,!}{(n-2)\,!\,(n+2)!}$

Solution :

(a) $C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots + nC_n$

$$= n + 2n\frac{(n-1)}{2!} + \frac{3n\,(n-1)\,(n-2)}{3!} + \ldots\ldots + n.1$$

$$= n\left\{1 + \frac{n-1}{1} + \frac{(n-1)\,(n-2)}{2!} + \ldots\ldots + 1\right\}$$

$$= n\,(1+1)^{n-1} = n\,2^{n-1}.$$

(b) $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$ ……(i)

Also, $(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots\ldots + C_n \ldots$ …(ii)

Multiplying (i) and (ii)

$$(1 + x)^{2n} = (C_0 + C_1x + \ldots + C_nx^n)\,(C_0x^n + C_1x^{n-1} + \ldots + C_n) \quad \ldots(iii)$$

Since (iii) is an identity, the coefficient of any power of x of the L.H.S. should be equal to the coefficient of the same power of x of the R.H.S.

The coefficient of x^{n-2} in the L. H. S. of (iii)

$$= C(2n, n+2) = \frac{2n!}{(n-2)!\,(n+2)!}.$$

Again the coefficient of x^{n-2} in the R.H.S. of (iii)

$$= C_0C_2 + C_1C_3 + C_2C_4 + \dots \dots \dots + C_{n-2}C_n$$

$$\therefore C_0C_2 + C_1C_3 + C_2C_4 + \dots \dots + C_{n-2}C_n = \frac{2n!}{(n-2)!\,(n+2)!}$$

EXERCISE

Expand each of the following by the binomial theorem and simplify.

1. (a) $(a+b)^7$ (b) $(2x-3y)^4$ (c) $(2x+y^2)^5$
   (d) $\left(\frac{x}{2}+\frac{2}{y}\right)^5$ (e) $\left(x^2-\frac{1}{x}\right)^5$

2. Find the stated terms in the following cases
   a) seventh term of $(2x+y)^{12}$ b) fourth term of $\left(2x^2+\frac{1}{x}\right)^8$
   c) fifth term of $\left(x-\frac{2}{x}\right)^7$

3. Find the general terms of the following
   a) $\left(x^2+\frac{1}{x}\right)^6$ b) $\left(\frac{a}{b}+\frac{b}{a}\right)^{2n+1}$

4. Find the coefficients of
   a) x^5 in the expansion of $\left(x+\frac{1}{2x}\right)^7$ b) x^2 in the expansion of $\left(x^3+\frac{a}{x}\right)^{10}$
   c) x^6 in the expansion of $\left(3x^2-\frac{1}{3x}\right)^9$

5. Find the term independent (free) of x in the expansion of
   a) $\left(x^2+\frac{1}{x}\right)^{12}$ b) $\left(2x+\frac{1}{3x^2}\right)^9$
   c) $\left(x+\frac{1}{x}\right)^{2n}$ d) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

6. a) Write down the fourth term in the expansion of $\left(px+\frac{1}{x}\right)^n$. If this term is independent of x, find the value of n. With this value of n, calculate the value of p given that the fourth term is $\frac{5}{2}$.

b) If the coefficient of x^{-1} in the expansion of $\left(x+\frac{k}{x^2}\right)^5$ is 90, find the value of k.

7. Find the middle term or terms in the expansion of

a) $\left(x+\frac{1}{x}\right)^{18}$ b) $\left(2x+\frac{1}{x}\right)^{17}$

c) $\left(\frac{x}{a}-\frac{a}{x}\right)^{2n+1}$ d) $\left(ax-\frac{1}{ax}\right)^{2n}$

8. Show that the middle term of the expansion of

a) $(1+x)^{2n}$ is $\frac{1\cdot3\cdot5\cdot\ldots\ldots\ldots\cdot(2n-1)}{n!}2^n x^n$

b) $(x-\frac{1}{x})^{2n}$ is $\frac{1\cdot3\cdot5\cdot\ldots\ldots\ldots\cdot(2n-1)}{n!}(-2)^n$

9. a) In the expansion of $(1+x)^{21}$, the coefficient of $(2r+1)$th term is equal to the coefficient of $(3r+2)$th term. Find r.

b) In the expansion of $(1+x)^{2n+1}$, the coefficients of x^r and x^{r+1} are equal. Find r.

10. Show that the coefficients of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1+x)^{2n-1}$.

11. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots\ldots\ldots + C_nx^n$, prove that

a) $C_1 - 2C_2 + 3C_3 - \ldots\ldots\ldots + n(-1)^{n-1}\cdot C_n = 0$

b) $C_0 + 2C_1 + 3C_2 + \ldots\ldots\ldots\ldots + (n+1)C_n = (n+2)2^{n-1}$

c) $C_0 + 3C_1 + 5C_2 + \ldots\ldots\ldots + (2n+1)C_n = (n+1)2^n$

d) $C_0 + 4C_1 + 7C_2 + 10C_3 + \ldots\ldots + (3n+1)C_n = (3n+2).2^{n-1}$

e) $\frac{C_1}{C_0} + \frac{2.C_2}{C_1} + \frac{3.C_3}{C_2} + \ldots\ldots + \frac{nC_n}{C_{n-1}} = \frac{n(n+1)}{2}$

f) $C_0C_n + C_1C_{n-1} + \ldots\ldots\ldots\ldots + C_nC_0 = \frac{2n!}{n!\,n!}$

g) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots\ldots\ldots\ldots + C_n^2 = \frac{2n!}{(n!)^2}$

h) $C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots + C_{n-1}.C_n = \frac{2n!}{(n+1)!\,(n-1)!}$

12. If the three consecutive coefficients in the expansion of $(1+x)^n$ be 165, 330, 462; find n.

13. If the four consecutive coefficients in the expansion of $(1+x)^n$ be a_1, a_2, a_3 and a_4, then prove that $\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$

A. List the coefficients of the terms in expansion of $(x+a)^0$, $(x+a)^1$ and $(x+a)^2$. Arrange these coefficients in a triangular form. Using Pascal's triangle, expand $(x+a)^6$ and $(2x+3a)^5$.

B. **A binomial expression with positive index n is given below**

$$\left(\frac{a}{x} - bx\right)^n$$

Using different values of a, b, n find the general term, middle term, term independent of x and the coefficient of x in the expansion of $\left(\frac{a}{x} - bx\right)^n$

Answer

1. (a) $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$
   (b) $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$
   (c) $32x^5 + 80x^4y^2 + 80x^3y^4 + 40x^2y^6 + 10xy^8 + y^{10}$
   (d) $\frac{1}{32}x^5 + \frac{5}{8}\frac{x^4}{y} + 5\frac{x^3}{y^2} + 20\frac{x^2}{y^3} + 40\frac{x}{y^4} + \frac{32}{y^5}$
   (e) $x^{10} - 5x^7 + 10x^4 - 10x + \frac{5}{x^2} - \frac{1}{x^5}$
2. a) $C(12, 6)\, 2^6x^6y^6$ b) $1792x^7$ c) $\frac{560}{x}$
3. a) $C(6, r)\, x^{12-3r}$ b) $C(2n + 1, r)\left(\frac{a}{b}\right)^{2n-2r+1}$
4. a) $\frac{7}{2}$ b) $120a^7$ c) 378
5. a) 9^{th} term = 495 b) 4^{th} term = $\frac{1792}{9}$ c) $(n + 1)$th term = $\frac{2n!}{(n!)^2}$ d) 7^{th} term = $\frac{7}{18}$
6. a) $n = 6, p = \frac{1}{2}$ b) $k = \pm 3$
7. a) $\frac{18!}{9!\,9!}$ b) $\frac{17!}{8!\,9!}2^9x, \frac{17!}{8!\,9!}\frac{2^8}{x}$ c) $(-1)^n\frac{(2n+1)!}{n!\,(n+1)!}\frac{x}{a}, (-1)^{n+1}\frac{(2n+1)!}{n!\,(n+1)!}\frac{a}{x}$
   d) $(-1)^n\frac{2n!}{n!\,n!}$
9. a) 4 b) n 11. 11

Application of Binomial Theorem

The binomial theorem has many important applications, the most useful of which is the determination of approximate values of certain algebraically as well as arithmetical quantities and sums of certain infinite series.

As an application of the binomial theorem, we can arrive at

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \dots$$
$$\dots + \frac{n(n-1)(n-2)\dots\dots(n-r-1)}{r!}x^r + \dots \dots$$

which terminates after $n + 1$, terms and when n is any positive integer. But when n is any real number different from positive integer, the expansion does not terminate and it is valid, only if

$|x| < 1$, and this expansion is known as binomial series. The proof is beyond the scope of this book. Also,

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots \qquad (|x| < 1)$$

Worked Out Examples

Example 1

Find the value of $(1.03)^{-5}$ correct to three significant figures.

Solution :

We have

$(1.03)^{-5} = (1 + 0.03)^{-5} \quad (|x| = 0.03 < 1)$

$= 1 + (-5)(0.03) + \frac{(-5)(-6)}{2!}(0.03)^2 + \frac{(-5)(-6)(-7)}{3!}(0.03)^3 + \ldots\ldots\ldots$

$= 1 - 0.15 + 0.0135 - 0.000945 + \ldots\ldots\ldots\ldots\ldots$

$= 0.862555 = 0.863.$

Example 2

Show that $\sqrt{8} = 1 + \frac{3}{4} + \frac{3\cdot5}{4\cdot8} + \frac{3\cdot5\cdot7}{4\cdot8\cdot12} + \ldots\ldots\ldots\ldots$

Solution :

Here $\sqrt{8} = \sqrt{2^3}$

$= 2^{3/2} = \left(\frac{1}{2}\right)^{-3/2} = \left(1 - \frac{1}{2}\right)^{-3/2}$

Applying binomial theorem for any index, we have

$\sqrt{8} = \left(1 - \frac{1}{2}\right)^{-3/2}$

$= 1 + \frac{3}{2}\left(\frac{1}{2}\right) + \frac{\frac{3}{2}\cdot\left(\frac{3}{2}+1\right)}{2!}\left(\frac{1}{2}\right)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}+1\right)\left(\frac{3}{2}+2\right)}{3!}\left(\frac{1}{2}\right)^3 + \ldots$

$= 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \ldots\ldots$

EXERCISE

1. Expand the following up to four terms

(a) $\frac{1}{1+x}$ (b) $\sqrt{1+x}$ (c) $\frac{1}{\sqrt{1-x}}$

2. Compute each of the following to three significant figures :

(a) $(1.01)^{-3}$ (b) $\sqrt{1.01}$ (c) $\sqrt[4]{1.02}$

(d) $\sqrt{26}$ (e) $\sqrt[3]{9}$ (f) $\sqrt{17}$ (g) $\sqrt[3]{999}$

3. Show that :

i) $1+\frac{1}{4}+\frac{1\cdot 4}{4\cdot 8}+\frac{1\cdot 4\cdot 7}{4\cdot 8\cdot 12}+\ldots\ldots\ldots\ldots \text{ to } \infty=(2)^{2/3}$

ii) $1-\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}-\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots\ldots\ldots\ldots \text{ to } \infty=\sqrt{\frac{2}{3}}$

A. Using binomial expansion find the approximate value of $\left(\frac{3}{2}\right)^{1/2}$ and $(3)^{-3/2}$.

Answer

1. (a) $1-x+x^2-x^3$ (b) $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$ (c) $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3$

2. (a) 0.971 (b) 1.005 (c) 1.005 (d) 5.099 (e) 2.080 (f) 4.123 (g) 9.997

Exponential and Logarithmic Series

Any function of the form $y = f(x) = a^x, \quad a > 0$

is called an *exponential function* in which the base a is constant and the index x is a variable. The inverse of an exponential function is called a *logarithmic function* which is denoted by $\log_a x$.

So, if $y = a^x$, we have $x = \log_a y$. There is a special type of exponential function e^x, where e is the limiting value $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

The value of e lies between 2 and 3 and is approximately 2.718282. The corresponding logarithmic function is called the *natural logarithmic function* and is denoted by $\log x$, base e being understood. The number e is known as *Euler's number* as it is initially introduced by Euler for the base of a natural logarithm.

The series corresponding to the exponential function is known as *exponential series* and the series corresponding to the logarithmic function is known as *logarithmic series*.

Expansion of e^x

We prove that for all values of x

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots+\frac{x^r}{r!}+\ldots \text{ to } \infty \qquad \ldots\ldots(1)$$

If $n > 1$ (so that $\frac{1}{n} < 1$), then we have by binomial theorem that

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\cdot\frac{1}{n} + \frac{nx\cdot(nx-1)}{2!}\cdot\frac{1}{n^2} + \frac{nx\cdot(nx-1)(nx-2)}{3!}\cdot\frac{1}{n^3} + \ldots\ldots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \ldots$$

Hence, when n is infinitely large,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \qquad \ldots\ldots (2)$$

Putting $x = 1$, we have from (2),

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots = e$$

Now, $\left(1+\frac{1}{n}\right)^{nx} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}$ which tends to e^x as $n \to \infty$

Making n infinitely large in (2), we have for all values of x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots + \frac{x^r}{r!} + \ldots\ldots \text{ to } \infty$$

This expansion is known as the exponential series.

Cor. 1: Since the series (1) is true for all values of x, by replacing x by $-x$ in the series (1), we have

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots + (-1)^r\frac{x^r}{r!} + \ldots\ldots\ldots$$

Cor. 2: When $x = 1$ and -1, the series (1) takes the following forms :

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots \quad \text{and} \quad e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots\ldots$$

The Value of e

a) e is not rational

If possible, let $e = \frac{m}{n}$ where m and n are positive integers; then

$$\frac{m}{n} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots + \frac{1}{n!} + \frac{1}{(n+1)!} + \ldots\ldots\ldots$$

Multiplying both sides by $n!$, we have

$$m\,(n-1)! = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \ldots\ldots\ldots + 1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots$$

$$= \text{an integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots\ldots$$

Evidently the term on the left side is an integer as m and $(n-1)$ are integers. All the terms on the right side are integers except $\frac{1}{n+1}, \frac{1}{(n+1)(n+2)}, \ldots\ldots\ldots$

Since each of the terms $\frac{1}{n+1}, \frac{1}{(n+1)(n+2)}, \ldots \ldots \ldots$ is positive and $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots \ldots$ is a proper fraction, for it is evidently greater than $\frac{1}{n+1}$ and less that the geometric sum of the series

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots \ldots \ldots \ldots \text{ which is } \frac{\frac{1}{n+1}}{1-\frac{1}{n+1}} = \frac{1}{n}$$

Thus an integer is equal to an integer plus a fraction which is absurd,. Therefore e is not rational.

b) e lies between 2 and 3

We know that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \ldots \ldots \quad + \frac{1}{r!} + \ldots \ldots \ldots \ldots$$

or, $$e = 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots \ldots \ldots \ldots + \frac{1}{r!} + \ldots \ldots \ldots \ldots$$

$\therefore \quad e > 2$ as all the terms after 2 are positive.

Since $r! = 1 \cdot 2 \cdot 3 \cdot \ldots \ldots \ldots \ldots r$

$> 1 \cdot 2 \cdot 2 \cdot \ldots \ldots \ldots \ldots r$ factors

$= 2^{r-1}$

$$\therefore \quad \frac{1}{r!} < \frac{1}{2^{r-1}}$$

$$\therefore \quad e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \ldots \ldots \ldots$$

$$= 1 + \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{r!} + \ldots\right)$$

$$< 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \ldots \ldots\right)$$

$$= 1 + \frac{1}{1-\frac{1}{2}} = 1 + 2 = 3$$

$$\therefore \quad e < 3.$$

So, $2 < e < 3$.

Expansion of a^x

To prove that if a is any positive number, then

$$a^x = 1 + \frac{x}{1!}\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \dots\dots + \frac{x^r}{r!}(\log_e a)^r + \dots \text{ to } \infty$$

for all values of x.

If a be any positive number and if k be a number such that

$a = e^k$

$\therefore \quad k = \log_e a.$

Now,

$$a^x = (e^k)^x = e^{kx}$$

$$= 1 + \frac{kx}{1!} + \frac{(kx)^2}{2!} + \dots\dots + \frac{(kx)^r}{r!} + \dots\dots \text{ to } \infty$$

$$= 1 + \frac{x}{1!}\log_e a + \frac{(x)^2}{2!}(\log_e a)^2 + \dots + \frac{(x)^r}{r!}(\log_e a)^r + \dots \text{ to } \infty$$

Note: The expansion is an exponential series.

The Logarithmic Series

Let x be a number lying between -1 and $+1$. Then the sum to infinity of the series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\dots + (-1)^{r-1}\frac{x^r}{r!} + \dots\dots \quad \text{is } \log_e(1+x).$$

The series has been found to be valid for $x = 1$, but not for $x = -1$.

Thus, for $-1 < x \le 1$,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\dots\dots \qquad \dots\dots(i)$$

The infinite series (i) is called the logarithmic series.

Cor 1 : If x be numerically less than unity, changing x to $-x$, we have

$$\therefore \quad \log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\dots \qquad \dots\dots(ii)$$

Also, from (i) and (ii), we have

$$\log_e(1+x) - \log_e(1-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots\right)$$

$$\therefore \quad \log_e\frac{1+x}{1-x} = \log_e(1+x) - \log_e(1-x)$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

Worked Out Examples

Example 1

Find the values of (i) $\frac{1}{2}(e + e^{-1})$ (ii) $\frac{1}{2}(e - e^{-1})$

Solution :

Since $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots$

Putting $x = 1$ and -1, we have

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots \quad \text{and} \quad e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots$$

Now $e + e^{-1} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots\right) + \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\right)$

$$= 2 + \frac{2}{2!} + \frac{2}{4!} + \ldots\ldots\ldots\ldots$$

$$= 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\ldots\ldots\ldots\right)$$

$\therefore \frac{1}{2}(e + e^{-1}) = 1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\ldots\ldots\ldots$

Again, $e - e^{-1} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots\right) - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\right)$

$$= \frac{2}{1!} + \frac{2}{3!} + \frac{2}{5!} + \ldots\ldots$$

$$= 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\right)$$

$\therefore \frac{1}{2}(e - e^{-1}) = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\ldots\ldots$

Example 2

Show that $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots\ldots\ldots\ldots \text{ to } \infty = \frac{1}{e}$

Solution :

$\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \ldots\ldots\ldots\ldots \text{ to } \infty$

$$= \frac{3-1}{3!} + \frac{5-1}{5!} + \frac{7-1}{7!} + \ldots\ldots\ldots\ldots\ldots \text{ to } \infty$$

$$= \frac{3}{3!} - \frac{1}{3!} + \frac{5}{5!} - \frac{1}{5!} + \frac{7}{7!} - \frac{1}{7!} + \ldots\ldots\ldots\ldots \text{ to } \infty$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \dots\dots\dots \text{ to } \infty$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots\dots\dots \text{ to } \infty$$

$$= e^{-1} = \frac{1}{e}$$

Example 3

Show that $\dfrac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots\dots\dots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots\dots\dots} = \dfrac{e-1}{e+1}$

Solution :

Since, $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots\dots\dots$

and, $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots\dots\dots$

So, we have

$$e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \dots\dots\dots = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \dots\dots\right)$$

$$\therefore \quad \frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots\dots$$

and also, $e - e^{-1} = \frac{2}{1!} + \frac{2}{3!} + \frac{2}{5!} + \dots\dots\dots$

$$\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots\dots\dots$$

Now, $\dfrac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots\dots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots\dots} = \dfrac{\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots\dots\dots\right) - 1}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \dots\dots\dots}$

$$= \frac{\frac{e + e^{-1}}{2} - 1}{\frac{e - e^{-1}}{2}} = \frac{e^2 + 1 - 2e}{e^2 - 1}$$

$$= \frac{(e-1)^2}{(e+1)(e-1)} = \frac{e-1}{e+1}.$$

Example 4

Show that $\frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+3}{4!} + \dots\dots = \frac{e}{2}$

Solution :

The n^{th} term of the series $= \dfrac{1+2+3+4+\dots\dots+n}{(n+1)!} = \dfrac{n(n+1)}{2(n+1)!} = \dfrac{1}{2(n-1)!}$

$$\text{1st. term} = \frac{1}{2}\cdot\frac{1}{0!}$$

$$\text{2nd term} = \frac{1}{2}\cdot\frac{1}{1!}$$

$$\text{3rd term} = \frac{1}{2}\cdot\frac{1}{2!}$$

… … … … … … …

By addition, the given series

$$= \frac{1}{2}\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots\dots\dots\dots\right)$$

$$= \frac{1}{2}\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\dots\dots\dots\right) = \frac{1}{2}e = \frac{e}{2}$$

Example 5

Sum to infinity the series $\dfrac{1^2}{1!}+\dfrac{2^2}{2!}+\dfrac{3^2}{3!}+\dots\dots\dots\dots$.

Solution :

Let t_n be the n^{th} term of the given series.

Then, $t_n = \dfrac{n^2}{n!} = \dfrac{n}{(n-1)!}$

or, $t_n = \dfrac{n-1+1}{(n-1)!}$

$\therefore$ $t_n = \dfrac{1}{(n-2)!}+\dfrac{1}{(n-1)!}$

$$t_1 = 0+\frac{1}{0!}$$

$$t_2 = \frac{1}{0!}+\frac{1}{1!}$$

$$t_3 = \frac{1}{1!}+\frac{1}{2!}$$

$$t_4 = \frac{1}{2!}+\frac{1}{3!}$$

… … … … … … …

Now, by addition, the sum of the given series

$$= \left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots\right)+\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right) = e+e = 2e$$

Example 6

Show that $\log_e 2 = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \dots\dots\dots\dots$

Solution :

Since $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \dots\dots\dots$

Putting $x = 1$ in the expansion, we have

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots\dots\dots$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \dots\dots$$

$$= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} \dots\dots\dots$$

Example 7

Prove that : $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \dots\dots\dots\dots = \log_e 2.$

Solution :

Here, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \dots\dots\dots$

$$= \frac{\frac{1}{2}}{1} + \frac{\frac{1}{2^2}}{2} + \frac{\frac{1}{2^3}}{3} + \dots\dots\dots\dots = -\left\{-\frac{\left(\frac{1}{2}\right)}{1} - \frac{\left(\frac{1}{2}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} - \dots\dots\right\}$$

$$= -\log_e\left(1 - \frac{1}{2}\right) = -\log_e\frac{1}{2}$$

$$= \log_e 2.$$

Example 8

If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots$, show that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots\dots\dots\dots$

Solution :

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\dots\dots\dots$$

or, $y = \log_e(1+x)$

$\therefore\ 1 + x = e^y$

or, $1 + x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots\dots\dots$

$\therefore\ x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \dots\dots\dots$

EXERCISE

Show that

1. $\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\right)\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\right)=1$

2. $\left(1+\frac{1}{2!}+\frac{1}{4!}+\ldots\ldots\right)^2-\left(1+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots\right)^2=1$

3. $\frac{2}{1!}+\frac{4}{3!}+\frac{6}{5!}+\ldots\ldots\ldots\ldots \text{ to } \infty = e$

4. $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots\ldots\ldots\ldots=1$

5. $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots\ldots=\frac{3e}{2}$

6. $\dfrac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots\ldots}{1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\ldots\ldots\ldots}=\frac{e^2+1}{e^2-1}$

7. $\frac{1}{1!}+\frac{1+3}{2!}+\frac{1+3+5}{3!}+\frac{1+3+5+7}{4!}+\ldots\ldots=2e$

8. Sum to infinity the following series

   i) $\frac{1\cdot2}{1!}+\frac{2\cdot3}{2!}+\frac{3\cdot4}{3!}+\ldots\ldots$ ii) $1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+\ldots\ldots$

   iii) $\left(1+\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}+\ldots\right)\left(1-\frac{1}{1\cdot2}+\frac{1}{1\cdot2\cdot3}-\ldots\right)$

   iv) $1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\ldots\ldots\ldots\ldots\ldots$

9. a) Show that $\sum_{n=1}^{\infty}\frac{n^2}{(n+1)!}=e-1.$

   b) Show that $\sum_{n=1}^{\infty}\frac{n^2}{(n-1)!}=5e.$

10. Prove that

    i) $\frac{1}{2\cdot3}+\frac{1}{4\cdot5}+\frac{1}{6\cdot7}+\ldots\ldots\ldots\ldots\ldots=1-\log_e 2$

    ii) $\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\ldots\ldots=\log\frac{3}{2}$

iii) $\left(\frac{1}{3}-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{3^2}+\frac{1}{2^2}\right)+\frac{1}{3}\left(\frac{1}{3^3}-\frac{1}{2^3}\right)+\ldots\ldots=0$

iv) $\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\ldots\ldots=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\ldots$

v) $1+\frac{1}{3\cdot2^2}+\frac{1}{5\cdot2^4}+\frac{1}{7\cdot2^6}+\ldots\ldots\ldots\ldots=\log_e 3$

11. a) If $y=x+\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\ldots\ldots$, show that $x=y-\frac{1}{2!}y^2+\frac{1}{3!}y^3-\frac{1}{4!}y^4+\ldots\ldots$

b) If $y=\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots$ to ∞, prove that $x=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ldots\ldots$ to ∞

c) $x=\frac{y}{1!}-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\ldots\ldots$, show that $y=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots\ldots$

A. Using logarithmic series, find the approximate values of $\log_e 2$, $\log_e 3$ and $\log_e 4$.

B. Using exponential series, find the approximate value of $\sqrt{e}$ and $\frac{1}{\sqrt{e}}$.

Answer

8. (i) $3e$ (ii) $3e$ (iii) $e+e^{-1}-2$ (iv) e^2-e

Chapter 3
Elementary Group Theory

Introduction

Set is one of the most important concept in modern mathematics. It alone conveys very limited information. It becomes more meaningful if we can relate or do something with or operate on its members or elements.

The simplest way is to relate or combine an element of a set with itself and get the same or a different element of the same set.

For instances, 1 *multiplied by* 1 yields 1; but 1 *added to* 1 yields 2, a different number.

We may also relate or combine an element of a set with a different element of the set and arrive at one of the numbers or completely different number.

For instances, 1 *multiplied by* 2 yields 2; but 1 *added to* 2 yields 3, a completely different number.

Note that, in all cases the resulting number belongs to the same original set or set under reference.

In the above examples, we have used what is known as *multiplication operation* and *addition operation* on the set of counting numbers. Such operations can also be performed on the set of integers Z i.e., the set consisting of negative numbers, zero and the set of positive integers:

$$\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$$

If any two elements or repeated consideration of the same element of a set gives rise to an element of the same set under a given operation, such an operation is known as a *binary operation* (*bi* means two).

A set together with a binary operation gives rise to what is known as an *algebraic structure* or *system*. An algebraic structure can be realized more concretely once some other characteristics of the operation are specified. Having done so, we arrive at the starting point of what is known as *Group* and *Group Theory*. Group theory originated from the study of polynomial equations sometime around 1830s. Three pioneers of Group Theory are E. Galois, A. Cayley and S. Lie.

Binary Operation on Sets of Integers

Counting numbers 1, 2, 3, …, … the number zero '0' and the negatives of the counting numbers …, –3, –2, –1 when taken together, form the set of integers. The set of integers is usually denoted by Z. The set Z becomes practically useful, once we can combine or operate on two (*bi* = two) integers (or, the same integer counted twice) in some or other way; and then form another integer. This demands a precisely defined *binary operation.*

a) Binary Operation

Any *rule* which assigns to each ordered pair of elements in the set Z of integers a *unique* element of Z, is called a *binary* (*bi* = two) *operation* on Z. In symbols, we may consider it as a function *f* defined by

$$f : Z \times Z \to Z$$

In practice, a variety of symbols such as

$*, \otimes, \oplus, \circ, \times, -, +$ are being used to denote a binary operation.

Written in terms of elements, the definition of a binary operation looks like:

"A binary operation $*$ on the set of integers Z associates each ordered pair of integers $(m, n) \in Z \times Z$ a unique element $c \in Z$."

In such a case, the set is said to be closed under the operations or the operation has a closure property.

Two of the most common binary operations on Z are the *addition operation*, denoted by '+', and the *multiplication operation* denoted by '×'

Addition: The operation of *addition* denoted by '+' says 'to each pair of integers *m* and *n*, there is an integer *x* such that $m + n = x$.

Here, the number *x* is called the *sum* of *m* and *n*.

For instances, $1 + 2 = 3$, $5 + 4 = 9$, $2 + (-2) = 0$ etc.

Multiplication: The second operation *multiplication* denoted by '×" or '.' says 'to each pair of integers *m* and *n*, there is an integer *y* such that $m \times n = y$'

For instances, $1 \times 2 = 2$, $5 \times 4 = 20$, $2 \times (-2) = -4$ etc.

Here, the number *y* is called the *product* of *m* and *n*. The product of *m* and *n* is also denoted by *mn*, *m*(*n*), (*m*)*n*, (*m*)(*n*) or simply *mn*.

b) Properties of binary operations:

Addition: Given an integer *n* and the number zero '0', the binary operation '+' of addition yields

$$n + 0 = 0 + n = n \qquad (3 + 0 = 0 + 3 = 3)$$

Here, the number '0' is called the *additive identity.*

Multiplication: Given an integer *n* and the number '1', the binary operation '×' of multiplication yields

$$n \times 1 = 1 \times n = n \qquad [3 \times 1 = 1 \times 3 = 3]$$

Here, the number '1' is called the *multiplicative identity.*

Additive Inverse (or negative): For each integer *n*, there is a *unique* integer, denoted by $-n$, such that

$$n + (-n) = (-n) + n = 0 \qquad [(3 + (-3) = (-3) + 3 = 0]$$

Here, the number $(-n)$ is called the *additive inverse* (or *negative of n*).

The associative properties of Addition and Multiplication: Given three integers m, n, and p if

$$m + (n + p) = (m + n) + p \quad \text{and} \quad m \times (n \times p) = (m \times n) \times p$$

$$[1 + (2 + 3) = (1 + 2) + 3 = 6 \quad \text{and} \quad 1 \times (2 \times 3) = (1 \times 2) \times 3 = 6]$$

then the addition and multiplication operation have the associative property respectively.

Commutative properties of addition and multiplication: Given two integers $m, n \in z$, if $m + n = n + m$ and $mn = nm$ then the addition and multiplication operation have the commutative property respectively.

Distributive properties: Given three integers m, n and p

$$m \times (n + p) = m \times n + m \times p \quad \text{and} \quad (m + n) \times p = m \times p + n \times p$$

$$[4 \times (2 + 3) = 4 \times 2 + 4 \times 3 = 20 \quad \text{and} \quad (4 + 2) \times 3 = 4 \times 3 + 2 \times 3 = 18]$$

c) Composition table or Operation table

Consider a finite set G with a binary operation denoted by addition (+) or multiplication (×). Then the binary operation on the elements of G can be specified or represented by a table known as the composition table or the operation table. Such a table was first prepared by Arthur Cayley, so this table is also known as Cayley's table by his name.

We can visualize this situation by studying the composition which is constructed with the help of binary operation defined on the set. The construction is convenient only when the set has only a small finite number of elements.

1. Binary Arithmetic: Let G = {0, 1}. Computation under binary arithmetic is carried out in the following way:

$$0 + 0 = 0; \qquad 0 + 1 = 0 \qquad 1 + 0 = 1; \qquad 1 + 1 = 0$$

Its tabular or Cayley's form is

+	0	1
0	0	1
1	1	0

2. Set Relation: Let G denote the set of subsets of the set {0, 1}. Here G = {ϕ, {0}, {1}, {0, 1}}. Consider the union relation as our binary operation. Then, it can be represented in the following form:

∪	ϕ	{0}	{1}	{0, 1}
ϕ	ϕ	{0}	{1}	{0, 1}
{0}	{0}	{0}	{0, 1}	{0, 1}
{1}	{1}	{0, 1}	{1}	{0, 1}
{0, 1}	{0, 1}	{0, 1}	{0, 1}	{0, 1}

3. Congruent Modulo m: If a positive integer a is divided by a positive integer m giving the positive integers k and b ($< m$) as the quotient and the remainder respectively then we call 'a congruent b modulo m' and is written as $a \equiv b$ (mod. m). Also, if a and b are two integers and $a - b$ is divisible by positive integer m, then $a \equiv b$ (mod. m)

For example: If 20 is divided by 3, the quotient is 6 and the remainder is 2, then we say that 20 congruent 2 modulo 3 i.e. $20 \equiv 2$ (mod. 3). Thus, $4 = 10 = 16 = 22 \equiv 1$ (mod. 3) because when each of the numbers 4, 10, 16 and 22 is divided by 3, the remainder in each case is 1.

Also, $8 + 5 = 13 = 4 \times 3 + 1 \equiv 1$ (mod. 3)

Here, the sum $8 + 5$ divided by 3 gives 4 as the quotient and 1, the remainder. The operation is called the addition modulo 3 and is represented by $+_3$.

Further, we have $8 \times 4 = 32 = 10 \times 3 + 2 \equiv 2$ (mod. 3)

Here, the product 8×4 divided by 3 gives 10 as the quotient and 2, the remainder. The operation is called the multiplication modulo 3 and is represented by $\times_3$.

Similarly, $4 + 8 \equiv 0$ (mod. 3) and $4 \times 9 \equiv 0$ (mod. 6)

Let us have the following composition table of the addition modulo 3 and the multiplication modulo 3 for the set G = {0, 1, 2}

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Worked Out Examples

Example 1

Name two binary operations other than the ones mentioned so far.

Solution:

a) Subtraction in the set of integers is a binary operation.

b) Division operation in the set of non-zero rational number is a binary operation.

Example 2

A binary operation $*$ is defined on the set of integers by

a) $m*n = m - n$

b) $m*n = 2mn$

Find $m*n$, if $m = 3$ and $n = 2$.

Solutions:

a) $m*n = m - n = 3 - 2 = 1$

b) $m*n = 2mn = 2 \times 3 \times 2 = 12$

Example 3

The set G = {1, ω, ω²} where ω represents the cube root of unity. Prepare Cayley's table representing the binary operation of ordinary multiplication ×.

Solution:

Since $\omega \times \omega^2 = \omega^3 = 1$, Cayley's table looks like

Cayley's table

×	1	ω	ω²
1	1	ω	ω²
ω	ω	ω²	1
ω²	ω²	1	ω

Example 4

Examine the set of negative integers for

a) Closure property under addition

b) Commutativity under multiplication

c) Associativity under subtraction

Solution:

The set of negative integers is denoted by Z^-.

a) Any $m, n \in Z^-$ $\Rightarrow$ $m + n \in Z^-$; $\therefore$ Z^- is closed.

b) Any $m, n \in Z^-$ $\Rightarrow$ $m\,n = n\,m$ $\therefore$ Z^- is commutative but not closed.

c) Consider $-2, -3, -4 \in Z^-$

$$\{-2-(-3)\}-(-4) = (-2+3)+4 = 5$$

and $(-2)-\{(-3)-(-4)\} = -2-1 = -3$

Thus, $\{-2-(-3)\}-(-4) \neq (-2)-\{(-3)-(-4)\}$

$\therefore$ Z^- is not associative.

Example 5

Test for the closure, associativity and commutativity properties in each case:

a) The operation defined by $m*n = \frac{1}{2}(m+n)$ on Z, $m, n \in$ Z.

b) The operation defined by $m \oplus n = m$ on Z, $m, n \in$ Z.

Solution:

a) Consider $1, 2 \in Z$

Then, $m*n = \frac{1}{2}(m+n)$

$\Rightarrow$ $1*2 = \frac{1}{2}(1+2) = \frac{3}{2} \notin Z$

$\therefore$ Z is not closed.

For associativity:

Consider 2, 4, 8 ∈ Z

$$2*(4*8) = 2*\{\frac{1}{2}(4+8)\}$$

$$= 2*6 = \frac{1}{2}(2+6) = 4$$

Again, $(2*4)*8 = \{\frac{1}{2}(2+4)\}*8$

$$= 3*8 = \frac{1}{2}(3+8) = \frac{11}{8} \notin Z$$

Hence the operation * defined on Z is not associative.

For commutative:

For $m, n \in Z$, $m*n = \frac{1}{2}(m+n)$

$$= \frac{1}{2}(n+m) = n*m$$

Thus, $m*n = n*m$ but may not belong to Z.

∴ the operation * is commutative but may not be closed.

b) Since, for $m, n \in Z$, $m \oplus n = m \in Z$, the operation ⊕ is closed.

Since, for m, n and $p \in Z$, $(m \oplus n) \oplus p = m \oplus p = m \in Z$

and $m \oplus (n \oplus p) = m \oplus n = m$

∴ the operation ⊕ is associative.

Since, for $m, n \in Z$, $m \oplus n = m \in Z$ and $n \oplus m = n$. But $m \neq n$. So the operation ⊕ is not commutative.

Example 6

Show that the multiplication is a binary operation on the set S = {–1, 0, 1} but the subtraction is not.

Solution:

First, we consider the operation of multiplication. So, we examine whether $xy \in S$ or not for all $x, y \in S$.

Since	$-1.0 = 0 \in S$,	$0.1 = 0 \in S$
	$1.(-1) = -1 \in S$,	$1.1 = 1 \in S$
and	$-1.(-1) = 1 \in S$,	$0.0 = 0 \in S$

So, multiplication is the binary operation.

Secondly, the operation is of subtraction. So, we examine whether $x - y \in S$ or not for all $x, y \in S$.

Since, $-1 - 0 = -1 \in S$, $1 - 0 = 1 \in S$

but $-1 - 1 = -2 \notin S$

$\therefore$ subtraction is not the binary operation.

Example 7

Let G be the set of subsets of the set {0, 1}. Show that the set G is closed under the operation of union.

Solution:

The subsets of G are ϕ, {0}, {1} and {0, 1}

$\therefore$ G = {ϕ, {0}, {1}, {0, 1}}

The union of any two elements of G is again an element of G as $\phi \cup \{0\} = \{0\} \in G$, $\{0\} \cup \{1\} = \{0, 1\} \in G$ etc.

$\therefore$ G is closed under the operation of union.

Example 8

If the binary operation $*$ on Q the set of rational numbers is defined by

$$a * b = a + b - ab \quad \text{for every } a, b \in Q$$

Show that $*$ is commutative and associative.

Solution:

i) '$*$' is commutative in Q because if $a, b \in Q$, then

$$a * b = a + b - ab = b + a - ba = b * a$$

ii) '$*$' is associative in Q because if $a, b \in Q$, then

$$\begin{aligned} a * (b * c) &= a * (b + c - bc) \\ &= a + (b + c - bc) - a(b + c - bc) \\ &= a + b - ab + c - (a + b - ab)\, c \\ &= (a * b) * c \end{aligned}$$

EXERCISE

1. A binary operation $*$ is defined on the set of integers by
   a) $m * n = m + n$ b) $m * n = m - n$
   c) $m * n = mn + m + n$
   Find $m * n$ if i) $m = 3$ and $n = 5$ ii) $m = 2$ and $n = -5$.

2. S is a given set and $a, b \in S$. Prove that the operation $*$ defined by
   a) $a * b = a + b$ on S = {−1, 0, 1} is not a binary operation.

b) $a * b = ab$ on S = {1, 2, 4} is not a binary operation.
c) $a * b = a + b$ on S = {2, 4, 6, 8, 10, ...} is a binary operation.
d) $a * b = a - b$ on S = set of integers is a binary operation.
e) $a * b = 3a + 5b$ on S = set of positive integers, is a binary operation.
f) $a * b = 5a - 3b$ on S = set of negative integers, is not a binary operation.
g) $a * b = a^b$ on S = set of integers, is not a binary operation.

3. Let S = {–1, 1} and × denotes the usual operation of multiplication. Represent it by Cayley's table. Show that the multiplication is a binary operation on S.

4. Examine the set of positive integers for
   a) closure property under addition.
   b) commutative property under multiplication.
   c) associative property under subtraction.

5. Test the closure, associative and commutative properties for each of the following cases
   a) the operation defined by $m * n = \frac{1}{2}(m - n)$ on Z, $m, n \in$ Z.
   b) the operation defined by $m * n = n$ on Z, $m, n \in$ Z.
   c) the operation defined by $m * n = m + n + 1$, $m, n \in$ Z.

A. Define a set S with $a, b \in$ S. Again define an operation * on S. Give two examples that shows
   a) the operation * on S is a binary operation
   b) the operation * on S is not a binary operation.

B. Consider a set of subsets of the set consisting of three elements and the opeation defined on it be the union of two sets. Examine whether the union operation defined on the set is a binary operation or not.

Answers

1. a) i) 8 ii) –3 b) i) –2 ii) 7 c) i) 23 ii) –13

3.

×	–1	1
–1	1	–1
1	–1	1

4. a) closed b) commutative c) not associative

5. a) not closed, not associative, not commutative b) closed, associative, not commutative
   c) closed, associative, commutative

Algebraic Structure

A set with one or more binary operations gives rise to what is commonly known as a *algebraic structure*. In particular, the set Z of integers under the addition operation '+' is an algebraic structure. It is commonly denoted by (Z, +). In the same way, the set of rational numbers Q under the usual multiplication operation '×', and denoted by (Q, ×), is another algebraic structure. A more complicated algebraic structure is the set of real numbers R together with the two usual operations: addition '+' and multiplication '×'. Such an algebraic structure is denoted by (R, +, ×). Algebraic structures with one or more binary operations are given special names depending upon additional properties involved.

An algebraic structure consisting of a set G under an operation $*$ on G, and denoted by (G, $*$), may enjoy one or more of the following characteristics:

Given $a, b, c, \ldots$ as the elements of the set G, the algebraic structure (G, $*$) may be

1. *Closed* if $a * b \in G$ for each $a, b \in G$.
2. *Commutative* if $a * b = b * a$, for each $a, b \in G$.
3. *Associative* if $(a * b) * c = a * (b * c)$, for each $a, b, c \in G$.
4. *Existence of identity element:* For each $a \in G$, if there exists an element $e \in G$, such that $a * e = a = e * a$ then e is called the *Identity element.*

   If $a * e = a$, then e is known as the right identity element and if $e * a = a$, then e is known as the left identity element of G for the operation $*$. If a set contains left as well as right identity element, then we say that it has the identity element.

   For example: If $0 + x = x$ for all $x \in G$, then 0 is the left identity element and $x + 0 = x$ for all $x \in G$, then 0 is the right identity element of G for the operation +. Similarly if $1.x = x$ and $x.1 = x$ for all $x \in G$, then 1 is said to be left identity and right identity element respectively of G for the operation ×.
5. *Existence of the inverse element*: For each $a \in G$, if there exists an element $a' \in G$ such that $a * a' = e = a' * a$ then a' is called the *inverse of the element a*.

   If $a * a' = e$, then a' is known as the right inverse of a and if $a' * a = e$, then a' is known as the left inverse of a under the operation $*$. For example: If $x + (-x) = 0$ for all $x \in G$, then $-x$ is the right additive inverse of x. Again if $(-x) + x = 0$ then $-x$ is the left additive inverse of x.

   Similarly, if $x.x^{-1} = 1$ for all $x \in G$ and $x \neq 0$ then x^{-1} is the right multiplicative inverse of x and if $x^{-1}.x = 1$ then x^{-1} is the left multiplicative inverse of x.

Worked Out Examples

Example 1

Given a set G = {0, 1} and a binary operation + defined by:

$0 + 0 = 0; \quad 0 + 1 = 1$

$1 + 0 = 1; \quad 1 + 1 = 0$

Cayley table

+	0	1
0	0	1
1	1	0

Find the additive identity and additive inverses of 0 and 1.

Solution:

a) Since $0 + 0 = 0$; $0 + 1 = 1 = 1 + 0$, so 0 is the additive identity,

b) Since $0 + 0 = 0$; $1 + 1 = 0$, so 0 is the additive inverse of 0, and 1 is the additive inverse of 1. In other words, every element of G is the additive inverse of itself.

Example 2

Given the algebraic structure (G, +) with G = {0, 1, 2, 3, 4, ...} and + is an operation of addition, find the identity elements of 2 and 3.

Solution:

Since $\quad 0 + 2 = 2 + 0 = 2 \in G$

and $\quad 0 + 3 = 3 + 0 = 3 \in G$

So 0 is the identity element of 2 and 3.

Example 3

Given the algebraic structure (G, ×) with G = {1, ω, ω²} where ω represents the complex cube root of unity, and × stands for the binary operation of ordinary multiplication of complex numbers, show that

a) 1 is the multiplicative identity

b) 1 is the multiplicative inverse of itself

c) ω and ω² are multiplicative inverses of each other.

Solution:

a) Since, $1 \times 1 = 1$, $1 \times \omega = \omega \times 1 = \omega$

and $1 \times \omega^2 = \omega^2 \times 1 = \omega^2$

so, 1 is the multiplicative identity.

b) Since $1 \times 1 = 1$, so 1 is the multiplicative inverse of itself.

c) Since $\omega \times \omega^2 = \omega^2 \times \omega = 1$

So ω is the inverse of ω^2 and ω^2 is the inverse of ω.

Example 4

Determine the identity element and inverse elements in each case given below:

a) Given an algebraic structure (G, •), with G = {–1, 1, –*i*, *i*}, where *i* stands for the imaginary unit; and • stands for usual multiplication operation.

b) Given an algebraic structure (Z, •), with the binary operation • defined by $m \cdot n = m + n - 1$ for all $m, n \in Z$.

Solution:

a) Since $(-1).1 = 1.(-1) = -1$

$1.1 = 1$

$-i.1 = 1.(-i) = -i$

and $i.1 = 1.i = i$

So 1 is the required identity element.

Again, $1.1 = 1, (-1).(-1) = 1$ and $(-i).i = i(-i) = 1$

So 1 is the inverse of 1, -1 is the inverse of -1. And $-i$ and i are the inverses of each other.

b) Suppose e is the required identity element under • defined by $m \bullet n = m + n - 1$ for all $m, n \in Z$.

Then, $m \bullet e = m + e - 1$

or, $m = m + e - 1$

So, $e = 1$

Since $m \in Z$, it suffices to obtain its inverse only. Let it be m'. Then,

$m \bullet m' = e = 1$

or, $m + m' - 1 = 1$

Hence, $m' = 2 - m$

Example 5

Solve: i) $3x = 1$ in Z_4 ii) $2x + 1 = 3$ in Z_5

Solution:

i) $3x = 1$ in Z_4

$\Rightarrow 3 \times_4 x = 1$

$\Rightarrow 3 \times_4 (3 \times_4 x) = 3 \times_4 1$

$\Rightarrow (3 \times_4 3) \times_4 x = 3$

$\Rightarrow 1 \times_4 x = 3$

$\therefore x = 3$

ii) $2x + 1 = 3$ in Z_5

$\Rightarrow 2 \times_5 x +_5 1 = 3$

$\Rightarrow 2 \times_5 x +_5 1 +_5 4 = 3 +_5 4$

$\Rightarrow 2 \times_5 x = 2$

$\Rightarrow 3 \times_5 (2 \times_5 x) = 3 \times_5 2$

$\Rightarrow (3 \times_5 2) \times_5 x = 1$

$\Rightarrow 1 \times_5 x = 1$

$\therefore x = 1$

EXERCISE

1. Given a set Z = {0, 1, 2, 3} and a binary operation $+_4$ is defined by the following Cayley's table.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Find the identity elements and the inverse elements of 2 and 3.

2. Let G = {0, 1, 2}. Form a composition table for G under addition modulo 3 and multiplication modulo 3. Find the identity elements and the inverse elements of 1 and 2 in each case.
3. Let G = {…, –3, –2, –1, 0, 1, 2, 3, …} and the operation defined on G be of addition. Find the identity and the inverse elements of 1 and 2.
4. Given the algebraic structure (G, ×) with G = {–1, 1} where × stands for the operation of multiplication, find the inverses of elements of G.
5. Determine the identity element and inverse elements of 3 and –2 in each case given below.
   a) Given an algebraic structure (Z, •) with binary operation • defined by $m \bullet n = m + n + 1$ for all $m, n \in Z$.
   b) Given an algebraic structure (G, •) with G = R – {1}, the set of real numbers without the unit number and • stands for the binary operation defined by $a \bullet b = a + b - ab$ for all $a, b \in G$.

Answers

1. Identity element = 0, inverse elements of 2 and 3 are 2 and 1 respectively.
2.

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Identity element = 0
Inverse elements of 1 and 2 are 2 and 1 respectively.

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Identity element = 1
Inverse elements of 1 and 2 are 1 and 2 respectively.

3. Identity elements of 1 and 2 = 0 and inverse elements of 1 and 2 are –1 and –2 respectively.
4. Inverse elements of –1 and 1 are –1 and 1 respectively

5. a) Identity element of 3 = –1, inverse element of 3 = –5, Identity element of –2 = –1, inverse element of –2 = 0

b) Identity element of 3 = 0, inverse element of 3 = $\frac{3}{2}$, Identity element of –2 = 0,

inverse element of –2 = $\frac{2}{3}$

Group

An algebraic structure (G, ∗), where G is a non-empty set with an operation '∗' defined on it, is said to be a *group*, if the operation ∗ satisfies the following axioms (called group axioms).

(G1) Closure Axiom. G is closed under the operation ∗,

i.e. $a * b \in G$ for all $a, b \in G$.

(G2) Associative Axiom. The binary operation ∗ is associative.

i.e., $(a * b) * c = a * (b * c)$ for $\forall\, a, b, c \in G$.

(G3) Identity Axiom. There exists an element $e \in G$, such that

$a * e = a = e * a$, for all $a \in G$.

The element e is called the identity of 'a' with respect to '∗' in G.

(G4) Inverse Axiom. Each element of G possesses inverse, i.e. for each element $a \in G$, there exists an element $b \in G$, such that $a * b = e = b * a$

The element b is then called the inverse of a with respect to '∗' and we often write $b = a^{-1}$. Thus a^{-1} is an element of G such that

$$a * a^{-1} = e = a^{-1} * a.$$

An algebraic structure with a set G under a binary operation ∗, and denoted by (G, ∗), is known as a *group* if it is associative, has an identity and an inverse element.

(Note: In the definition of a group sometimes the closure property is also mentioned. Once we mention operation ∗ as a binary operation this becomes redundant.)

a) Finite and infinite group

A group may contain a *finite* or an *infinite* number of elements. It is said to be *finite* or *infinite* according as the number of elements is finite or infinite. The number of elements in a group is often called the *order of the group*. It is denoted by | G | or o(G).

The set S = {1, –1} is a *finite* group under multiplication; and its order is 2. But, the set of integers is an *infinite* group under addition.

b) Trivial group

A group consisting of only one element is called a *trivial group*. The set G = {0} with usual addition operation is a trivial group; and the set H = {1} with usual multiplication is a trivial group under multiplication.

c) Abelian group

A group (G, $*$) is said to be an abelian group if $a * b = b * a$ for all $a, b \in G$.

Groups with elements other than numbers

Since a set may consist of any kind of non-material or material objects, and a binary operation can also be defined in various ways, we can define groups whose elements are non-numerical or blocks of numbers or matrices, or physical movements like translation, rotation etc. We now discuss some simple cases that involve blocks of numbers or non-numerical objects or processes.

Matrix Group

The set of square matrices of a given order is known to possess the associative property, the zero matrix as the identity matrix and the negative of a matrix as the inverse of the matrix. This means the set of all square matrices of a given order is a group. Such a group is known as the matrix group of the given order. A matrix group of order 2 under matrix addition is generally denoted by

$$M_2 = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, +\right) \quad \text{where } a, b, c, d \in R$$

Furthermore, the set of non-singular square matrices of a given order under matrix multiplication is known to be associative, has the unit matrix as the identity matrix, and the inverse of the given matrix as the inverse element. This means the set of non-singular square matrices forms a group under matrix multiplication.

Worked Out Examples

Example 1

Show that the set Z of integers does not form a group under the operation defined as $x * y = x - y$ for every $x, y \in Z$.

Solution:

Here, $Z = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\}$

Since the difference $x - y$ of any two integers is also an integer so belongs to Z, hence the operation is closed.

For associativity

$(2 * 3) * 4 = (2 - 3) * 4 = (-1) * 4 = -1 - 4 = -5$

and $2 * (3 * 4) = 2 * (3 - 4) = 2 * (-1) = 2 - (-1) = 3$

$2 * (3 * 4) \neq (2 * 3) * 4$

i.e. associativity is not satisfied.

Hence Z does not form a group under the given operation.

Example 2

Given the algebraic structure (G, ×), with G = {1, ω, ω^2} where ω represents the complex cube root of unity, and × stands for the binary operation of ordinary multiplication of complex numbers, show that (G, ×) is a group.

Solution:

a) Since the product of any two elements of G is also an element of G, so, the operation × is closed.

b) $1 \times (\omega \times \omega^2) = 1 \times \omega^3 = 1 \times 1 = 1$

and $(1 \times \omega) \times \omega^2 = \omega \times \omega^2 = \omega^3 = 1$

$\therefore \quad 1 \times (\omega \times \omega^2) = (1 \times \omega) \times \omega^2$

Associativity is satisfied.

c) Since $1 \times 1 = 1$, $1 \times \omega = \omega \times 1 = \omega$ and $1 \times \omega^2 = \omega^2 \times 1 = \omega^2$

1 is the multiplicative identity.

d) Since 1 × 1 = 1, 1 is the multiplicative inverse of itself.

And, since $\omega \times \omega^2 = 1$, $\omega^2 \times \omega = 1$, so ω and ω^2 are multiplicative inverses of each other.

Hence (G, ×) is a group.

Example 3

Let G = Q − {1}, the set of all rational numbers without the unit number. Suppose an operation ∗ defined on G is given by $a * b = a + b - ab$. Show that the system is a group.

Solution:

i) Here, $a * b = a + b - ab$ is obviously a rational number Q. It cannot be 1.

If $a + b - ab = 1$, then $(a - 1)(b - 1) = 0$

or, $a = 1$ or both 1, which is not possible.

So, ∗ is closed; and the operation is binary.

ii) For $a, b, c \in G$, $(a * b) * c = (a + b - ab) * c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b + c - ab - ac - bc + abc$

and $a * (b * c) = a * (b + c - bc) = a + b + c - bc - a(b + c - bc)$

$= a + b + c - ab - ac - bc + abc$

Hence, $(a * b) * c = a * (b * c)$, (associativity)

iii) Since $(a * 0) = a + 0 - a.0 = a$ and $(0 * a) = 0 + a - 0.a = a$

0 is the identity element.

iv) Suppose b is the inverse of a. Then $(a * b) = 0$, the identity element. Hence

$(a * b) \quad = a + b - ab = 0$

or, $b = \frac{a}{a-1} \in G$, because $a \neq 1$. i.e., the inverse of a is $\frac{a}{a-1}$.

Hence, from (i) – (iv), G is a group.

Example 4

Let G = {0, 1, 2, 3}. Form a Cayley's table for G under the multiplication modulo 4 ($\times_4$). Show that $\times_4$ satisfies closure property and associative property but G does not form a group under $\times_4$.

Solution:

Cayley's table for G under the multiplication modulo 4 ($\times_4$) is presented below

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

From the table, we see that the product of any two elements of G under multiplication modulo 4 is also in G. Hence $\times_4$ satisfies closure property.

For associativity

$$1 \times_4 (2 \times_4 3) = 1 \times_4 2 = 2$$

Also, $(1 \times_4 2) \times_4 3 = 2 \times_4 3 = 2$

$\therefore$ $1 \times_4 (2 \times_4 3) = (1 \times_4 2) \times_4 3$

This result is true for every element of G.

$\therefore$ $\times_4$ satisfies associative property.

From the third row and third column of the table, we see that 1 is the identity element.

No element of G multiplying 0 modulo 4 will give the identity element 1. So, 0 has no inverse element.

$\therefore$ G does not form a group under $\times_4$.

Example 5

Let S be the set of matrices A, B, C and D i.e. S = {A, B, C, D} where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\quad D = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Under matrix multiplication, prove the following results

a) $BC \in S$.

b) A is an identity element.

c) C is the inverse of itself.

d) B and D are the inverses of each other.

Solution:

a) $BC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = D \in S$

b) $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = B$

Similarly AC = C, AD = D, AA = A

∴ A is the identity element.

c) $CC = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A$

∴ C is the inverse of itself.

d) $BD = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A$

Similarly DB = A

∴ B and D are inverses of each other.

Example 6

Show that the set of all positive rational numbers form an abelian group under the composition defined by $a \circ b = \frac{ab}{2}$.

Solution:

Let Q^+ be the set of all positive rational numbers. We define the operation ∘ on Q^+ by

$$a \circ b = \frac{ab}{2} \quad \text{for all } a, b \in Q^+$$

a) Closure property: Since $a, b \in Q^+$, so $\frac{ab}{2} \in Q^+$

Hence Q^+ is closed under ∘.

b) Associativity: Suppose $a, b, c \in Q^+$

Then, $a \circ (b \circ c) = a \circ \left(\frac{bc}{2}\right) = \frac{1}{2} a \left(\frac{bc}{2}\right) = \frac{1}{4} abc$

Again, $(a \circ b) \circ c = \left(\frac{ab}{2}\right) \circ c = \frac{1}{2}\left(\frac{ab}{2} c\right) = \frac{1}{4} abc$

∴ $a \circ (b \circ c) = (a \circ b) \circ c$

c) Identity Element: Let e be an identity element.

If $e \in Q'$, then

$e \circ a = a$

Also, $e \circ a = \frac{ea}{2}$

$\therefore \quad a = \frac{ea}{2} \quad \Rightarrow \quad a(1 - \frac{e}{2}) = 0$

Since $a \neq 0$, so $e = 2$.

$\therefore$ 2 is an identity element.

d) Inverse element: Let b be an inverse element of a. If $b \in Q^+$

then $a \circ b = e = 2$

Also, $a \circ b = \frac{ab}{2}$

$\therefore \quad \frac{ab}{2} = 2 \quad \Rightarrow \quad b = \frac{4}{a}$

$\therefore \frac{4}{a}$ is the inverse element of a.

Since Q^+ satisfy all conditions of a group, so Q^+ represents a group under the operation $\circ$.

Also, $a \circ b = \frac{ab}{2} = \frac{ba}{2} = b \circ a$

Q^+ satisfies commutative property as well.

$\therefore (Q^+, \circ)$ is an abelian group.

EXERCISE

1. State whether the following statements are true or false. If false, give at least one reason
   a) the order of the group (G, •) with G = {–1, 1, –*i*, *i*} where *i* stands for the imaginary unit and • stands for the operation multiplication is infinite.
   b) The group ({1, –1}, •) under the operation • of usual multiplication is of order 2.
   c) The set {–2, –1, 0, 1, 2} is a group under multiplication.
   d) The set {2, 4, 6, 8, 10} is a group under addition.
   e) $G = \{2^m : m \in z\}$ forms a group under multiplication.
2. Show that the set S = {–1, 0, 1} does not form a group with respect to addition operation.
3. Prove that the set of natural numbers does not form a group with respect to addition operation.
4. Show that the set T = {–1, 1} forms a group under the addition operation.
5. Let G = {1, –1, *i*, –*i*} and the operation be of multiplication operation.
   a) show that G is closed under multiplication
   b) show that the operation is associative

c) show that identity element and the inverses exist.

d) can you conclude that G forms a group?

6. Show that the set of integers Z forms a group under the operation of addition.

7. A binary operation ∗ defined on the set S = {a, b, c} is presented in the following Cayley's table

∗	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Show that (S, ∗) forms a group.

8. Let G = {0, 1, 2, 3}. Show that G forms a group under the addition modulo 4.

9. If G = {a, b} forms a group under the operation ∗ with an identity element a, prepare a Cayley's table for G.

10. A set of matrices of the form $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where θ is a number, is given.

a) Show that the operation of matrix multiplication is closed.

b) Show that A_0 is the identity element of A_θ.

c) Show that $A_{-\theta}$ is the inverse element of A_θ.

11. If the set of all positive rational numbers with composition defined by $a \circ b = \frac{ab}{3}$ forms a group, find the identity element and the inverse of a.

12. Show that the set of all positive rational numbers forms an abelian group under the composition defined by $a \circ b = \frac{ab}{4}$.

A. S = {–1, 1, –i, i} is the set of fourth roots of units. Complete the following Cayley's table

×	–1	1	–i	i
–1	…	…	…	…
1	…	…	…	…
–i	…	…	…	…
i	…	…	…	…

a) Examine whether the multiplication operation defined above satisfies the closure property, associativity. Also examine whether there exist identity and inverse elements or not.

b) Does the set form a goup under multiplication?

c) Is the group an abelian?

Answers

1. a) No, it is of order 4.
   b) Yes, the group contains two elements.
   c) No, closure property is not satisfied.
   d) No, no identity element exists.

5. d) Yes.

9.

*	a	b
a	a	b
b	b	a

12. $e = 3, \frac{9}{a}$

Elementary Properties of Group

We know the definition of a group. It says that a group must have an identity element and every element must have an inverse. But there is no mention about the number of identities and the number of inverses of an element. For the investigation about these matters and others we shall prove some theorems. These theorems and others reveal some properties of groups.

Theorem I.

In a group there is one and only one identity element. (or in a group the identity element is unique).

Proof.

Let e be an identity element in a group (G, ∘). If possible, let e′ be another identity element. Then we have

$e \circ e' = e' \circ e = e'$, considering e as the identity element

Also $e \circ e' = e' \circ e = e$, considering e′ as the identity element.

Hence e = e′, i.e. there is one and only one identity element.

Theorem II.

Every element in a group (G, ∘) has unique inverse.

Proof.

We know that an element a in a group has an inverse a^{-1} such that

$a^{-1} \circ a = e = a \circ a^{-1}$

Suppose a' is another inverse of a. So we have

$a' \circ a = e = a \circ a'$

Now, $a' = e \circ a'$, as e is the identity element

$= (a^{-1} \circ a) \circ a'$ as $e = a^{-1} \circ a$

$= a^{-1} \circ (a \circ a')$ by associativity law

$= a^{-1} \circ e$ as $a \circ a' = e$

$= a^{-1}$ as e is the identity element.

This shows that the inverse element is unique.

Theorem III.

Cancellation law. If a, b, c are the element of a group (G, $\circ$) and $a \circ b = a \circ c$, then $b = c$. Also, if $b \circ a = c \circ a$, then $b = c$.

Proof.

As $a \in$ (G, $\circ$), a has the inverse a^{-1} such that $a^{-1} \circ a = a \circ a^{-1} = e$

We have $\quad a \circ b = a \circ c \quad$ ......(i)

Multiply both sides of (i) by a^{-1} on the left. Then,

$a^{-1} \circ (a \circ b) = a^{-1} \circ (a \circ c)$

or, $\quad (a^{-1} \circ a) \circ b = (a^{-1} \circ a) \circ c \quad$ by associativity law

or, $\quad e \circ b = e \circ c \quad$ because $a^{-1} \circ a = e$

or, $\quad b = c \quad$ as e is the identity element.

Similarly, we can show that if $b \circ a = c \circ a$, then $b = c$.

Theorem IV.

If $a, b \in$ (G, $\circ$), then $\quad$ i) $\quad (a \circ b)^{-1} = b^{-1} \circ a^{-1} \quad$ ii) $\quad (a^{-1})^{-1} = a$

Proof.

(i) We have

$(a \circ b) \circ (b^{-1} \circ a^{-1})$

$= ((a \circ b) \circ b^{-1}) \circ a^{-1} \quad$ by associativity law

$= (a \circ (b \circ b^{-1})) \circ a^{-1}$

$= (a \circ e) \circ a^{-1} \quad$ as $b \circ b^{-1} = e$

$= a \circ a^{-1}$

$= e$

Similarly, we can show that $(b^{-1} \circ a^{-1}) \circ (a \circ b) = e$

$\therefore a \circ b$ is the invese of $b^{-1} \circ a^{-1}$.

i.e. $\quad (a \circ b)^{-1} = b^{-1} \circ a^{-1}$

ii) We have $\quad a^{-1} \circ a = e$

Multiplying both sides on the left by $(a^{-1})^{-1}$ we get

$(a^{-1})^{-1} \circ (a^{-1} \circ a) = (a^{-1})^{-1} \circ e$

or, $\quad ((a^{-1})^{-1} \circ a^{-1}) \circ a = (a^{-1})^{-1} \quad$ by associativity law

or, $\quad e \circ a = (a^{-1})^{-1}$

or, $\quad a = (a^{-1})^{-1}$

Theorem V.

If a and b are the elements of a group (G, $\circ$), then

$a \circ x = b \quad$ and $\quad x \circ a = b \quad$ have unique solutions in (G, $\circ$).

Proof.

Let a be an element of the group (G, ∘). So a^{-1}, the inverse element of a, is also in (G, ∘) and $a^{-1} \circ a = a \circ a^{-1} = e$.

Now, $a \circ x = b$

$a^{-1} \circ (a \circ x) = a^{-1} \circ b$ Multiplying both sides by a^{-1} on the left.

or, $(a^{-1} \circ a) \circ x = a^{-1} \circ b$ Associativity law

or, $e \circ x = a^{-1} \circ b$

or, $x = a^{-1} \circ b$

This is the required solution.

To show the uniqueness of the solution, we suppose that x_1 and x_2 are the solutions, then

$a \circ x_1 = b$ and $a \circ x_2 = b$

$\therefore$ $a \circ x_1 = a \circ x_2$ each being equal to b

or, $a^{-1} \circ (a \circ x_1) = a^{-1} \circ (a \circ x_2)$

or, $(a^{-1} \circ a) \circ x_1 = (a^{-1} \circ a) \circ x_2$

or, $e \circ x_1 = e \circ x_2$

or, $x_1 = x_2$

So the solution is unique.

Similarly, we can show that $x \circ a = b$ has a unique solution in (G, ∘).

Worked Out Examples

Example 1

Prove that the additive and multiplicative identities in the set of real numbers are unique.

Solution:

i) The additive identity for R is denoted by 0. If it is not unique, let $e \in$ R be another additive identity. Then,

$0 + e = e$ when 0 is considered as the additive identity,

$0 + e = 0$ when e is considered as the additive identity.

Hence $e = 0$, i.e., the additive identity is unique.

ii) The multiplicative identity for R is denoted by 1. If it is not unique, let $e \in$ R denote another multiplicative identity. Then,

$1 \times e = e$, assuming 1 as the multiplicative identity.

$1 \times e = 1$, assuming e as the multiplicative identity.

Hence $e = 1$, i.e., the multiplicative identity is unique.

Example 2

If $a, b \in$ G and G is abelian, show that $(ab)^2 = a^2b^2$

Solution:

Since G is abelian so

$ab = ba$

Now, $(ab)^2 = (ab)(ab) = a(ba)b$

$= a(ab)b = (aa)(bb)$

$= a^2b^2$

EXERCISE

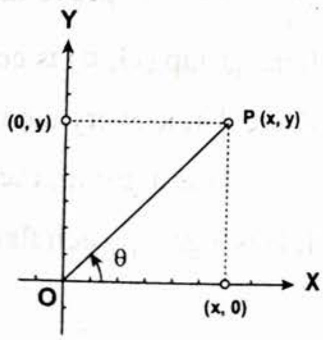
1. If a and b are the elements of a group (G, $*$) such that
   a) $a * b = b$, prove that $a = e$ b) $a * b = e$, prove that $b = a^{-1}$.
2. If the group (G, $\circ$) is commutative show that $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$, for all $a, b \in$ G.
3. Prove that if every element of a group G is its own inverse, then G is abelian.
4. If (G, $\circ$) is a group, then the group equation $x \circ x = x$ has a unique solution $x = e$.
5. If G is a group such that $(ab)^2 = a^2b^2$ for all $a, b \in$ G, prove that G is an abelian group.

Chapter 4
Complex Number

We have discussed about the complex number, its conjugate, absolute value and the geometrical interpretation in grade XI. Now, here we deal De-Moivre theorem, Euler's formula in complex number and then related problems.

Polar Form of a Complex Number

Let $z = (x, y) = x + iy$ be a complex number. It can be represented by a point P in the complex plane with cartesian coordinates (x, y). Let θ be the angle in the standard position with OP as its terminal arm, and r the length of the line segment OP.



So, we have

$$x = r\cos\theta, \qquad y = r\sin\theta$$

Thus, the complex number, $z = x + iy$, may be written in the following trigonometric form (polar form)

$$z = r(\cos\theta + i\sin\theta)$$

where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}, x \neq 0$.

Actually $r = |z|$ is the modulus of z and the angle θ is called the **amplitude** or the **argument of** z and is written as amp (z) or arg (z).

Since sin θ and cos θ are both periodic with a period 2π or 360°, the complex number

$$z = x + iy = r(\cos\theta + i\sin\theta)$$

may be written in the general form as

$$z = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)], \quad \text{if } \theta \text{ is in radians,}$$
$$= r[\cos(\theta + n\,360°) + i\sin(\theta + n\,360°)], \quad \text{if } \theta \text{ is in degrees, } n \text{ is an integer}$$

Euler's formula

The relation between the complex numbers in the exponential form and the trigonometrical form is known as Euler's formula in the complex number. It is defined by

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ is the complex number chose magnitude is r and the amplitude is θ. The formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

is also called the complex number in the Euler's form.

Products and Quotients in Polar Form

One of the important uses of the polar (or trigonometric) form of complex numbers is in the computation of products of complex numbers. It provides us a quick and efficient method for calculating the product and hence that of the quotient. Other important uses include the computation of powers and roots of complex numbers.

Theorem

The product and quotient of two complex numbers

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

are given by

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

and $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Proof:

$$z_1z_2 = r_1(\cos\theta_1 + i\sin\theta_1).\ r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$$
$$= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)], \text{ thus proving the first part.}$$

To prove the second part, we have

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)[\cos(-\theta_2) + i\sin(-\theta_2)]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$

Alternative method:

The above theorem can also be proved by the use of Euler's formula with an easy and short way.

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$$

Then, $\quad z_1.z_2 = r_1.e^{i\theta_1}.\ r_2e^{i\theta_2}$

$$= r_1r_2\, e^{i(\theta_1+\theta_2)}$$
$$= r_1r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\} \quad \ldots\ldots(i)$$

Again, $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$

$$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$
$$= \frac{r_1}{r_2} \{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\} \quad \ldots\ldots(ii)$$

(i) and (ii) are the two results to be proved.

From the above results, we have the following relations.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1}$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}$

give $|z_1| = r_1$, $|z_2| = r_2$

and $\text{amp}(z_1) = \theta_1$, $\text{amp}(z_2) = \theta_2$

Now, $z_1z_2 = r_1r_2\, e^{i(\theta_1+\theta_2)} = r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

gives $|z_1z_2| = r_1r_2 = |z_1||z_2|$

and $\text{amp}(z_1z_2) = \theta_1 + \theta_2 = \text{amp}(z_1) + \text{amp}(z_2)$

Again, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \dfrac{r_1}{r_2}\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\}$

gives, $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

and $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \text{amp}(z_1) - \text{amp}(z_2)$

The product and the quotient of two complex numbers can be represented as follows:

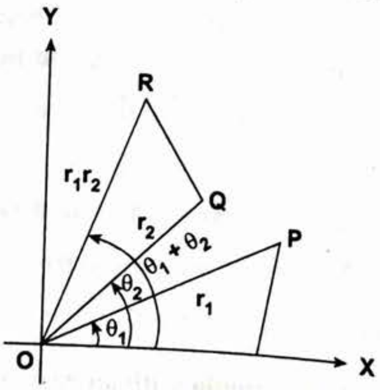


Fig. (i)

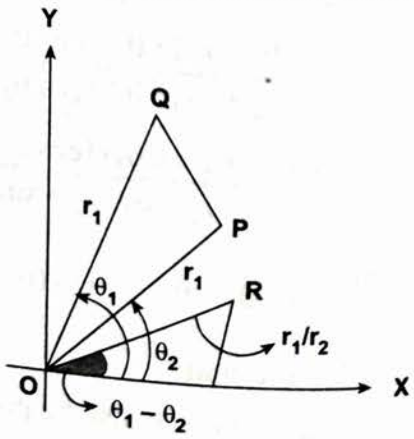


Fig. (ii)

If the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ with magnitudes r_1, r_2 and the amplitudes θ_1, θ_2 be represented in the complex plane by the points

P(r_1, θ_1) and Q(r_2, θ_2) then the complex number z_1z_2 with magnitude r_1r_2 and the amplitude $\theta_1 + \theta_2$ will be represented by the point R such that $\angle XOR = \theta_1 + \theta_2$ and $OR = r_1r_2$.

In the same way, the complex number $\frac{z_1}{z_2}$ with magnitude $\frac{r_1}{r_2}$ and amplitude $\theta_1 - \theta_2$ will be represented by the point R such that

$$\angle XOR = \theta_1 - \theta_2 \quad \text{and} \quad OR = \frac{r_1}{r_2}.$$

Integral Powers and Roots of Complex Numbers

The product of a complex number z by itself, i.e., $z. z$ is denoted by z^2 and is called the square or second power of z. The cube or the third power of z, denoted by z^3, is defined by $z^3 = z^2. z$. In general, for any positive integer n, the nth power of a complex number z is defined by

$$z^n = z^{n-1}. z \quad \text{and} \quad z^o = 1.$$

Obviously, the nth power of a complex number is also a complex number. Let us denote it by w, so that

$$z^n = w.$$

Conversely, if $w \neq 0$, and if n is a positive integer, then any complex number z whose n^{th} power is w, is known as the n^{th} root of w. In other words, any complex number z such that

$$z^n = w, \quad w \neq 0, \quad n = 1, 2, 3, ...$$

is known as the n^{th} root of w. The n^{th} root of w is usually denoted by

$$w^{1/n} \quad \text{or} \quad \sqrt[n]{w}.$$

The computation of the n^{th} power and the n^{th} root of a complex number may be carried out with comparative case with the help of a theorem known as De Moivre's theorem, which uses the polar (or trigonometric) form of a complex number.

The method of finding the n^{th} roots of a complex number will be given after De-Moivre's Theorem.

De Moivre's Theorem

If n is any positive integer,

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Proof.

Obviously, when $n = 1$,

$[r(\cos\theta + i\sin\theta)]^1 = r(\cos\theta + i\sin\theta)$.

For $n = 2$, we have

$[r(\cos\theta + i\sin\theta)]^2$

$$= r^2(\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta)$$

$= r^2 (\cos^2\theta - \sin^2\theta + i\, 2 \sin\theta \cos\theta)$

$= r^2 (\cos 2\theta + i \sin 2\theta)$.

Thus the theorem is true for $n = 1$ and $n = 2$. We prove the theorem by induction.

Let us assume that the theorem is true for some positive integer k.

By assumption,

$$[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta).$$

Multiplying both sides by $r(\cos\theta + i\sin\theta)$, we get

$$[r(\cos\theta + i\sin\theta)]^{k+1} = r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= r^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)]$$
$$= r^{k+1}[\cos(k+1)\theta + \sin(k+1)\theta],$$

which shows that the theorem is true for $n = k + 1$ whenever it is true for $n = k$. But we know that it is true for $n = 1$ and $n = 2$. When it is true for $n = 2$, the above proof shows that it is true for $n = 3$. Continuing this way, we come to the conclusion that the theorem is true for every positive integer n. This completes the proof.

In fact, the theorem is true not only for a positive integer but also for any real number n. We shall assume this without proof.

Let us now apply this theorem to compute the integral powers of complex numbers.

Alternative proof of De-Moivre's theorem

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, …… and $z_n = r_n(\cos\theta_n + i\sin\theta_n)$, n be complex numbers. Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$

We have $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $z_3 = r_3 e^{i\theta_3}$, …… and $z_n = r_n e^{i\theta_n}$

Now,

$$z_1 z_2 z_3 \dots z_n = r_1 e^{i\theta_1}, r_2 e^{i\theta_2} . r_3 e^{i\theta_3} \dots\dots . r_n e^{i\theta_n}$$
$$= (r_1 r_2 r_3 \dots r_n)\, e^{i(\theta_1 + \theta_2 + \theta_3 + \dots\dots + \theta_n)}$$
$$= r_1 r_2 r_3 \dots r_n \{\cos(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n)$$

If $z_1 = z_2 = z_3 = \dots z_n = r_z$ (say)

$r_1 = r_2 = r_3 = \dots = r_n = r$ and $\theta_1 = \theta_2 = \theta_3 = \dots = \theta_n = \theta$

then $z . z . z \dots z$ (n times)

$$= (r . r . r \dots . r)\{\cos(\theta + \theta + \theta + \dots + \theta) + i\sin(\theta + \theta + \theta + \dots + \theta)\}$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$\Rightarrow \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta)$$

which completes the proof of De-Moivre's theorem.

De-Moivre's theorem can also be proved more shortly in the following way.

If $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = z\,.\,z\,.\,z \ldots z \text{ (n times, n being positive integer)}$$
$$= r(\cos\theta + i\sin\theta)\,.\,r(\cos\theta + i\sin\theta) \ldots\ldots r(\cos\theta + i\sin\theta)$$
$$= r^{i\theta}\,.\,re^{i\theta}\,.\,re^{i\theta} \ldots\ldots re^{i\theta} \qquad \text{(n times)}$$
$$= r^n e^{i(\theta+\theta+\theta+\ldots\ldots+\theta)}$$
$$= r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

n^{th} root of a given complex number

Let $z = r(\cos\theta + i\sin\theta)$ be the given complex number whose n^{th} root is required.

Let $w = R(\cos ø + i\sin ø)$ be the n^{th} root of the complex number z.

Then, $\quad w^n = z$

$\Rightarrow \quad \{R(\cos ø + i\sin ø)\}^n = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad R^n(\cos nø + i\sin nø) = r(\cos\theta + i\sin\theta)$

Since the two complex numbers are equal, so

$$R^n = r \quad \Rightarrow \quad R = r^{1/n} = \sqrt[n]{r}$$

and $\cos nø + i\sin nø = \cos\theta + i\sin\theta$

$\Rightarrow \quad \cos nø = \cos\theta$ and $\sin nø = \sin\theta$

$\Rightarrow \quad nø = 2k.\pi + \theta \qquad$ or, $\quad k.360° + \theta$

$$\therefore \quad ø = \frac{k.360° + \theta}{n}$$

$$\therefore \quad w = R(\cos ø + i\sin ø)$$
$$= \sqrt[n]{r}\left\{\cos\frac{k.360° + \theta}{n} + i\sin\frac{k.360° + \theta}{n}\right\}$$

For $k = 0, 1, 2, 3, \ldots\ldots, n-1$, we get n different roots of z. For other integral value of k will give the root obtained earlier i.e. roots will be repeated.

Thus the k^{th} root of z denoted by z_k is given by

$$z_k = \sqrt[k]{r}\left\{\cos\frac{k.360° + \theta}{n} + i\sin\frac{k.360° + \theta}{n}\right\} \qquad k = 0, 1, 1, 2, 3, \ldots\ldots, n-1$$

Worked Out Examples

Example 1.

Express $2 + 2\sqrt{3}\,i$ in the polar form and Euler form.

Solution.

Let $2 + 2\sqrt{3}\,i = r(\cos\theta + i\sin\theta)$. Then by the definition of the equality of two complex numbers

$r\cos\theta = 2$

and $r\sin\theta = 2\sqrt{3}$.

This gives $r^2 = 4 + 12 = 16$,

or $r = 4$ (positive value only).

Also, $\cos\theta = \dfrac{1}{2}$ and $\sin\theta = \dfrac{\sqrt{3}}{2}$,

giving $\theta = 60°$

Hence $2 + 2\sqrt{3}i = 4(\cos 60° + i\sin 60°)$

$= 4(\cos \pi/3 + i\sin \pi/3)$

$= 4e^{i\pi/3}$

Example 2

Find the value of $\dfrac{2(\cos 70° + i\sin 70°)}{\cos 10° + i\sin 10°}$ using Euler's formula.

Solution :

$$\frac{2(\cos 70° + i\sin 70°)}{\cos 10° + i\sin 10°} = \frac{2(\cos 7\pi/18 + i\sin 7\pi/18)}{(\cos \pi/18 + i\sin \pi/18)}$$

$$= \frac{2e^{i\,7\pi/18}}{e^{i\,\pi/18}} = 2e^{i(7\pi/18 - \pi/18)}$$

$$= 2e^{i\pi/3} = 2(\cos \pi/3 + i\sin \pi/3)$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 1 + i\sqrt{3}$$

Example 3

Prove that $\dfrac{\cos 8\theta + i\sin 8\theta}{(\cos\theta + i\sin\theta)^6} = \cos 2\theta + i\sin 2\theta$

Solution :

$$\frac{\cos 8\theta + i\sin 8\theta}{(\cos\theta + i\sin\theta)^6}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^6} \quad \text{(using De-Moivre's theorem)}$$

$$= (\cos\theta + i\sin\theta)^2$$

$$= \cos 2\theta + i\sin 2\theta$$

Example 4

Using De-Moivre's theorem, evaluate $(1 - \sqrt{3}\,i)^6$.

Solution :

Let $1-\sqrt{3}\,i = r(\cos\theta + i\sin\theta)$

Equating real and imaginary parts,

$$r\cos\theta\,(x) = 1,\ r\sin\theta\,(y) = -\sqrt{3}$$

$$r = \sqrt{(1)^2 + (-\sqrt{3})^2} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$

$\therefore \quad \theta = 300°$

$\therefore \quad 1-\sqrt{3}\,i = 2(\cos 300° + i\sin 300°)$

Now, $(1-\sqrt{3}\,i)^6 = \{2\cos(300° + i\sin 300°)\}^6$

$= 2^6\{\cos 1800° + i\sin 1800°\}$

$= 2^6\{\cos 0° + i\sin 0°\}$

$= 2^6 = 64$

Example 5

Show that $z^3 = 1$, if $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Solution.

Let $z = x + iy = r(\cos\theta + i\sin\theta) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Here, $x = -\frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

Then, $r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$.

$\tan\theta = \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$, or $\theta = 120°$ $\left[\begin{array}{l}\text{in the second quadrant}\\ \text{as } x \text{ is negative and}\\ y \text{ is positive}\end{array}\right]$

Thus, $z = \cos 120° + i\sin 120°$.

Again, $z^3 = (\cos 120° + i\sin 120°)^3$

$= \cos 3(120°) + i\sin 3(120°)$

$= \cos 360° + i\sin 360°$

$= 1.$

Another important use of De Moivre's theorem is in the computation of the roots of complex numbers. We shall illustrate the method with some examples.

Example 6

Find the square roots of $2 + 2\sqrt{3}i$

Solution :

Let $z = 2 + 2\sqrt{3}i$

Here, $x = 2$ and $y = 2\sqrt{3}$

To write z in polar form we note that

$$r = \sqrt{2^2 + (2\sqrt{3})^2} = 4$$

and $\tan\theta = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$ or, $\theta = 60°$

$\therefore$ in polar form

$$z = 4(\cos 60° + i \sin 60°)$$

$$= 4[\cos(60° + n\,360°) + i \sin(60° + n\,360°)] \text{ in general polar form}$$

$$\therefore \quad \sqrt{z} = z^{1/2} = \sqrt{4}\,[\cos(60° + n\,360°) + i \sin(60° + n\,360°)]^{1/2}$$

$$= 2[\cos(30° + n\,180°) + i \sin(30° + n\,180°)] \quad \text{where } n = 0, 1$$

(Using De Moivre's theorem)

When $n = 0$, $z^{1/2} = 2(\cos 30° + i \sin 30°) = \sqrt{3} + i$

When $n = 1$, $z^{1/2} = 2(\cos 210° + i \sin 210°) = -\sqrt{3} - i$

For $n = 2, 3, \ldots$ the values obtained above will repeat, as the angles differ by multiples of 360°.

Note: The square roots can be obtained directly using the formula given in Art. 11.11

Example 7

If z be a complex number, prove that $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$ and $\text{amp}\left(\dfrac{1}{z}\right) = -\text{amp}(z)$

Solution :

Let $z = r(\cos\theta + i \sin\theta)$

Then, $|z| = r$ and $\text{amp}(z) = \theta$

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r(\cos\theta + i\sin\theta)} \times \frac{\cos\theta - i\sin\theta}{(\cos\theta - i\sin\theta)}$$

$$= \frac{\cos\theta - i\sin\theta}{r(\cos^2\theta + \sin^2\theta)}$$

$$= \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

Now, $\left|\frac{1}{z}\right| = \frac{1}{r} = \frac{1}{|z|}$

Again, amp $\left(\frac{1}{z}\right) = -\theta = -$ amp (z)

Alternatively

$$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r\,e^{i\theta}} = \frac{1}{r}\,e^{-i\theta}$$

$$= \frac{1}{r}\{\cos(-\theta) + i\sin(-\theta)\}$$

Now, $\left|\frac{1}{z}\right| = \frac{1}{r} = \frac{1}{|z|}$

and amp $\left(\frac{1}{z}\right) = -\theta = -$ amp (z)

Example 8

Solve : $z^6 + 1 = 0$

Solution:

$$z^6 + 1 = 0$$

$$\Rightarrow \quad z^6 = -1$$

$$\Rightarrow \quad z^6 = \cos 180° + i\sin 180°$$

$$\Rightarrow \quad z = (\cos 180° + i\sin 180°)^{1/6}$$

$$= \{\cos(k.360° + 180°) + i\sin(k.360° + 180°)\}^{1/6}$$

$$= \cos\frac{k.360° + 180°}{6} + i\sin\frac{k.360° + 180°}{6}, \qquad k = 0, 1, 2, 3, 4, 5$$

$$= \cos(k.60° + 30°) + i\sin(k.60° + 30°)$$

When $k = 0$, $z = \cos 30° + i\sin 30° = \frac{\sqrt{3}}{2} + i\frac{1}{2} = \frac{\sqrt{3}+i}{2}$

$k = 1$, $z = \cos 90° + i\sin 90° = 0 + i\,.\,1 = i$

$k = 2$, $z = \cos 150° + i\sin 150° = -\frac{\sqrt{3}}{2} + i\,.\,\frac{1}{2} = \frac{-\sqrt{3}+i}{2}$

$k = 3$, $z = \cos 210° + i\sin 210° = -\frac{\sqrt{3}}{2} - i\,.\,\frac{1}{2} = \frac{-\sqrt{3}-i}{2}$

$k = 4$, $z = \cos 270° + i\sin 270° = 0 + i(-1) = -i$

$k = 5$, $z = \cos 330° + i\sin 330° = \frac{\sqrt{3}}{2} - i\,.\,\frac{1}{2} = \frac{\sqrt{3}-i}{2}$

which are the 6 roots of the given equation.

Example 9

If $z = \cos\theta + i\sin\theta$, prove that $z^n + \frac{1}{z^n} = 2\cos n\theta$

Solution :

$z = \cos\theta + i\sin\theta$

Then, $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

and $\frac{1}{z^n} = z^{-n} = \cos n\theta - i\sin n\theta$

Now, $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

EXERCISE

1. Express the following complex numbers in the Euler's form:
   (a) $2 + 2i$ (b) $\sqrt{3} + i$ (c) $2i$ (d) $i - \sqrt{3}$
   (e) $1 - i$ (f) $\frac{(1+i)}{(1-i)}$ (g) $\frac{i}{(1+i)}$ (h) $\sqrt{\frac{1-i}{1+i}}$
2. Express the following complex numbers in the Euler's form:
   (a) $3(\cos 60° + i\sin 60°)$ (b) $3(\cos 120° + i\sin 120°)$
   (c) $2(\cos 150° + i\sin 150°)$ (d) $2\cos(-45°) + i\,2\sin(-45°)$.
3. Simplify using Euler's formula
   a) $(\cos 32° + i\sin 32°)(\cos 13° + i\sin 13°)$
   b) $(\sin 40° + i\cos 40°)(\cos 40° + i\sin 40°)$
   c) $\frac{\cos 80° + i\sin 80°}{\cos 20° + i\sin 20°}$
   d) $\frac{(\cos 3\theta + i\sin 3\theta)(\cos\theta - i\sin\theta)}{(\cos\theta + i\sin\theta)^2}$
4. Apply De Moivre's Theorem to compute
   a) $[2(\cos 15° + i\sin 15°)]^6$
   b) $[3(\cos 120° + i\sin 120°)]^3$
   c) $(\cos 18° + i\sin 18°)^5$
   d) $[\cos 9° + i\sin 9°]^{40}$
   e) $(1 + i)^{20}$
   f) $(-1 + i)^{14}$
   g) $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^7$
   i) i^2
5. Using De–Moivre's Theorem, find the square roots of
   a) $4 + 4\sqrt{3}i$
   b) $-1 + \sqrt{3}i$
   c) $-2 - 2\sqrt{3}i$
   d) $2i$
   e) $-i$

6. Determine the cube roots of -1.

7. Solve the following equations
   a) $z^4 = 1$ b) $z^6 = 1$ c) $z^4 + 1 = 0$ d) $z^3 = 8i$

8. Find the fourth roots of $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

9. If $\bar{z}$ be the conjugate of the complex number z, prove that $\operatorname{Arg}(\bar{z}) = 2\pi - \operatorname{Arg}(z)$.

10. If $z = \cos\theta + i\sin\theta$, prove that $z^n - \frac{1}{z^n} = 2\sin n\theta i$.

A. Using Euler's formula, prove that
   a) $i^2 = -1$
   b) i) $\cos(x + y) = \cos x \cos y - \sin x \sin y$
      ii) $\sin(x + y) = \sin x \cos y + \cos x \sin y$

B. Using De-Moivre's theorem, prove that $i^2 = -1$.

Answers

1. (a) $2\sqrt{2}\, e^{i\pi/4}$ (b) $2\, e^{i\pi/6}$ (c) $2\, e^{i\pi/2}$ (d) $2\, e^{i5\pi/6}$ (e) $\sqrt{2}\, e^{i7\pi/4}$ (f) $e^{i\pi/2}$
   (g) $\frac{1}{\sqrt{2}} e^{i\pi/4}$ (h) $e^{i7\pi/4}$
2. (a) $3\, e^{i\pi/3}$ (b) $3\, e^{i2\pi/3}$ (c) $2\, e^{i5\pi/6}$ (d) $2\, e^{-i\pi/4}$
3. a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ b) i c) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ d) 1
4. a) $64i$ b) 27 c) i d) 1 e) -2^{10} f) $2^7 i$ g) $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ h) -1
5. a) $\pm(\sqrt{6} + i\sqrt{2})$ b) $\pm\frac{1}{\sqrt{2}}(1 + i\sqrt{3})$ c) $\pm(-1 + i\sqrt{3})$ d) $\pm(1 + i)$ e) $\pm\frac{1 - i}{\sqrt{2}}$
6. $-1, \frac{1}{2}(1 + i\sqrt{3}), \frac{1}{2}(1 - i\sqrt{3})$
7. a) $\pm 1, \pm i$ b) $\pm 1, \frac{1}{2}(1 \pm i\sqrt{3}), \frac{1}{2}(-1 \pm i\sqrt{3})$ c) $\pm\frac{1 + i}{\sqrt{2}}, \pm\frac{1 - i}{\sqrt{2}}$
   d) $\sqrt{3} + i, -\sqrt{3} + i, -2i$
8. $\pm\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right), \pm\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$

The Cube Roots of Unity

Let z be the cube roots of unity i.e. 1. Then,

$z^3 = 1$

$\Rightarrow z^3 = 1 + i.0$

$\Rightarrow z^3 = \cos 0° + i \sin 0°$ (expressing in polar form)

$$\Rightarrow z = (\cos 0° + i \sin 0°)^{1/3}$$
$$= \{\cos (k.360° + 0°) + i \sin (k.360° + 0°)\}^{1/3}$$
$$= \cos \frac{k.360°}{3} + i \sin \frac{k.360°}{3}, \quad k = 0, 1, 2$$
$$= \cos (k.120°) + i \sin (k.120°)$$

When $k = 0$, $z = \cos 0° + i \sin 0° = 1 + i.0 = 1$

$k = 1$, $z = \cos 120° + i \sin 120° = -\frac{1}{2} + i.\frac{\sqrt{3}}{2} = \frac{1 + \sqrt{3}i}{2}$

$k = 2$, $z = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{-1 - \sqrt{3}i}{2}$

So the three cube roots of unity are 1, $\frac{-1 + \sqrt{3}i}{2}$ and $\frac{-1 - \sqrt{3}i}{2}$.

The first one is a real number and the other two are imaginary or complex numbers, and these are often known as the **imaginary cube roots** of unity, any one of which is denoted by the greek letter ω (omega).

Properties of the Cube Roots of Unity

i) Each imaginary cube root of unity is the square of the other.

For, $\left(\frac{-1 + \sqrt{3}i}{2}\right)^2 = \frac{1 - 2\sqrt{3}i + 3i^2}{4}$

$$= \frac{1 - 2\sqrt{3}i - 3}{4} = \frac{-1 - \sqrt{3}i}{2}$$

and $\left(\frac{-1 - \sqrt{3}i}{2}\right)^2 = \frac{1 + 2\sqrt{3}i + 3i^2}{4}$

$$= \frac{-2 + 2\sqrt{3}i}{4} = \frac{-1 + \sqrt{3}i}{2}$$

Thus if we write ω for any one of the imaginary cube roots, the other will be ω^2. Hence the three cube roots of unity are 1, ω, ω^2.

ii) The product of the two imaginary cube roots of unity is equal to 1.

For $\omega \cdot \omega^2 = \frac{-1 + \sqrt{3}i}{2} \times \frac{-1 - \sqrt{3}i}{2} = \frac{1 - 3i^2}{4} = \frac{4}{4} = 1$

As a direct consequence, we have

$\omega^3 = \omega \cdot \omega^2 = 1$, $\omega^{3n} = 1$ for any integral value of n.

Also $\omega^2 = \frac{1}{\omega}$ and $\omega = \frac{1}{\omega^2}$

Thus, one imaginary cube root of unity is the reciprocal of the other.

iii) The sum of the three cube roots of unity is zero.

For $1 + \omega + \omega^2 = 1 + \frac{-1 + \sqrt{3}i}{2} + \frac{-1 - \sqrt{3}i}{2}$

$$= \frac{2 - 1 + \sqrt{3}i - 1 - \sqrt{3}i}{2} = 0$$

Thus, we have two important relations

$$1 + \omega + \omega^2 = 0 \quad \text{and} \quad \omega^3 = 1$$

It may be noted here that any integral power of ω will reduce to 1, ω or ω^2.

For examples, $\omega^4 = \omega^3 \cdot \omega = \omega$, $\quad \omega^5 = \omega^3 \cdot \omega^2 = \omega^2$

$\omega^6 = 1$, $\quad \omega^{20} = \omega^{18} \cdot \omega^2 = (\omega^3)^6 \omega^2 = \omega^2$

$\omega^{-1} = \frac{1}{\omega} = \frac{\omega^3}{\omega} = \omega^2$ $\quad \omega^{-10} = \frac{1}{\omega^{10}} = \frac{1}{\omega^9 \cdot \omega} = \frac{1}{\omega} = \omega^2$, etc.

Here we solve only the problems related to the cube roots of unity.

Worked Out Examples

Example 1

Show that $(1 - \omega + \omega^2)^4 + (1 + \omega - \omega^2)^4 = -16$

Solution :

Since $1 + \omega + \omega^2 = 0$,

we have $1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$

$\therefore$ L.H.S. $= (1 - \omega + \omega^2)^4 + (1 + \omega - \omega^2)^4$

$= (-2\omega)^4 + (-2\omega^2)^4$

$= 16\omega^4 + 16\omega^8$

$= 16(\omega^4 + \omega^8)$

$= 16(\omega + \omega^2)$

$= 16(-1) = -16$

Example 2

Prove that : $\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2} = \omega$

Solution :

$\frac{a + b\omega + c\omega^2}{b + c\omega + a\omega^2}$

$= \frac{a\omega^3 + b\omega + c\omega^2}{b + c\omega + a\omega^2}$ $\quad (\because \omega^3 = 1)$

$= \frac{\omega(a\omega^2 + b + c\omega)}{b + c\omega + a\omega^2} = \omega$

EXERCISE

1. If ω be a complex cube root of unity, show that
   a) $(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3=0$
   b) $(2+\omega+\omega^2)^3+(1+\omega-\omega^2)^8-(1-3\omega+\omega^2)^4=1$
   c) $(1-\omega+\omega^2)^4(1+\omega-\omega^2)^4=256$
   d) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=9.$
   e) $\dfrac{a+b\omega+c\omega^2}{a\omega+b\omega^2+c}+\dfrac{a+b\omega+c\omega^2}{a\omega^2+b+c\omega}=-1$
2. a) If $\alpha=\frac{1}{2}(-1+\sqrt{-3})$ and $\beta=\frac{1}{2}(-1-\sqrt{-3})$, show that $\alpha^4+\alpha^2\beta^2+\beta^4=0$
   b) If α and β are the complex cube roots of unity, prove that $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=0$
3. Prove that
   a) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^6=2$ b) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^4+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^4=-1$
4. If $x=a+b$, $y=a\omega+b\omega^2$, $z=a\omega^2+b\omega$ show that
   i) $x+y+z=0$ ii) $xyz=a^3+b^3$ iii) $x^3+y^3+z^3=3(a^3+b^3)$

Chapter 5
Polynomial Equations and Natural Numbers

Introduction

A function f defined by

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n, \ (a_0 \neq 0) \qquad \ldots(1)$$

where n is non-negative integer, $a_0, a_1, \ldots\ldots, a_{n-1}, a_n$ are all constants is called a **rational integral function** or **polynomial of degree n in x**. The constants $a_0, a_1, \ldots\ldots, a_n$ are respectively called the **coefficients** of $x^n, x^{n-1}, \ldots\ldots, x^0 = 1$, and each of $a_0x^n, a_1x^{n-1}, \ldots\ldots, a_n$ is called a **term** of the polynomial. The term a_0x^n is called the **leading term** of the polynomial. Particular examples of polynomials written as functions are

$$g(x) = 2x - 4, \qquad G(x) = x^2 + 2x - 3, \qquad h(x) = 5x^3 - 3x^2 + x - 1, \text{ etc.}$$

A polynomial is sometimes called a **quantic**. The quantics of various successive degrees have special names. For instances, the polynomials of degrees one, two, three and four are respectively called **linear (first degree), quadratic (or quadric), cubic and biquadratic (or quartic).** Thus the polynomials defined by

$$g(x) = ax + b, \qquad G(x) = ax^2 + bx + c,$$
$$h(x) = ax^3 + bx^2 + cx + d \quad \text{and} \quad H(x) = ax^4 + bx^3 + cx^2 + dx + e$$

are linear, quadratic, cubic and biquadratic respectively.

If for a certain value a of x, $f(a) = 0$, the value $x = a$ is called a **zero** of the polynomial defined by (1). For instance, the value $x = 2$ is a zero of the linear function (polynomial) defined by $g(x) = 2x - 4$, since

$$g(2) = 2.2 - 4 = 0.$$

Note that there is no other zero (i.e. no other value of x which makes it zero). Similarly, it is easy to verify that $x = 1$ and $x = -3$ are the zeros of the quadratic defined by $G(x) = x^2 + 2x - 3$. Here the number of zeros is exactly two. Thus we notice that a polynomial may have one or more zeros depending upon the degree of the polynomial.

Polynomial Equations

A polynomial of degree n in x defined by

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}\,x + a_n \qquad \ldots\ldots(1)$$

where n is a non-negative integer and $a_0, a_1, \ldots, a_n$ are all constants may vanish for one or more values of x.

Suppose f(x) = 0 for some values of x, then

$$f(x) = 0 \quad \ldots\ldots(2)$$

is called a general equation of degree n in x and the values of x for which (2) is true are known as the solutions of the equation. This equation is sometimes known as a polynomial equation of degree n in x. Some special types of polynomial equations are :

(a) Linear equation: $ax + b = 0$

(b) Quadratic equation: $ax^2 + bx + c = 0$

(c) Cubic equation: $ax^3 + bx^2 + cx + d = 0$

and (d) Biquadratic equation: $ax^4 + bx^3 + cx^2 + dx + e = 0$

Regarding polynomial equations, we assume without proof, the following fundamental theorem of algebra :

'Every Equation has at least a root'

The proof of this theorem is beyond the level of the present text. On the basis of this theorem and the 'Factor Theorem', we shall now prove a proposition.

Theorem 1

Every equation of degree n in x has n roots, and no more.

Proof.

Let the given equation be denoted by f(x) = 0,

where $f(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_{n-1}x + a_n$.

From the 'Fundamental Theorem of Algebra', the equation f(x) = 0 has a root, real or imaginary. Let this be denoted by α_1; then, by the 'Factor Theorem', $x - \alpha_1$ is a factor of f(x), so that

$$f(x) = (x - \alpha_1)\,\phi_1(x),\ \phi_1(x) = a_0x^{n-1} + \ldots\ldots$$

where $\phi_1(x)$ is a polynomial of degree $n - 1$. Again, the equation $\phi_1(x) = 0$ has a root, real or imaginary; let it be denoted by α_2 then $x - \alpha_2$ is factor of $\phi_1(x)$, so that

$$\phi_1(x) = (x - \alpha_2)\,\phi_2(x),\ \phi_2(x) = a_0x^{n-2} + \ldots\ldots$$

where $\phi_2(x)$ is polynomial of degree $n - 2$.

Thus $f(x) = (x - \alpha_1)(x - \alpha_2)\,\phi_2(x)$.

Proceeding in this way, we obtain

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots\ldots (x - \alpha_n)$$

Hence the equation f(x) = 0 has n solutions (i.e. roots), since f(x) vanishes when x has any of the values

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_n.$$

Also the equation cannot have more than n roots; for if x has any value different from any of the quantities α_1, α_2, α_3,......, α_n, all the factors on the right are different from zero, and therefore f(x) cannot vanish for that value of x.

This theorem may be used to deduce that

i) A biquadratic equation in x has exactly four roots and no more.

ii) A cubic equation has three and only three roots.

iii) A quadratic equation has two and only two roots.

It may also be used in the investigation of the relations between the roots and the coefficients of any equation.

Quadratic Equation

Various properties of quadratic equation may be derived directly from those of the general equation of degree n. But, in many occasions, we do not need the general theory of equation of degree n. It is generally felt that a systematic study of the theory of quadratic equation throws sufficient lights on the general theory. We shall therefore consider the quadratic equation in a greater detail in the present and the following sections:

Theorem 1

The two roots of a quadratic equation,

$$ax^2 + bx + c = 0 \qquad (a \neq 0),$$

are $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

Proof.

Let us divide both sides of

$$ax^2 + bx + c = 0$$

by the coefficient of x^2 and transpose the constant to the right side of the equation, so we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Adding $\dfrac{b^2}{4a^2}$ to both sides of the equation,

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

or $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

Taking square roots we get

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a},$$

or $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

The use of the plus and the minus signs gives the two roots of the quadratic equation. This completes the proof.

Theorem 2

The quadratic equation $ax^2 + bx + c = 0$ cannot have more than two roots.

Proof.

For, if possible, let α, β, γ be three different roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0).$$

Then, since each of these values must satisfy the equation, we have

$$a\alpha^2 + b\alpha + c = 0 \qquad \ldots\ldots (1)$$

$$a\beta^2 + b\beta + c = 0 \qquad \ldots\ldots (2)$$

$$a\gamma^2 + b\gamma + c = 0 \qquad \ldots\ldots (3)$$

From (1) and (2), by subtraction,

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$$

Since $\alpha \neq \beta$, divide out by $\alpha - \beta$; then

$$a(\alpha + \beta) + b = 0$$

Similarly, from (2) and (3)

$$a(\beta + \gamma) + b = 0$$

Hence by subtraction,

$$a(\alpha - \gamma) = 0;$$

which is impossible, since, by, hypothesis $a \neq 0$, and α is not equal to γ. Hence there cannot be more than two different roots.

Cor. A quadratic equation has two and only two roots.

Combining theorems 1 and 2, we get the required proof.

(Note : An argument similar to the one used in the case of the general equation of degree n can also be used.)

Nature of the Roots of a Quadratic Equation

Let the two roots of the quadratic equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

be denoted by α and β, so that

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(Remember that the constants a, b, c are rational numbers).

The expression $b^2 - 4ac$, under the radical sign, occurs in both the roots. Its value depends on the coefficients a, b and c. There are three possibilities:

I. If $b^2 - 4ac > 0$, (the quantity under the radical sign is positive), then the roots are real and unequal.

In particular, if $b^2 - 4ac$ is positive as well as a perfect square, the roots are rational and unequal, provided a, b, c are rational.

II. If $b^2 - 4ac = 0$, then the roots are real and equal, each being $-\frac{b}{2a}$.

III. If $b^2 - 4ac < 0$, then the roots are imaginary and unequal.

Thus computing the value of $b^2 - 4ac$, it is possible to determine the nature of the quadratic equation without actually solving the equation. This quantity is therefore known as the **discriminant** of the quadratic equation.

Two conclusions that can be derived from the above discussion are as follows:

(a) **In a quadratic equation with rational coefficients, irrational roots always occur in conjugate pair.** If $b^2 - 4ac$ is positive but not a perfect square, then $\sqrt{b^2 - 4ac}$ will be irrational. Since $\sqrt{b^2 - 4ac}$ occurs in both roots α and β, so if we put

$$-\frac{b}{2a} = p \qquad \text{and} \qquad \frac{\sqrt{b^2 - 4ac}}{2a} = \sqrt{q},$$

then $\alpha = p + \sqrt{q}$ and $\beta = p - \sqrt{q}$

are the two roots, both of which are irrational, each being the conjugate of the other. Hence in a quadratic equation with rational coefficients, the irrational roots occur in pair.

(b) In a quadratic equation with real coefficients, imaginary (complex) roots always occur in pair. If $b^2 - 4ac$ is negative, then $\sqrt{b^2 - 4ac}$ will be imaginary. Since $\sqrt{b^2 - 4ac}$ occurs in both roots α and β, so if we put

$$-\frac{b}{2a} = p \qquad \text{and} \qquad \frac{\sqrt{b^2 - 4ac}}{2a} = iq$$

then $\alpha = p + iq$ and $\beta = p - iq$

are the two roots, both of which are imaginary (complex), each being the conjugate of the other. Hence in a quadratic equation with real coefficients, the imaginary (complex) roots occur in pair.

Worked Out Examples

Example 1

Determine the nature of the roots of $2x^2 - 3x - 2 = 0$.

Solution.

Since $a = 2$, $b = -3$ and $c = -2$,

$b^2 - 4ac = 25 > 0$, the roots are real, rational and unequal.

Example 2

Prove that the roots of $2x^2 - 6x + 7 = 0$ are imaginary.

Solution.

Here $a = 2$, $b = -6$ and $c = 7$, so that

$$b^2 - 4ac = -20.$$

Hence the result.

Example 3

If the equation $x^2 + (k + 2)x + 2k = 0$ has equal roots, find k.

Solution.

Here $a = 1$, $b = k + 2$ and $c = 2k$. The condition for equal roots gives

$$(k + 2)^2 - 4.1.2k = 0$$

or $\quad k^2 - 4k + 4 = 0$

$\therefore \quad k = 2$

Example 4

If the equation $(1 + m^2)x^2 + 2mcx + c^2 - a^2 = 0$ has equal roots, show that $c^2 = a^2(1 + m^2)$

Solution :

Comparing the given equation with $Ax^2 + Bx + C = 0$

we have $A = 1 + m^2$, $B = 2mc$ and $C = c^2 - a^2$

$$B^2 - 4AC = 4m^2c^2 - 4.(1 + m^2)(c^2 - a^2)$$
$$= 4m^2c^2 - 4c^2 + 4a^2 - 4m^2c^2 + 4m^2a^2$$
$$= -4c^2 + 4a^2 + 4m^2a^2$$

For equal roots, $B^2 - 4AC = 0$

or, $\quad -4c^2 + 4a^2 + 4m^2a^2 = 0$

$$c^2 = a^2(1 + m^2)$$

Example 5

If the roots of the equation $x^2 - 2cx + ab = 0$ be real and unequal, prove that the roots of $x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0$ will be imaginary.

Solution :

The given equations are

$$x^2 - 2cx + ab = 0 \quad \text{...... (i)}$$

and $\quad x^2 - 2(a + b)x + a^2 + b^2 + 2c^2 = 0 \quad$ ...... (ii)

Discriminant of (i) $= (-2c)^2 - 4.1.ab$

$$= 4c^2 - 4ab$$
$$= 4(c^2 - ab) > 0$$

as the roots are real and unequal. ...... (iii)

Again, the discriminant of (ii)

$$= \{-2(a+b)\}^2 - 4.1.(a^2+b^2+2c^2)$$
$$= 4(a+b)^2 - 4(a^2+b^2+2c^2)$$
$$= 4\{a^2+2ab+b^2-a^2-b^2-2c^2\}$$
$$= -8(c^2-ab) < 0 \quad \text{(from (iii))}$$

Hence the roots of (ii) are imaginary.

EXERCISE

1. Determine the nature of the roots of each of the following equations:
   a) $x^2 - 6x + 5 = 0$ b) $x^2 - 4x - 3 = 0$
   c) $x^2 - 6x + 9 = 0$ d) $4x^2 - 4x + 1 = 0$
   e) $2x^2 - 9x + 35 = 0$ f) $4x^2 + 8x - 5 = 0$
2. For what values of p will the equation $5x^2 - px + 45 = 0$ have equal roots ?
3. If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find k.
4. For what value of a will the equation $x^2 - (3a-1)x + 2(a^2-1) = 0$ have equal roots ?
5. If the roots of the equation $(a^2+b^2)x^2 - 2(ac+bd)x + (c^2+d^2) = 0$ are equal, then $\frac{a}{b} = \frac{c}{d}$.
6. Show that the roots of the equation $(a^2-bc)x^2 + 2(b^2-ca)x + (c^2-ab) = 0$ will be equal, if either $b = 0$, or $a^3+b^3+c^3-3abc = 0$.
7. If a, b, c are rational and $a+b+c = 0$, show that the roots of $(b+c-a)x^2 + (c+a-b)x + (a+b-c) = 0$ are rational.
8. Prove that the roots of the equation $(x-a)(x-b) = k^2$ are real for all values of k.
9. Show that the roots of the equation $x^2 - 4abx + (a^2+2b^2)^2 = 0$ are imaginary.
10. If the roots of the quadratic equation $qx^2 + 2px + 2q = 0$ are real and unequal, prove that the roots of the equation $(p+q)x^2 + 2qx + (p-q) = 0$ are imaginary.

Answers

1. a) real, rational and unequal, b) real, irrational and unequal, c) real, rational and equal
   d) real, rational and equal. e) imaginary and unequal. f) real, rational and unequal
2. $p = \pm 30$ 3. $k = 1$ or 4 4. $a = 3$

Relations between Roots and Coefficients

Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

We have, by addition

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}$$

$$= -\frac{2b}{2a} = -\frac{b}{a} \quad = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$$

Again, by multiplication, we have

$$\alpha\beta = \frac{(-b + \sqrt{b^2 - 4ac})(-b - \sqrt{b^2 - 4ac})}{4a^2}$$

$$= \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$$

$\therefore$ the sum of the roots $= -\frac{b}{a}$, the product of the roots $= \frac{c}{a}$.

Note: In a quadratic equation where the coefficient of x^2 is unity,

(i) the sum of the roots is equal to the coefficient of x with its sign changed;

(ii) the product of the roots is equal to the constant term.

In any equation the constant term (i.e. the term without x) is frequently called the absolute term.

Special Roots

Under the following conditions the given quadratic equation will have the special roots :

i) Roots equal in magnitude but opposite in sign :

Two roots will be equal in magnitude but opposite in magnitude if there sum is zero

i.e. $\alpha + \beta = -\frac{b}{a} = 0$

$\therefore \quad b = 0$

ii) Reciprocal roots :

The two roots will be reciprocal to each other if their product is 1

i.e. $\alpha\beta = \frac{c}{a} = 1$

$\therefore \quad c = a$

iii) One root zero

If one root is zero, then product of the roots is zero.

i.e. $\frac{c}{a} = 0 \quad \Rightarrow \quad c = 0$

iv) Both roots zero

If both roots are zero, then

$$\alpha + \beta = 0 \qquad \Rightarrow \quad -\frac{b}{a} = 0$$

and $\alpha\beta = 0 \qquad \Rightarrow \quad \frac{c}{a} = 0$

$\therefore b = 0$ and $c = 0$

Formation of a Quadratic Equation

Suppose $ax^2 + bx + c = 0$ is the required equation, and α and β are the given roots.

The required equation may be written as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0,$$

or, $$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad (\because \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a})$$

Hence any quadratic equation may be expressed in the form

x^2 – (sum of the roots) x + product of the roots = 0.

Symmetric Functions of Roots

Symmetric functions of the roots of a quadratic equation are those functions in which the two roots are so involved that the function is unaltered when the two roots are interchanged. For instance, the sum $\alpha + \beta$ and the product $\alpha\beta$ of the roots, we have considered in the last two sections, are symmetric functions of α and β. Further examples of symmetric functions are

$$\frac{1}{\alpha} + \frac{1}{\beta}, \qquad \alpha^2 + \beta^2, \qquad \frac{\alpha + \beta}{\alpha\beta}, \qquad \alpha^2\beta^2, \text{ etc.}$$

One of the interesting characteristics of symmetric functions in α and β is that they can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$. Hence a symmetric function in α and β can be expressed in terms of the coefficients of the equation.

For example, if α and β are the roots of $x^2 - px + q = 0$, then

$$\alpha + \beta = p \qquad \text{and} \qquad \alpha\beta = q.$$

From these values, it is easy to show that

(i) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= p^2 - 2q, \; \alpha^2\beta^2 = q^2$

(ii) $\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta)$

$= p\{(\alpha + \beta)^2 - 3\alpha\beta\}$

$= p(p^2 - 3q)$

and $\alpha^3\beta^3 = q^3$

Example 1

Form the equation whose roots are 2, – 3.

Solution.

We have sum of roots = –1, product of roots = –6.

Hence the required equation is $x^2 + x - 6 = 0$

Example 2

Form the equation whose one root is $2 + \sqrt{3}$.

Solution.

Since the irrational roots occur in pair, so if one root = $2 + \sqrt{3}$, then other root will be $2 - \sqrt{3}$.

Sum of the roots = $2 + \sqrt{3} + 2 - \sqrt{3} = 4$

Product of the roots = $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1$

Now the required quadratic equation is $x^2 - 4x + 1 = 0$

Example 3

Form a quadratic equation whose roots are the squares of the roots of $4x^2 + 8x - 5 = 0$.

Solution.

Let α and β be the roots of this equation, then

$$\alpha + \beta = -2 \quad \text{and} \quad \alpha\beta = -\frac{5}{4}$$

Since the roots of the required equation are the squares of α and β, we have :

Sum of the roots = $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 4 + \frac{5}{2} = \frac{13}{2}$

Product of the roots = $\alpha^2\beta^2 = \frac{25}{16}$

Hence the required equation is

$$x^2 - \frac{13}{2}x + \frac{25}{16} = 0$$

or $$16x^2 - 104x + 25 = 0$$

Example 4

Find the value of k so that the equation $(3k + 1)x^2 + 2(k + 1)x + k = 0$ may have reciprocal roots.

Solution :

Comparing the given equation with $ax^2 + bx + c = 0$, we have

$$a = 3k + 1, \quad b = 2(k + 1), \quad c = k$$

For reciprocal roots,

$$c = a$$

$$\Rightarrow \quad k = 3k + 1$$

$$\Rightarrow \quad -2k = 1$$

$$\therefore \quad k = -\frac{1}{2}$$

Example 5

Show that 1 is a root of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$. Also, find the other root.

Solution :

Putting $x = 1$ in the given equation, we have $b - c + c - a + a - b = 0$ which is true, so 1 is the root of the given equation.

If α be the other root, then

$$\text{Product of the roots} = \frac{a-b}{b-c}$$

$$\Rightarrow \quad 1.\alpha = \frac{a-b}{b-c}$$

$$\therefore \quad \alpha = \frac{a-b}{b-c}$$

Example 6

If α and β are the roots of the equation $2x^2 - 3x - 5 = 0$, form a quadratic equation whose roots are $2\alpha + \frac{1}{\beta}$ and $2\beta + \frac{1}{\alpha}$.

Solution :

From the given quadratic equation,

$$\alpha + \beta = -\frac{(-3)}{2} = \frac{3}{2}, \qquad \alpha.\beta = -\frac{5}{2}$$

For the required quadratic equation:

$$\text{sum of the roots} = 2\alpha + \frac{1}{\beta} + 2\beta + \frac{1}{\alpha}$$

$$= 2(\alpha + \beta) + \left(\frac{\alpha + \beta}{\alpha\beta}\right) = 2\left(\frac{3}{2}\right) + \left(\frac{3/2}{-5/2}\right) = \frac{12}{5}$$

$$\text{Product of the roots} = \left(2\alpha + \frac{1}{\beta}\right)\left(2\beta + \frac{1}{\alpha}\right)$$

$$= 4\alpha\beta + 4 + \frac{1}{\alpha\beta} = 4\left(-\frac{5}{2}\right) + 4 - \frac{2}{5} = -\frac{32}{5}$$

The required quadratic eqution is

$x^2 - \frac{12}{5}x - \frac{32}{5} = 0$ or, $5x^2 - 12x - 32 = 0$

Example 7

If one root of the equation $x^2 - px + q = 0$ be twice the other, show that $2p^2 = 9q$.

Solution.

If α is a root of the equation $x^2 - px + q = 0$ then the other root is 2α; and

$\alpha + 2\alpha = p$ and $\alpha.2\alpha = q$

or $3\alpha = p,$ and $2\alpha^2 = q$

$2\alpha^2 = q \Rightarrow 2.\left(\frac{p}{3}\right)^2 = q$

Hence $2p^2 = 9q.$

Example 8

Find the condition that the roots of the quadratic equation $ax^2 + cx + c = 0$ may be in the ratio $m: n$

Solution.

If α and β are the roots of the equation $ax^2 + cx + c = 0$, then

$\alpha + \beta = \frac{-c}{a}$ and $\alpha\beta = \frac{c}{a}$

By given $\alpha : \beta = m : n.$

We know $\frac{\alpha + \beta}{\sqrt{\alpha\beta}} = \frac{-c/a}{\sqrt{\frac{c}{a}}}$

or $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = -\sqrt{\frac{c}{a}}$

Hence $\sqrt{\frac{m}{n}} + \sqrt{\frac{n}{m}} + \sqrt{\frac{c}{a}} = 0$

is the required condition.

Example 9

If α and β are the roots of the equation $x^2 - px + q = 0$, find the equation whose roots are $\alpha^2\beta^{-1}$ and $\beta^2\alpha^{-1}$.

Solution :

From the given quadratic equation,

$\alpha + \beta = -(-p) = p$, $\alpha.\beta = q$

For the required quadratic equation:

sum of the roots $= \alpha^2\beta^{-1} + \beta^2\alpha^{-1}$

$$= \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{p^3 - 3qp}{q}$$

product of the roots $= \alpha^2\beta^{-1}.\beta^2\alpha^{-1} = \alpha\beta = q$

Now, the required quadratic equation is

$$x^2 - \frac{p^3 - 3pq}{q}x + q = 0$$

or, $qx^2 - p(p^2 - 3q)x + q^2 = 0$

EXERCISE

1. Form the equation whose roots are
   a) $3, -2$ b) $-5, 4$ c) $\sqrt{3}, -\sqrt{3}$
   d) $\frac{1}{2}(-1+\sqrt{5}), \frac{1}{2}(-1-\sqrt{5})$ e) $-3+5i, -3-5i$ f) $a+ib, a-ib$
2. a) Find a quadratic equation whose roots are twice the roots of $4x^2 + 8x - 5 = 0$
   b) Find a quadratic equation whose roots are the reciprocals of the roots of $3x^2 + 5x - 2 = 0$.
   c) Find a quadratic equation whose roots are greater by h than the roots of $x^2 - px + q = 0$
   d) Form a quadratic equation whose roots are the squares of the roots of $3x^2 - 5x - 2 = 0$
3. Find the quadratic equation with rational coefficient one of whose roots is
   a) $4 + 3i$ b) $\frac{1}{5+3i}$ c) $2 + \sqrt{3}$
4. Find the value of k so that the equation
   a) $2x^2 + kx - 15 = 0$ has one root $= 3$
   b) $3x^2 + kx - 2 = 0$ has roots whose sum is equal to 6
   c) $2x^2 + (4-k)x - 17 = 0$ has roots equal but opposite in sign
   d) $3x^2 - (5+k)x + 8 = 0$ has roots numerically equal but opposite in sign.
   e) $3x^2 + 7x + 6 - k = 0$ has one root equal to zero
   f) $4x^2 - 17x + k = 0$ has the reciprocal roots.
   g) $4x^2 + kx + 5 = 0$ has roots whose difference is $\frac{1}{4}$.

5. Show that –1 is a root of the equation $(a + b - 2c)x^2 + (2a - b - c)x + (c + a - 2b) = 0$. Find the other root.

6. Find the value of m for which the equation $(m + 1)x^2 + 2(m + 3)x + (2m + 3) = 0$ will have (a) reciprocal roots (b) one root zero.

7. If the roots of the equation $x^2 + ax + c = 0$ differ by 1, prove that $a^2 = 4c + 1$.

8. If α and β be the roots of the equation $x^2 - x - 6 = 0$, find the equation whose roots are
   a) $\alpha^2\beta^{-1}$ and $\beta^2\alpha^{-1}$
   b) $\alpha + \dfrac{1}{\beta}$ and $\beta + \dfrac{1}{\alpha}$

9. If α and β are the roots of the equation $ax^2 + bx + c = 0$, find the equation whose roots are
   a) $\alpha\beta^{-1}$ and $\beta\alpha^{-1}$
   b) α^3 and β^3
   c) $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$
   d) the reciprocal of the roots of the given equation.

10. a) If the roots of the equation $ax^2 + bx + c = 0$ be in the ratio of 3 : 4, prove that $12b^2 = 49ac$.
    b) If one root of the equation $ax^2 + bx + c = 0$ be four times the other root, show that $4b^2 = 25ac$.
    c) For what value of m, the equation $x^2 - mx + m + 1 = 0$ may have its roots in the ratio 2:3.

11. a) If α and β are the roots of $px^2 + qx + q = 0$, prove that $\sqrt{\dfrac{\alpha}{\beta}} + \sqrt{\dfrac{\beta}{\alpha}} + \sqrt{\dfrac{q}{p}} = 0$
    b) If the roots of the equation $lx^2 + nx + n = 0$ be in the ratio $p : q$, prove that
$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$$

12. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other, prove that $b^3 + a^2c + ac^2 = 3abc$.

Answers

1. a) $x^2 - x - 6 = 0$ b) $x^2 + x - 20 = 0$ c) $x^2 - 3 = 0$ d) $x^2 + x - 1 = 0$ e) $x^2 + 6x + 34 = 0$ f) $x^2 - 2ax + (a^2 + b^2) = 0$
2. a) $x^2 + 4x - 5 = 0$ b) $2x^2 + 5x - 3 = 0$ c) $x^2 - (p + 2h)x + (q + ph + h^2) = 0$ d) $9x^2 - 37x + 4 = 0$
3. a) $x^2 - 8x + 25 = 0$ b) $34x^2 - 10x + 1 = 0$ c) $x^2 - 4x + 1 = 0$
4. a) $k = -1$ b) $k = -18$ c) $k = 4$ d) $k = -5$ e) $k = 6$ f) $k = 4$ g) $k = \pm 9$
5. $\dfrac{2b - c - a}{a + b - 2c}$
6. a) -2 b) $-\dfrac{3}{2}$
8. a) $6x^2 + 19x - 36 = 0$ b) $6x^2 - 5x - 25 = 0$
9. a) $acx^2 - (b^2 - 2ac)x + ac = 0$ b) $a^3x^2 + b(b^2 - 3ca)x + c^3 = 0$ c) $a^4x^2 - 2a^2(b^2 - 2ca)x + b^2(b^2 - 4ac) = 0$ d) $cx^2 + bx + a = 0$
10. c) 5 or $-\dfrac{5}{6}$

Miscellaneous Results

In this section, we shall obtain conditions under which two given quadratic equations may have one root common or both roots common.

a) One Root Common

Let $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ be two equations. Suppose that α is a root common to both the equations. Then

$$a\,\alpha^2 + b\,\alpha + c = 0$$

$$a'\,\alpha^2 + b'\,\alpha + c' = 0$$

By the rule of cross-multiplication

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}$$

This gives $\alpha = \dfrac{bc' - b'c}{ca' - c'a}$ and also $\alpha = \dfrac{ca' - c'a}{ab' - a'b}$

Combining the two, we have the required condition

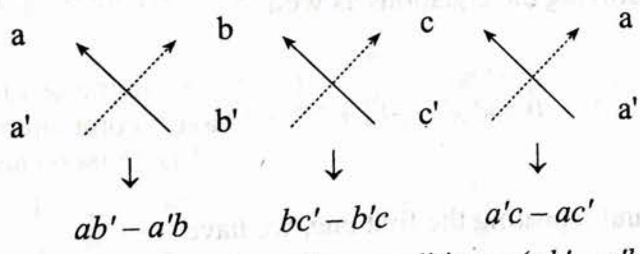
$$(bc' - b'c)\,(ab' - a'b) = (ca' - c'a)^2$$

and the common root is

$$\frac{bc' - b'c}{ca' - c'a} \qquad \text{or} \qquad \frac{ca' - c'a}{ab' - a'b}.$$

The condition of one root common of the two given quadratic equations can be remembered in the following way:

(List the coeffs. and repeat the first)



The left hand expression of the condition = $(ab' - a'b)\,(bc' - b'c)$

and the right hand expression of the condition = $(a'c - ac')^2$

b) Two Roots Common

If the quadratic equations have both roots common and if α and β be the common roots, then

$$\alpha + \beta = -\frac{b}{a} = -\frac{b'}{a'} \qquad \text{or} \qquad \frac{a}{a'} = \frac{b}{b'}$$

$$\text{and} \quad \alpha\beta = \frac{c}{a} = \frac{c'}{a'} \qquad \text{or} \qquad \frac{a}{a'} = \frac{c}{c'}$$

Hence $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$,

which is the required condition for the equations to have both roots common.

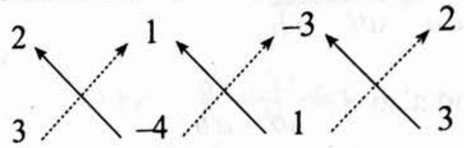
Worked Out Examples

Example 1

Show that the following two quadratic equations $2x^2 + x - 3 = 0$ and $3x^2 - 4x + 1 = 0$ have one root common.

Solution :

Writing the coefficients in order and repeating the first one, we have



The left hand expression of the condition

$= (2 \times -4 - 3 \times 1)\,(1 \times 1 - (-4) \times (-3))$

$= (-11)\,(-11)$

$= 121$

The right hand expression of the condition

$= (-3 \times 3 - 1 \times 2)^2 = 121$

Since the two results are equal, so the two equations have one root common.

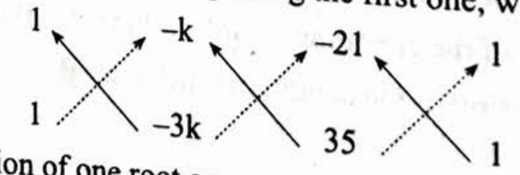
Note: we can prove the result by solving the equations as well.

Example 2

Find the value of k so that $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ have a common root.

Solution :

Writing the coefficients in order and repeating the first one, we have



Now, using the condition of one root common, we have

$(-3k + k).(-35k - 63k) = (-21 - 35)^2$

$\Rightarrow \quad (-2k).(-98k) = (-56)^2$

$\Rightarrow \quad 196k^2 = 56 \times 56$

$\Rightarrow \quad k^2 = 16$

$\therefore \quad k = \pm 4$

Example 3

If the quadratic equations $x^2 + qx + pr = 0$ and $x^2 + rx + pq = 0$ have one root common, prove that $p + q + r = 0$

Solution :

If α be a common root, then

$$\alpha^2 + q\alpha + pr = 0 \quad \ldots\ldots \text{(i)}$$

$$\alpha^2 + r\alpha + pq = 0 \quad \ldots\ldots \text{(ii)}$$

$$- \quad - \quad - \quad -$$

Subtracting, $(q - r)\alpha - p(q - r) = 0$

$(q - r)(\alpha - p) = 0$

$\therefore \quad \alpha = p$

Substituting the value of α in (i) we have, $p + q + r = 0$

Example 4

Find the condition that the roots of $ax^2 + bx + c = 0$ be the reciprocals of roots of $a'x^2 + b'x + c' = 0$.

Solution:

$$ax^2 + bx + c = 0 \quad \ldots\ldots \text{(i)}$$

$$a'x^2 + b'x + c' = 0 \quad \ldots\ldots \text{(ii)}$$

If α, β be the roots of equation (i), then the roots of (ii) are $\frac{1}{\alpha}, \frac{1}{\beta}$.

From equation (i), $\alpha + \beta = -\frac{b}{a}$ and $\alpha.\beta = \frac{c}{a}$

From equation (ii)

$$\frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b'}{a'}$$

$$\Rightarrow \frac{\alpha + \beta}{\alpha\beta} = -\frac{b'}{a'}$$

$$\Rightarrow \frac{-b/a}{c/a} = -\frac{b'}{a'}$$

$$\Rightarrow \frac{b}{c} = \frac{b'}{a'}$$

$$\therefore \frac{b}{b'} = \frac{c}{a'} \quad \ldots\ldots \text{(I)}$$

Again, $\frac{1}{\alpha} \cdot \frac{1}{\beta} = \frac{c'}{a'}$

$$\frac{1}{c/a} = \frac{c'}{a'}$$

$$\Rightarrow \frac{a}{c} = \frac{c'}{a'}$$

$$\Rightarrow \frac{a}{c'} = \frac{c}{a'} \qquad ......(II)$$

From (I) and (II)

$$\frac{b}{b'} = \frac{c}{a'} = \frac{a}{c'}$$

which is the required condition.

EXERCISE

1. Show that each pair of the following equations has a common root
   a) $x^2 - 8x + 15 = 0$ and $2x^2 - x - 15 = 0$
   b) $3x^2 - 8x + 4 = 0$ and $4x^2 - 7x - 2 = 0$
2. Find the value of p so that each pair of the equations may have one root common.
   a) $4x^2 + px - 12 = 0$ and $4x^2 + 3px - 4 = 0$
   b) $2x^2 + px - 1 = 0$ and $3x^2 - 2x - 5 = 0$
3. If the quadratic equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be either $\frac{pq' - p'q}{q - q'}$ or $\frac{q - q'}{p' - p}$.
4. If the equations $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ have a common root, prove that either $p = q$, or $p + q + 1 = 0$.
5. If the quadratic equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then either $a + b + c = 0$ or $a = b = c$.
6. Prove that if the equations $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root, their other roots will satisfy $x^2 + ax + bc = 0$.

A. A quadratic equation of the form $ax^2 + bx + c = 0$ is given. Giving different values of a, b, c solve it. Show that the equation has two roots.

Next is, taking the two values of the variable obtained by solving above as the two roots, form a quadratic equation.

Show that the equation obtained is the given original equation.

Answers

2. (i) $p = \pm 2$ (ii) $p = 1$ or $-\frac{41}{15}$

Additional Questions

1. Consider the quadratic equation $ax^2 + bx + c = 0$. Under what conditions are the roots
   i) real and unequal; ii) equal; iii) imaginary
   Find a quadratic equation whose roots are the reciprocals of the roots of $3x^2 - 5x - 3 = 0$.

2. Find the condition that $ax^2 + bx + c = 0$ may have
   i) one root reciprocal of the other
   ii) roots equal in magnitude but opposite in sign
   iii) both roots positive
   iv) both roots negative

3. Prove that the roots of the equation
   $$x^2 + (2k-1)x + k^2 = 0 \quad \text{are real if } k \le \tfrac{1}{4}.$$

4. Show that the roots of the equation
   $$2(a^2 + b^2)x^2 + 2(a + b)x + 1 = 0$$
   can never be real unless $a = b$.

5. Show that the roots of the quadratic equation $(b - c)x^2 + 2(c - a)x + (a - b) = 0$ are always real.

6. For all $a, b, c \in R$, prove that the roots of the quadratic equation
   $$(a^2 + b^2)x^2 + 2(bc + ad)x + (c^2 + d^2) = 0$$
   are imaginary. Also, prove that the roots will be real only when $\frac{a}{b} = \frac{d}{c}$.

7. Determine the value of m for which $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

8. If the ratio of the roots of $ax^2 + bx + c = 0$ be equal to that of the roots of $a'x^2 + b'x^2 + c' = 0$ prove that $\frac{b^2}{b'^2} = \frac{ac}{a'c'}$

9. If –4 is a root of the equation $x^2 + px - 4 = 0$ and the equation $x^2 + px + q = 0$ has equal roots, find the value of q.

10. For what value of k is one root of the equation $x^2 + 3x - 6 = k(x - 1)^2$ double the other ?

11. For what values of k are the roots of the equation $3x^2 - 2kx + k = 0$ in the ratio of 3 : 1 ?

12. The sum of the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero. Prove that the product of the roots is $-\frac{1}{2}(a^2 + b^2)$

13. If the sum of the roots of the equation $ax^2 + bx + c = 0$ be equal to the sum of their squares, show that $2ac = ab + b^2$.

14. The sum of the roots of a quadratic equation is 1 and the sum of their squares is 13. Find the equation.

15. If α, β be the roots of $ax^2 + bx + c = 0$, obtain an equation whose roots are

    i) $(a\alpha + b)^{-1}$ and $(a\beta + b)^{-1}$ ii) $\sqrt{\frac{\alpha}{\beta}}$ and $\sqrt{\frac{\beta}{\alpha}}$

16. If the roots of the equation $lx^2 + nx + n = 0$ be in the ratio $p : q$, find the value of
$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}.$$

17. If the difference of the roots of $x^2 + 2px + q = 0$ be equal to the difference of the roots of $x^2 + 2qx + p = 0$ prove that $p + q + 1 = 0$.

18. Find the condition that a root of $ax^2 + bx + c = 0$ is the reciprocal of one root of
$$a'x^2 + b'x + c' = 0$$

Answers

2. $3x^2 + 5x - 3 = 0$
4. i) $c = a$ ii) $b = 0$
   iii) a and c should be of same sign and b of opposite sign and discriminant positive.
   iv) a, b, c should be of the same sign

9. $\frac{7}{4}$ or $-\frac{11}{8}$ 9. $\frac{9}{4}$ 10. -24 or 3 11. 4 14. $x^2 - x - 6 = 0$

15. i) $acx^2 - bx + 1 = 0$ ii) $\sqrt{ac}\,x^2 + bx + \sqrt{ac} = 0$ 16. $-\sqrt{\frac{n}{l}}$

18. $(cc' - aa')^2 = (a'b - b'c)(ab' - bc')$

Natural Numbers

Sum of the Natural Numbers

The numbers 1, 2, 3, … … … are said to be the natural numbers. Now we derive some of the formulae for the sum of the first n natural numbers, the sum of the squares of first n natural numbers and the sum of the cubes of first n natural numbers.

(i) Sum of the first n natural numbers

The first n natural numbers are 1, 2, 3, … …… …, n

Let $S_n = 1 + 2 + 3 + \dots \dots \dots + n$

$= \frac{n}{2}[2\cdot 1 + (n - 1)\cdot 1]$ $[S_n = \frac{n}{2}\{2a + (n - 1)d\}]$

$= \frac{n(n + 1)}{2}$

$\therefore\ S_n = \frac{n(n + 1)}{2}$

(ii) Sum of the first *n* even natural numbers

The first n even natural numbers are 2, 4, 6, … ……n terms

Let $S_n = 2 + 4 + 6 + \ldots \ldots\ldots \ldots$ to n terms

$= 2(1 + 2 + 3 + \ldots \ldots\ldots$ to n terms)

$= 2\cdot\dfrac{n(n+1)}{2}$

$= n\,(n+1)$

(iii) Sum of the first *n* odd natural numbers

The first n odd natural numbers are 1, 3, 5, … … …, n terms

Let $S_n = 1 + 3 + 5 + \ldots \ldots \ldots$ to n terms

$= \dfrac{n}{2}[\,2\cdot 1 + (n-1)\cdot 2]$ $\quad\left[S_n = \dfrac{n}{2}\{2a + (n-1)\,d\}\right]$

$= n\,(1 + n - 1) = n^2$

(iv) Sum of the squares of the first *n* natural numbers

Let $S_n = 1^2 + 2^2 + 3^2 + \ldots \ldots\ldots + n^2,$

We know $r^3 - (r-1)^3 = 3r^2 - 3r + 1$

This is an identity and is true for all values of r. Putting r equal to 1, 2, 3, … ……, n respectively

we have $1^3 - 0^3 = 3\cdot 1^2 - 3\cdot 1 + 1$

$2^3 - 1^3 = 3\cdot 2^2 - 3\cdot 2 + 1$

$3^3 - 2^3 = 3\cdot 3^2 - 3\cdot 3 + 1$

$4^3 - 3^3 = 3\cdot 4^2 - 3\cdot 4 + 1$

… …… … … … ……

$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding we get,

$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$

or, $n^3 = 3S_n - 3\cdot\dfrac{1}{2}\,n\,(n+1) + n$

or, $3S_n = n^3 - n + \dfrac{3n(n+1)}{2}$

$= n\,(n^2 - 1) + \dfrac{3n(n+1)}{2}$

$= n\,(n+1)\,(n-1) + \dfrac{3n(n+1)}{2}$

$$= \frac{1}{2} n (n + 1) (2n + 1)$$

$$\therefore \quad S_n = \frac{n (n + 1) (2n + 1)}{6}$$

(v) Sum of the cubes of the first *n* natural numbers

Let $S_n = 1^3 + 2^3 + 3^3 + \dots \dots + n^3$

We know, $r^4 - (r - 1)^4 = 4r^3 - 6r^2 + 4r - 1$

Putting $r = 1, 2, 3, \dots \dots, n$ in succession, we have

$$1^4 - 0^4 = 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1$$
$$2^4 - 1^4 = 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1$$
$$3^4 - 2^4 = 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1$$
$$4^4 - 3^4 = 4 \cdot 4^3 - 6 \cdot 4^2 + 4 \cdot 4 - 1$$
$$\dots \dots \dots \dots \dots \dots \dots$$
$$n^4 - (n - 1)^4 = 4 \cdot n^3 - 6 \cdot n^2 + 4 \cdot n - 1$$

Adding $n^4 = 4(1^3 + 2^3 + 3^3 + \dots + n^3) - 6(1^2 + 2^2 + \dots + n^2) + 4(1 + 2 + 3 + \dots \dots + n) - n$

$$n^4 = 4S_n - 6 \frac{n(n + 1)(2n + 1)}{6} + \frac{4n(n + 1)}{2} - n$$

or, $4S_n = n^4 + n + n (n + 1) (2n + 1) - 2n (n + 1)$

$$= n (n^3 + 1) + n (n + 1) (2n + 1) - 2n (n + 1)$$
$$= n (n + 1) (n^2 - n + 1) + n (n + 1) (2n + 1) - 2n (n + 1)$$
$$= n (n + 1) (n^2 - n + 1 + 2n + 1 - 2)$$
$$= n (n + 1) (n^2 + n)$$
$$= n^2 (n + 1)^2$$

or, $S_n = \dfrac{n^2 (n + 1)^2}{4}$

$$\therefore \quad S_n = \sum n^3 = \frac{n^2 (n + 1)^2}{4} = \left(\frac{n (n + 1)}{2}\right)^2$$

Cor. $1^3 + 2^3 + 3^3 + \dots \dots \dots + n^3 = \left(\dfrac{n (n + 1)}{2}\right)^2$

and $1 + 2 + 3 + \dots \dots \dots + n = \dfrac{n (n + 1)}{2}$,

it is clear that the sum of the cubes of the first *n* natural numbers is the square of the sum of the first *n* natural numbers.

Sum of the series using the method of successive differences

Sometimes, the terms of the given series may not be in A.S. or G.S. But the difference of the successive terms of the given series will be in A.S. or G.S. and then we can easily find the *n*th term and the corresponding sum of *n* terms as well.

Let us see the following example $\quad 1 + 5 + 12 + 23 + \ldots\ldots$

The terms of the above series are neither in A.S. nor in G.S.

Let $\quad S = 1 + 5 + 12 + 22 + \ldots\ldots + t_n$

Also, $\quad S = \quad 1 + 5 + 12 + \ldots\ldots + t_{n-1} + t_n$

$$\begin{array}{l} - \quad - \quad - \quad - \quad - \quad - \\ \hline 0 = 1 + 4 + 7 + 10 + \ldots\ldots n^{th} \text{ term} - t_n \end{array}$$

$\therefore \quad t_n = 1 + 4 + 7 + 10 + \ldots\ldots$

Thus, we see that the terms after getting the difference of the successive terms are in A.S. Now we can find the n^{th} term and the corresponding sum easily.

Worked Out Examples

Example 1

Sum to *n* terms the series: $2\cdot3 + 3\cdot4 + 4\cdot5 + \ldots\ldots\ldots$

Solution:

Here, the r^{th} term of 2, 3, 4, ……… $= r + 1$

and the r^{th} term of 3, 4, 5, ……… $= r + 2$

Hence, the r^{th} term of the given series is

$$t_r = (r + 1)(r + 2) = r^2 + 3r + 2$$

Thus, sum of *n* terms of the given series

$$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} (r^2 + 3r + 2)$$

$$= \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + 2n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n$$

$$= \frac{n(n^2 + 6n + 11)}{3}.$$

Example 2

Find the n^{th} term and then the sum of the first *n* terms of the series

$$1 + 3 + 6 + 10 + \ldots\ldots\ldots$$

Solution:

Let t_n be the n^{th} term and S_n the sum of the first n terms of

$$1 + 3 + 6 + 10 + \dots\dots$$

then, $S_n = 1 + 3 + 6 + 10 + \dots\dots + t_{n-1} + t_n$

Also, $S_n = \quad 1 + 3 + 6 + \dots\dots + t_{n-2} + t_{n-1} + t_n$

Subtraction yields, $0 = 1 + 2 + 3 + \dots\dots + (t_n - t_{n-1}) - t_n$

or, $t_n = 1 + 2 + 3 + \dots$ to n terms $= \dfrac{n(n+1)}{2} = \dfrac{1}{2}n^2 + \dfrac{1}{2}n$

Hence, $S_n = \frac{1}{2}\sum n^2 + \frac{1}{2}\sum n$

$$= \frac{1}{2}(1^2 + 2^2 + 3^2 + \dots\dots + n^2) + \frac{1}{2}(1 + 2 + 3 + \dots\dots + n)$$

$$= \frac{1}{2}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n(n+1)\left\{\frac{(2n+1)}{3} + 1\right\}$$

$$= \frac{1}{4}n(n+1)\frac{(2n+1+3)}{3}$$

$$= \frac{n(n+1)(n+2)}{6}.$$

Example 3

Sum to n terms of the following series: $1 + (1 + 2) + (1 + 2 + 3) + \dots\dots$

Solution:

$t_n = n^{th}$ term of the given series $= 1 + 2 + 3 + \dots\dots + n = \dfrac{n(n+1)}{2}$

Now, sum to n term of given series is

$$\sum t_n = \frac{1}{2}\sum n^2 + \frac{1}{2}\sum n$$

$$S_n = \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\cdot\frac{n(n+1)}{2}$$

$$= \frac{1}{4}n(n+1)\left\{\frac{2n+1}{3} + 1\right\}$$

$$= \frac{1}{12}n(n+1)\,.\,2(n+2)$$

$$= \frac{1}{6}n(n+1)(n+2)$$

EXERCISE

1. Given the nth term of the series, find the sum to n terms.
   a) $n + n^2$ b) $n^2 - 3n$ c) $n(n + 1) - 2$ d) $3^n - 4n^3$

2. Find the nth term and then the sum of the first n terms of each of the following series:
   a) $1.1 + 2.2 + 3.3 + 4.4 + \ldots\ldots$ b) $1.3 + 2.4 + 3.5 + \ldots\ldots$
   c) $1.3 + 3.5 + 5.7 + \ldots\ldots$ d) $2.5 + 4.8 + 6.11 + \ldots\ldots$
   e) $5.3 + 7.5 + 9.7 + \ldots\ldots$ f) $2 + 6 + 12 + 20 + \ldots\ldots$
   g) $2 + (2 + 3) + (2 + 3 + 4) + \ldots\ldots$ h) $1 + (1 + 3) + (1 + 3 + 5) + \ldots\ldots$

3. a) Find the general term and then the sum of first n terms of the series
   $1.n + 2.(n - 1) + 3.(n - 2) + \ldots\ldots$
   b) The natural numbers are grouped as follows:
   (1), (2, 3), (4, 5, 6), (7, 8, 9, 10), ……
   Find the first term of the nth group.

4. Sum to n terms of the following series
   a) $1^2.1 + 2^2.3 + 3^2.$ b) $3.1^2 + 4.2^2 + 5.3^2 + \ldots\ldots$
   c) $1^2.2 + 2^2.3 + 3^2.4$ d) $1.2^2 + 3.4^2 + 5.6^2 + \ldots\ldots$
   e) $1^2 + 3^2 + 5^2 + \ldots\ldots$ f) $1^3 + 3^3 + 5^3 + \ldots\ldots$

5. Find the nth term and then sum of n terms of the following series
   a) $2 + 4 + 8 + 14 + 22 + \ldots\ldots$ b) $3 + 7 + 13 + 21 + 31 + \ldots\ldots$
   c) $2 + 5 + 10 + 17 + 26 + \ldots\ldots$ d) $5 + 11 + 19 + 29 + 41 + \ldots\ldots$
   e) $5 + 7 + 13 + 31 + 85 + \ldots\ldots$ f) $2 + 5 + 14 + 41 + \ldots\ldots$

A. The first three terms of a series are given below

$$t_1 = 2.3^2 - 8$$
$$t_2 = 2.4^2 - 8$$
$$t_3 = 2.5^2 - 8$$
$$t_4 = \ldots\ldots$$
$$t_5 = \ldots\ldots$$

   a) complete the 4th and 5th terms.
   b) Write the nth term
   c) Find the sum to n terms of the series.

Answer

1. a) $\frac{1}{3}n(n + 1)(n + 2)$ b) $\frac{1}{3}n(n + 1)(n - 4)$ c) $\frac{1}{3}n(n - 1)(n + 4)$
   d) $\frac{3}{2}(3^n - 1) - n^2(n + 1)^2$

2. a) $n^2, \frac{1}{6}n(n+1)(2n+1)$ b) $n(n+2), \frac{1}{6}n(n+1)(2n+7)$ c) $4n^2 - 1, \frac{1}{3}n(4n^2 + 6n - 1)$

d) $6n^2 + 4n, n(n+1)(2n+3)$ e) $4n^2 + 8n + 3, \frac{1}{3}n(4n^2 + 18n + 23)$

f) $n(n+1), \frac{1}{3}n(n+1)(n+2)$ g) $\frac{1}{2}n(n+3), \frac{1}{6}n(n+1)(n+5)$ h) $n^2, \frac{1}{6}n(n+1)(2n+1)$

3. a) $r(n-r+1), \frac{1}{6}n(n+1)(n+2)$ b) $\frac{1}{2}(n^2 - n + 2)$

4. a) $\frac{1}{6}n(n+1)(3n^2 + n - 1)$ b) $\frac{1}{12}n(n+1)(3n^2 + 11n + 4)$ c) $\frac{1}{12}n(n+1)(n+2)(3n+1)$

d) $\frac{2}{3}n(n+1)(3n^2 + n - 1)$ e) $\frac{1}{3}n(4n^2 - 1)$ f) $n^3(2n^2 - 1)$

5. a) $n^2 - n + 2, \frac{1}{3}n(n^2 + 5)$ b) $n^2 + n + 1, \frac{1}{3}n(n^2 + 3n + 5)$ c) $n^2 + 1, \frac{1}{6}n(2n^2 + 3n + 7)$

d) $n^2 + 3n + 1, \frac{1}{3}n(n+2)(n+4)$ e) $3^{n-1} + 4, \frac{1}{2}(3^n + 8n - 1)$ f) $\frac{1}{2}(3^n + 1), \frac{1}{4}(2n - 3 + 3^{n+1})$

Mathematical Induction

We know that the product of two consecutive natural numbers is an even number. Thus if n and $n + 1$ be two consecutive natural numbers, then their product $n(n + 1)$ is an even number. This is the general result. For $n = 5$, the product = 5.6 = 30 which is an even number. This is the particular result obtained from the general one. This type of process of getting the particular result from the general one is known as the method of deduction.

Again, let us see the following two digit numbers.

24, 32, 40, 68

Each of the above two digit numbers is exactly divisible by 2. But from the above results, we cannot conclude that all two digits numbers are exactly divisible by 2 as 31 is also a two digit number but not divisible by 2.

If the above results are to be shown true, we would have to present its validity either by verifying the above type of results for all two digit numbers or by using some mathematical process.

The process of getting the general result from the particular one is known as the method of induction.

Thus there are two ways of proving the results. One is the direct method known as the deductive method in which the results are proved using the established axioms, definitions or the theorems already proved. But second is the indirect method known as the inductive method where the results are proved by making observations or experiments and drawing conclusion on their basis.

Mathematial Induction

Many mathematical theorems or formulae which are complicated to prove by direct method, are proved easily by indirect method known as the mathematical induction. Its meaning will be clear from the following explanation.

Firstly, we prove that theorem or formulae for $n = 1$. When the theorem is true for $n = 1$, we shall prove that it is also true for $n = 1 + 1 = 2$. In the same way, we prove that it is also true for $2 + 1 = 3$ and so on. Then, we conclude that the theorem or result is true for all values of $n \in N$.

The most important word used in this section is the "statement". In this section, the result or the formula to be proved is termed as "statement". A statement involving the natural number is denoted by $P(n)$ where $n \in N$.

For example: The sum of two consecutive natural numbers is odd. This is a statement. So, it is denoted by $P(n)$: the sum of two consecutive natural numbers is odd.

Principle of Mathematical Induction

The principle of mathematical induction states that if P(n) be the statement and if

i) P(1) is true

ii) P(k + 1) is true whenever P(k) is true

then $P(n)$ is true for all $n \in N$..

Working rules for the use of Principle of Mathematical Induction

In solving a problem with the use of principle of mathematical induction, the following steps are to be used.

1. Denote the given statement (i.e. the result to be proved) by $P(n)$.
2. Show that the given statement is true for $n = 1$ i.e. P(1) is true.
3. Assume that the given statement is true for $n = k$ i.e. assume $P(k)$ is true.
4. Show that the statement is true for $n = k + 1$ when it is true for $n = k$ i.e. show that $P(k + 1)$ is true when $P(k)$ is true.
5. Draw a conclusion that the statement is true for all $n \in N$.

Worked Out Examples

Example 1

Let $P(n)$ be the statement "$n(n + 1)$ is divisible by 2". Are P(1), P(2) and P(3) true?

Solution:

$P(n)$: $n(n + 1)$ is divisible by 2

P(1): $1(1 + 1) = 1.2 = 2$ P(2): $2(2 + 1) = 2.3 = 6$

P(3): 3.(3 + 1) = 3.4 = 12

Each of P(1), P(2) and P(3) is divisible by 2. Hence P(1), P(2) and P(3) are true.

Example 2

Prove by the principle of mathematical induction that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

Let P(n) be the given statement. Then,

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

When n = 1: L.H.S. = $1^2 = 1$,

$$\text{R.H.S.} = \frac{1(1+1)(2.1+1)}{6} = 1$$

∴ L.H.S. = R.H.S. i.e. P(1) is true.

Let P(*k*) be true for m ∈ N. Then

$$P(k): 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \quad \ldots\ldots \text{(i)}$$

Now, we shall show that P(*k* + 1) is true when P(*k*) is true. For this, we add $(k+1)^2$ on both sides of (i)

Now, $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left\{\frac{k(2k+1)}{6} + (k+1)\right\}$$

$$= (k+1)\left(\frac{2k^2 + k + 6k + 6}{6}\right)$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

which shows that P(*k* + 1) is true whenever P(*k*) is true. Hence by the principle of mathematical induction P(n) is true for all n ∈ N.

Example 3

Prove by the method of induction that

$$1.3 + 2.4 + 3.5 + \ldots + n.(n+2) = \frac{n(n+1)(2n+7)}{6}$$

Solution:

Let P(n) be the given statement. Then,

$$P(n): 1.3 + 2.4 + 3.5 + \ldots + n.(n + 2) = \frac{n(n + 1)(2n + 7)}{6}$$

When n = 1:

L.H.S. = 1.3 = 3

$$\text{R.H.S.} = \frac{1.(1 + 1)(2.1 + 7)}{6} = 3$$

$\therefore$ L.H.S. = R.H.S. i.e. P(1) is true

Let $P(k)$ be true for $k \in N$. That is,

$$P(k): 1.3 + 2.4 + 3.5 + \ldots + k.(k + 2) = \frac{k(k+1)(2k+7)}{6} \quad \ldots\ldots (i)$$

Now, we shall show that $P(k + 1)$ is true when $P(k)$ is true. For this, we add $(k + 1)(k + 3)$ on both sides of (i).

Now, $1.3 + 2.4 + 3.5 + \ldots + k.(k + 2) + (k + 1)(k + 3)$

$$= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) \qquad \text{(From (i))}$$

$$= (k+1)\left\{\frac{k(2k+7)}{6} + k + 3\right\}$$

$$= \frac{(k+1)(2k^2 + 7k + 6k + 18)}{6}$$

$$= \frac{(k+1)(2k^2 + 13k + 18)}{6}$$

$$= \frac{(k+1)(k+2)(2k+9)}{6}$$

$$= \frac{(k+1)(\overline{k+1} + 1)(2(k+1) + 7)}{6}$$

which shows that $P(k + 1)$ is true whenever $P(k)$ is true. Hence, by the principle of mathematical induction P(n) is true for all $n \in N$.

Example 4

Applying principle of mathematical induction, prove that n(n + 1)(2n + 1) is divisible by 6 for all $n \in N$.

Solution:

Let P(n) be the given statement. Then

P(n): n(n + 1)(2n + 1) is divisible by 6.

When n = 1: n(n + 1)(2n + 1) = 1(1 + 1)(2.1 + 1) = 6 which is divisible by 6. So, P(1) is true.

Let us assume that P(k) is true for $k \in$ N. Then,

$P(k)$: $k(k + 1)(2k + 1)$ is divisible by 6 ...... (i)

Now, we shall show that P(k + 1) is true when P(k) is true

i.e. we show that $(k + 1)(k + 2)(2(k + 1) + 1)$ is divisible by 6

Now, $(k + 1)(k + 2)(2(k + 1) + 1)$

$= (k + 1)(k + 2)(2k + 1 + 2)$

$= \{(k + 1)k + (k + 1)2\}\{(2k + 1) + 2\}$

$= k(k + 1)(2k + 1) + 2k(k + 1) + 2(k + 1)(2k + 1) + 4(k + 1)$

$= k(k + 1)(2k + 1) + 2(k + 1)\{k + 2k + 1 + 2\}$

$= k(k + 1)(2k + 1) + 2(k + 1)(3k + 3)$

$= k(k + 1)(2k + 1) + 6(k + 1)^2$

which is divisible by 6 as the first term is divisible by 6 by (i) and the second term is the multiple of 6.

This shows that P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathemtical induction, P(n) is true for all n $\in$ N.

Example 5

Prove by induction method that $2^{3n} - 1$ is divisible by 7

Solution:

Let P(n) be the given statement. Then,

P(n): $2^{3n} - 1$ is divisible by 7

When n = 1:

$2^{3n} - 1 = 2^3 - 1 = 8 - 1 = 7$ which is divisible by 7.

$\therefore$ P(1) is true.

Let us suppose that P(k) is true for $k \in$ N.

i.e. P(k): $2^{3k} - 1$ is divisible by 7 ...... (i)

Now, we shall show that P(k + 1) is true when P(k) is true.

i.e. we show that $2^{3(k+1)} - 1$ is divisible by 7

Now, $2^{3(k+1)} - 1 = 2^{3k+3} - 1$

$= 2^{3k}\,2^3 - 1 = 2^{3k}.8 - 1$

$= 2^{3k}\,8 - 8 + 8 - 1$

$= 8(2^{3k} - 1) + 7$

which is divisible by 7 as the first term is divisible by 7 by (i)

This relation shows that P(k + 1) is true whenever P(k) is true. Hence, by the principle of mathematical induction P(n) is true for all n $\in$ N.

Example 6

Prove that

a) $n^2 > 2n + 1$ for all $n \geq 3$ b) $3^n < (n + 2)!$

Solution:

a) Let P(n) be the given statement. Then

$P(n) : n^2 > 2n + 1$" for all $n \geq 3$

When $n = 3$, $n^2 = (3)^2 = 9$

and $2n + 1 = 2 \times 3 + 1 = 7$ which tends to $n^2 > 2n + 1$

$\therefore$ P(3) is true.

When $n = k$: Let us suppose that P(n) si true for $n = k \geq 3$.

That is, $P(k) = k^2 > 2k + 1$ ……(i)

When $n = k + 1$: Now we shall show that P(k + 1) is true when P(k) is true. That is,

$(k + 1)^2 > 2(k + 1) + 1$

Now, $(k + 1)^2 = k^2 + 2k + 1$

$> 2k + 1 + 2$

$= 2(k + 1) + 1$

This result shows that P(k + 1) is true when P(k) is true. Hence by the principle of mathematical induction P(n) is true for all $n \geq 3 \in N$.

b) $3^n < (n + 2)!$

Let P(n) be the given statement. Then

$P(n) : 3^n < (n + 2)!$

When $n = 1$ $3^n = 3^1 = 3$

and $(n + 2)! = (1 + 2)! = 6$

which tends to $3^n < (n + 2)!$

When $n = k$: Let us suppose that P(k + 1) is true when P(k) is true. That is

$3^k < (k + 2)!$ ……(i)

Now we shall show that P(k + 1) is true when P(k) is true.

That is, $3^{k+1} < (k + 1 + 2)! = (k + 3)!$

Now, $3^{k+1} = 3^k.3$

$> (k + 2)!.3$ (using (i))

$> (k + 2)!\,(k + 3)$ ($\because k + 3 \geq 4$)

$= (k + 3)!$

$= \{(k + 2) + 1\}!$

Which shows that P(k + 1) is true whenever P(k) is true. Hence by the principle of mathematical induction P(n) is true for all $n \in N$.

EXERCISE

1. a) If P(n) is the statement "$(n + 1)(n + 2)$ is an odd number". Find P(1), P(2) and P(3). Are they true?
   b) If P(n) is the statement "$n^3 + n$ is divisible by 2". Write P(1) and P(2). Are they true?
   c) If P(n) is the statement "$n^2 \geq 2^n$" show that P(1) is false and P(2), P(3) are true.

2. Using the principle of mathematical induction, show the following statements for all natural numbers (n):
   i) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
   ii) $1 + 3 + 5 + \ldots + (2n - 1) = n^2$
   iii) $2 + 4 + 6 + \ldots + 2n = n(n + 1)$
   iv) $2 + 5 + 8 + \ldots + (3n - 1) = \frac{n(3n+1)}{2}$
   v) $1^2 + 3^2 + 5^2 + \ldots + (2n - 1)^2 = \frac{n(2n-1)(2n+1)}{3}$
   vi) $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

3. Applying the principle of mathematical induction, show the following statements for all natural numbers (n):
   i) $1.2 + 2.3 + 3.4 + \ldots + n.(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$
   ii) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
   iii) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$
   iv) $2 + 2^2 + 2^3 + \ldots + 2^n = 2(2^n - 1)$

4. Prove by the method of induction that
   i) $n^2 + n$ is an even number
   ii) $n^3 + 2n$ is divisible by 3
   iii) $n(n + 1)(n + 2)$ is a multiple of 6

5. Using principle of mathematical induction prove that
   i) $3^{2n} - 1$ is divisible by 8
   ii) $x^n - y^n$ is divisible by $x - y$

6. Applying principle of mathematical induction, prove that
   a) $3n \geq 2n + 1$
   b) $(n + 3)^2 \geq 2n + 7$
   c) $(1 + x)^n \geq 1 + nx$ for $x > -1$
   d) $2^n < n!$ for $n \geq 4$

A. If a = first term, r = common ratio, n = no. of terms and S_n = sum to n terms, then

$$S_1 = t_1 = a = \frac{a(r-1)}{r-1}$$

$S_2 = t_1 + t_2 = a + ar = \frac{a(r^2 - 1)}{r - 1}$

$S_3 = t_1 + t_2 + t_3 = \ldots\ldots$

$S_4 = t_1 + t_2 + t_3 + t_4 = \ldots\ldots$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$S_n = t_2 + t_2 + \ldots\ldots + t_n = \ldots\ldots$

Observe the above operation and fill in the gaps.

Using principle of mathematical induction, prove the following relations

$$t_1 + t_2 + t_3 + \ldots\ldots + t_n = S_n = \frac{a(r^n - 1)}{r - 1}$$

$\Rightarrow$ .... = ...

•

Chapter 6

System of Linear Equations

Introduction

We come across equations, when we try to solve some problems in mathematics. These equations may be of one or more variables. The solutions of the equations give the solutions of the problems. So it is quite natural that we should have knowledge of different methods of solving equations. The methods, we consider here, are row equivalent method, Cramer's rule and matrix method.

Systems of Linear Equations

A simple example of *a linear equation* in two variables x and y is

$$3x - y = 5$$

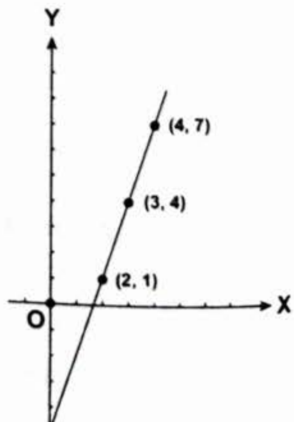
It is evident that solutions of this equation are ordered pairs such as (2, 1), (3, 4), (4, 7), etc. That is, each of the ordered pairs (2, 1), (3, 4), (4, 7), etc. satisfies the equation. For instance

$3.[3] - [4] = 5$. (true).

Moreover, there are infinite number of such pairs. If we plot each of these pairs, they will lie in a straight line. That is, the graph of the equation $3x - y = 5$ is a straight line. If we are interested in finding various pairs satisfying the given equation, we may write it in the form

$$y = 3x - 5,$$

which gives y in terms of x.



Given a value to x in this equation, a corresponding value of y can be found and each pair of values furnishes a solution of the equation.

The conjunction (pair) of two such linear equations in two variables is referred to as *a system*.

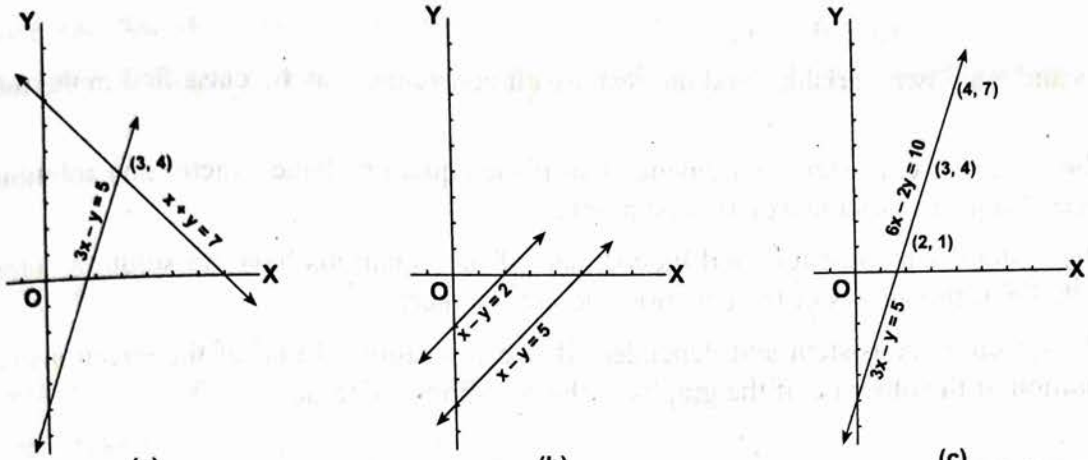
Three typical examples of systems are:

(a) $3x - y = 5$
$x + y = 7$

(b) $x - y = 2$
$x - y = 5$

(c) $3x - y = 5$
$6x - 2y = 10$

The graphs of these equations are shown in (Fig. a, b, c.)



(a) (b) (c)

We shall now discuss the three cases somewhat elaborately.

Case I. Intersecting Lines

The graphs of the first pair $3x - y = 5$ and $x + y = 7$ intersect at the point (3, 4). This means that both equations have the common solution $x = 3, y = 4$. Although there is an infinite number of solution sets which separately satisfy these equations, the solution $x = 3$, $y = 4$ is the only solution which satisfies both of them at the same time. This is why such a system of equations is called *simultaneous equations*.

Two linear equations in two variables which have at least one solution in common are said to be *consistent*. If consistent linear equations have just one solution in common, they are said to be *independent*. The system of equations $x + y = 7$ and $3x - y = 5$ are not only consistent but also independent.

Case II. Parallel Lines

The graphs of the second pair $x - y = 2$ and $x - y = 5$ are *parallel lines*, that is they never meet. No set of values of x and y which satisfies one equation will satisfy the other equation. In other words, they have no common solution. These equations are *inconsistent* and *independent*. We can see why they are so called. For one equation states that the difference of two numbers x and y is 2 while the other states that their difference is 5, and two such numbers whose difference is both 2 and 5 at the same time do not exist.

Case III. Coincident Lines

The graphs of the pair of equations $3x - y = 5$ and $6x - 2y = 10$ *coincide* (i.e. one fits exactly on the other). Obviously, these equations are consistent, that is, a set of values of x and y which satisfies the first equation satisfies the second also. Moreover, these equations are *dependent*, because one of them can be obtained from the other just by multiplying both sides by a constant. In the above case, the second can be obtained from the first by multiplying both sides of the first by 2.

In the light of the discussion we had so far, a system of linear equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

where x and y are two variables and the rest are all constants, may be classified in the following way:

I. The system is consistent and independent if the equations have exactly one solution, i.e. if their graphs intersect in exactly one point.

II. The system is inconsistent and independent if the equations have no solution common to both, i.e. if the graphs of the equations do not intersect.

III. The system is consistent and dependent if every solution of one of the equations is also a solution of the other i.e. if the graphs of the equations coincide.

EXERCISE

1. By drawing graphs or otherwise, classify each of the following systems:

(a) $4x - 3y = -6$, $-4x + 2y = 16$

(b) $9x - 2y = -4$, $3x + 4y = 1$

(c) $-6x + 4y = 10$, $3x - 2y = -5$

(d) $3x - 4y = 1$, $6x - 8y = 7$

(e) $25x - 15y = 45$, $-5x - 3y = 24$

Answers

1. (a) Consistent and Independent
(b) Consistent and Independent
(c) Consistent and Dependent
(d) Inconsistent and Independent
(e) Consistent and Independent

The system of equations considered so far consists of two variables only. Similar discussion holds good if the number of variable increased to three or more. An example of a linear equation in the three variables x, y and z is

$$3x + y - z = 14.$$

Solutions of this equation consist of the ordered triples such as (5, 6, 7) which satisfies the equation. Here (5, 6, 7) is just one solution such that

$$3.[5] + [6] - [7] = 14 \text{ (True)}.$$

One set of such triples is {(5, 0, 1),(5, 1, 2),(5, 3, 4) ......}. It is easy to verify that each of these triples satisfies the linear equation

$$x + y - z = 4$$

also. We thus observe that a system of two equations in three variables have an infinite number of solutions. But, if we further consider a third linear equation such as

$$x + y + z = 6,$$

then we notice that the triple which satisfies the third also is (5, 0, 1).

Actually, there is no other triple of numbers which satisfies the system of three equations:

$$3x + y - z = 14$$
$$x + y - z = 4$$
$$x + y + z = 6$$

We shall, as before, refer to such a system of equations to be consistent and independent.

Solutions of System of Linear Equations

One of the most fundamental techniques of solving a system like

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

is the technique of elimination. We can illustrate this technique on the linear system

$$x - 2y = 3$$
$$2x + y = 1$$

If we add 2 times the second equation to the first equation, we obtain

$$5x = 5 \quad \text{or} \quad x = 1$$

If we add −2 times the first equation to the second equation, we obtain

$$5y = -5 \quad \text{or} \quad y = -1$$

So we conclude that the ordered pair (1, −1) is the solution. It is not difficult to notice that in solving the above linear system, we actually work with constants (coefficients) and not with the variables. In the other words, there is no need to continue writing the variables x and y, since one actually computes with the coefficients and the numbers in the right side of the equations. For this type of computation, it is helpful to list just the coefficient only in the form of a rectangular array, and enclose them within square brackets or parentheses. For instance,

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

is such an arrangement and is called the *matrix of coefficients* or *coefficient matrix* of the system.

$$x - 2y = 3$$
$$2x + y = 1$$

For a system

$$5x - 2y = -2$$
$$2x + 5y = 24$$

the square matrix

$$\begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$$

is the *coefficient matrix*. If, in this matrix, we include a third column consisting of the numbers on the right sides of the equations, we get another matrix of the form

$$\begin{bmatrix} 5 & -2 & -2 \\ 2 & 5 & 24 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 5 & -2 & : & -2 \\ 2 & 5 & : & 24 \end{bmatrix}$$

This matrix is called the *augmented matrix* of the system.

Let us now see how the idea of the matrices can be used in solving system of the linear equations. We shall present a comparative sketch of the various operations that we perform while solving the system by usual method (Addition method) and those which we perform with the entries (or elements) of rows of the augmented matrix. For instance, consider the following example.

Example 1.

Solve the system

$$-x + y = -9$$
$$x - 3y = 5$$

Addition method	Row equivalent matrix method
Addition method $-x + y = -9$ $x - 3y = 5$	**Row equivalent matrix method** $\begin{bmatrix} -1 & 1 & : & -9 \\ 1 & -3 & : & 5 \end{bmatrix}$
Interchange first and second equations $x - 3y = 5$ $-x + y = -9$	**Interchange first and second rows** $\begin{bmatrix} 1 & -3 & : & 5 \\ -1 & 1 & : & -9 \end{bmatrix}$
Add the first equation to the second $x - 3y = 5$ $-2y = -4$	**Add the first row to the second** $\begin{bmatrix} 1 & -3 & : & 5 \\ 0 & -2 & : & -4 \end{bmatrix}$
Multiply the second equation by $-\frac{1}{2}$ $x - 3y = 5$ $y = 2$	**Multiply the second row by** $-\frac{1}{2}$ $\begin{bmatrix} 1 & -3 & : & 5 \\ 0 & 1 & : & 2 \end{bmatrix}$
Multiply the second equation by 3 and add to the first $x = 11$ $y = 2$ The solution is (11, 2)	**Multiply the second row by 3 and add it to the first** $\begin{bmatrix} 1 & 0 & : & 11 \\ 0 & 1 & : & 2 \end{bmatrix}$ The solution is (11, 2)

Note: In the *row equivalent matrix method*, the most important point to be remembered is that we always try to get a matrix in the form

$$\begin{bmatrix} 1 & 0 & : & p \\ 0 & 1 & : & q \end{bmatrix}$$

called the ***reduced form*** in which the coefficient matrix becomes a unit matrix, by which we mean a square matrix in which every entry along the leading diagonal (or principal diagonal) is unity and all other entries are zero. For instance, the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is a 2×2 unit matrix (or identity matrix).

The matrices obtained in the above procedure are known as *row equivalent matrices*.

Elementary row operations:

Row-equivalent matrices are obtained by performing one or more of the following elementary row operations :

(i) Interchange of any two rows

(ii) Addition of one row to another

(iii) Multiplication of any row by a non-zero number

(iv) Multiplication of any row by a non-zero number and addition of the resulting row to another.

Let us now illustrate this procedure with more examples.

Worked Out Examples

Example 1

Solve the system $\quad 3x + 2y = -9$

$\qquad\qquad\qquad\quad 2x - 3y = -6$

by using row-equivalent matrices.

Solution.

Here the augmented matrix is

$$\begin{bmatrix} 3 & 2 & : & -9 \\ 2 & -3 & : & -6 \end{bmatrix}$$

Multiply the first row by $\frac{1}{3}$ to obtain

$$\begin{bmatrix} 1 & 2/3 & : & -3 \\ 2 & -3 & : & -6 \end{bmatrix}$$

Multiply the first row by -2 and add to the second (or perform $R_2 - 2R_1$)

$$\begin{bmatrix} 1 & 2/3 & : & -3 \\ 0 & -13/3 & : & 0 \end{bmatrix}$$

Multiply the second row by $-\frac{3}{13}$

$$\begin{bmatrix} 1 & 2/3 & : & -3 \\ 0 & 1 & : & 0 \end{bmatrix}$$

Multiply the second row by $-\frac{2}{3}$ and add it to the first row (or perform $R_1 - \frac{2}{3}R_2$)

$$\begin{bmatrix} 1 & 0 & : & -3 \\ 0 & 1 & : & 0 \end{bmatrix}$$

This gives us the solution

$x = -3, y = 0$ i.e. (–3, 0).

Let us condense this procedure as follows:

$$\begin{bmatrix} 3 & 2 & : & -9 \\ 2 & -3 & : & -6 \end{bmatrix}$$

$$\leftrightarrow \begin{bmatrix} 1 & 2/3 & : & -3 \\ 2 & -3 & : & -6 \end{bmatrix} \quad R_1 \to \tfrac{1}{3}R_1$$

$$\leftrightarrow \begin{bmatrix} 1 & 2/3 & : & -3 \\ 0 & -13/3 & : & 0 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\leftrightarrow \begin{bmatrix} 1 & 2/3 & : & -3 \\ 0 & 1 & : & 0 \end{bmatrix} \quad R_2 \to -\tfrac{3}{13}R_2$$

$$\leftrightarrow \begin{bmatrix} 1 & 0 & : & -3 \\ 0 & 1 & : & 0 \end{bmatrix} \quad R_1 \to R_1 - \tfrac{2}{3}R_2$$

This gives us the solution

$x = -3, y = 0$ i.e. (–3, 0).

Note: The symbol $\leftrightarrow$ means "is equivalent to" and is used as we proceed by an elementary row operation from one step to the next.

For a system of three equations in three variables we try to get augmented matrix in the following reduced form:

$$\begin{pmatrix} 1 & 0 & 0 & : & p \\ 0 & 1 & 0 & : & q \\ 0 & 0 & 1 & : & r \end{pmatrix}$$ from which we obtain the solution (p, q, r).

Example 2

Solve the system

$$2x - y + z = -1$$
$$x - 2y + 3z = 4$$
$$4x + y + 2z = 4$$

Solution

Addition method	Row – equivalent matrix method
$2x - y + z = -1$ $x - 2y + 3z = 4$ $4x + y + 2z = 4$	$\begin{bmatrix} 2 & -1 & 1 & : & -1 \\ 1 & -2 & 3 & : & 4 \\ 4 & 1 & 2 & : & 4 \end{bmatrix}$
Interchange the first two equations $x - 2y + 3z = 4$ $2x - y + z = -1$ $4x + y + 2z = 4$	Interchange the first two rows $\begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 2 & -1 & 1 & : & -1 \\ 4 & 1 & 2 & : & 4 \end{bmatrix}$
Multiply the first equation by –2 and add to second $x - 2y + 3z = 4$ $3y - 5z = -9$ $4x + y + 2z = 4$	Multiply the first row by –2 and add to the second $\begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 0 & 3 & -5 & : & -9 \\ 4 & 1 & 2 & : & 4 \end{bmatrix}$
Multiply the first equation by –4 and add to the third $x - 2y + 3z = 4$ $3y - 5z = -9$ $9y - 10z = -12$	Multiply the first row by –4 and add to the third $\begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 0 & 3 & -5 & : & -9 \\ 0 & 9 & -10 & : & -12 \end{bmatrix}$
Multiply the second equation by $\frac{1}{3}$ $x - 2y + 3z = 4$ $y - \frac{5z}{3} = -3$ $9y - 10z = -12$	Multiply the second row by $\frac{1}{3}$ $\begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 0 & 1 & -5/3 & : & -3 \\ 0 & 9 & -10 & : & -12 \end{bmatrix}$
Multiply the second equation by –9 and add to the third $x - 2y + 3z = 4$ $y - \frac{5z}{3} = -3$ $5z = 15$	Multiply the second row by –9 and add to the third $\begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 0 & 1 & -5/3 & : & -3 \\ 0 & 0 & 5 & : & 15 \end{bmatrix}$
Multiply the third equation by $\frac{1}{5}$ $x - 2y + 3z = 4$ $y - \frac{5z}{3} = -3$ $z = 3$	Multiply the third row by $\frac{1}{5}$ $\begin{bmatrix} 1 & -2 & 3 & : & 4 \\ 0 & 1 & -5/3 & : & -3 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$

Multiply the third equation by –3 and $+\frac{5}{3}$, and then add to the first and second equations respectively $x - 2y = -5$ $y = 2$ $z = 3$	Multiply the third row by –3 and $\frac{5}{3}$ and then add to the first and second row respectively $\begin{bmatrix} 1 & -2 & 0 & : & -5 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$
Multiply the second equation by 2 and add to the first $x = -1$ $y = 2$ $z = 3$ Hence the solution is (–1,2,3).	Multiply the second row by 2 and add to the first $\begin{bmatrix} 1 & 0 & 0 & : & -1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$ Hence the solution is (–1,2,3)

Example 3

Solve the system
$$\begin{aligned} 2x - y + 4z &= -3 \\ x \quad - 4z &= 5 \\ 6x - y + 2z &= 10 \end{aligned}$$
by using row–equivalent matrices.

Solution.

Here the second equation does not contain y and so we find some difficulty in writing the augmented matrix. To remove this difficulty, we may write the system as

$$\begin{aligned} 2.x - 1.y + 4.z &= -3 \\ 1.x + 0.y - 4.z &= 5 \\ 6.x - 1.y + 2.z &= 10 \end{aligned}$$

The augmented matrix is

$$\begin{bmatrix} 2 & -1 & 4 & : & -3 \\ 1 & 0 & -4 & : & 5 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

Interchange the first two rows

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 2 & -1 & 4 & : & -3 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

Perform $R_2 - 2R_1$ and $R_3 - 6R_1$

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix}$$

Perform $R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & 0 & 14 & : & -7 \end{bmatrix}$$

Perform $(-1)R_2$ and $\frac{1}{14}R_3$

$$\begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix}$$

Perform $R_1 + 4R_3$ and $R_2 + 12R_3$

$$\begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 7 \\ 0 & 0 & 1 & : & -\frac{1}{2} \end{bmatrix}$$

This gives us the solution $x = 3, y = 7, z = -\frac{1}{2}$, i.e. $(3, 7, -\frac{1}{2})$.

Let us condense this procedure as follows.

$$\begin{bmatrix} 2 & -1 & 4 & : & -3 \\ 1 & 0 & -4 & : & 5 \\ 6 & -1 & 2 & : & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 2 & -1 & 4 & : & -3 \\ 6 & -1 & 2 & : & 10 \end{bmatrix} \quad R_1 \times R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & -1 & 26 & : & -20 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & -1 & 12 & : & -13 \\ 0 & 0 & 14 & : & -7 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -4 & : & 5 \\ 0 & 1 & -12 & : & 13 \\ 0 & 0 & 1 & : & -1/2 \end{bmatrix} \quad \begin{array}{l} R_2 \to (-1)R_2 \\ R_3 \to \frac{1}{14} R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 3 \\ 0 & 1 & 0 & : & 7 \\ 0 & 0 & 1 & : & -\frac{1}{2} \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 + 4R_3 \\ R_2 \to R_2 + 12R_3 \end{array}$$

This gives us the solution $x = 3, y = 7, z = -\frac{1}{2}$, i.e. $(3, 7, -\frac{1}{2})$.

A condensed flow chart of this process

Start

↓

Given three equations in three unknown

↓

Form the augmented matrix

↓

Is the matrix in the form $\begin{bmatrix} 1 & 0 & 0 & : & p \\ 0 & 1 & 0 & : & q \\ 0 & 0 & 1 & : & r \end{bmatrix}$ — No → Use the elementary row operations

↓ Yes

The solution is $x = p, y = q, z = r$.

↓

Stop

EXERCISE

1. Solve, by using both addition method and row–equivalent matrices, the following system of linear equations:

(a) $5x - 2y = -2$
$2x + 5y = -24$

(b) $x - y = 2$
$2x + 3y = 9$

(c) $x + y - 2z = 7$
$2x - 3y - 2z = 0$
$x - 2y - 3z = 3$

2. Solve the follo…
(a) $8x - 3y =$ …
$2x + 6y =$ …
(d) $3x - 2y =$ …
$5x + 3y =$ …
(g) $x + y = 5$ …
$2x + 3y =$ …
(j) $3x - 2y =$ …
$-x + 4y$ …

3. Use the row–…
(a) $x - y +$ …
$x - 2y +$ …
$2x - 2y$ …
(c) $x + y +$ …
$x + 2y$ …
$x + 3y$ …
(e) $9y - 5x$ …
$x + z =$ …
$z + 2y$ …
(g) $x - y +$ …
$x + y +$ …
$3x - 4$ …
(i) $2y + 6$ …
$3y + 6$ …
$2x + 4$ …

1. (a) (–2, –4)
2. (a) (–2,5)
(f) (2,3)
3. (a) (0,2,1)
(f) (–3,3, –1…

2. Solve the following systems by using row–equivalent matrices:

(a) $8x - 3y = -31$, $2x + 6y = 26$
(b) $3x - 3y = 11$, $9x - 2y = 5$
(c) $x = 2y + 3$, $3x - 5y = 8$
(d) $3x - 2y = 8$, $5x + 3y = 7$
(e) $5x - 3y = -2$, $4x + 2y = 5$
(f) $6x - 4y = 0$, $x + y - 5 = 0$
(g) $x + y = 5$, $2x + 3y = 12$
(h) $5x - 3y = 20$, $2x + 3y = 8$
(i) $2x + 12y = 16$, $3x + 10y = 8$
(j) $3x - 2y = -8$, $-x + 4y = 6$

3. Use the row–equivalent matrices to solve the system:

(a) $x - y + 2z = 0$, $x - 2y + 3z = -1$, $2x - 2y + z = -3$
(b) $x + 2y - 3z = 9$, $2x - y + 2z = -8$, $3x - y - 4z = 3$
(c) $x + y + z = 1$, $x + 2y + 3z = 4$, $x + 3y + 7z = 13$
(d) $3x - 5z = -7$, $3x + 5y = 3$, $3z - 3y = 2$
(e) $9y - 5x = 3$, $x + z = 1$, $z + 2y = 2$
(f) $x + 4y + 3z = 6$, $3x + 9y = 18$, $-5x - 6y + 2z = -5$
(g) $x - y + z = -3$, $x + y + z = 1$, $3x - 4y - z = 1$
(h) $x - y - z = -2$, $5x + 10z = 20$, $10y - 20z = 10$
(i) $2y + 6z = 2$, $3y + 6z = 6$, $2x + 4y + 6z = 22$
(j) $x + 4y + z = 18$, $3x + 3y - 2z = 2$, $-4y + z = -7$

Answers

1. (a) (–2, –4) (b) (3,1) (c) (1,2, –2)
2. (a) (–2,5) (b) (–1/3,–4) (c) (1, –1) (d) (2, –1) (e) (1/2, 3/2) (f) (2,3) (g) (3,2) (h) (4,0) (i) (–4,2) (j) (–2,1)
3. (a) (0,2,1) (b) (–1,2, –2) (c) (1, –3,3) (d) (–1/9,2/3,4/3) (e) (3,2, –2) (f) (–3,3, –1) (g) (2,2, –3) (h) (2,3,1) (i) (6,4, –1) (j) (1,3,5)

Application of Matrices and Determinants

a) A system of linear equations in two or more variables appear in various branches of pure, applied and applicable mathematics. Representation of such a system in the form of a *matrix equation* and use of *matrix methods* have proved to be of great help in solving systems of linear equations in two or more variables. To illustrate this beautiful technique, we consider a system of linear equations in two variables.

Let the system of the linear equations in two variables x and y be

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Collecting the coefficients of the variables and the constants separately in the order in which they occur in the equations and enclosing them within parentheses, we have the matrices

$$A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Also, writing the two variables x and y as a column matrix, we have

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then, by virtue of the definition of the product and equality of the two matrices, the system of the linear equations may be written in the compact and elegant matrix form:

$$AX = C,$$

or, $$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

or $$\begin{pmatrix} a_1x + b_1y \\ a_2x + b_2y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

which implies $$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Extension of the above process to a system of linear equations in three or more variables is obvious.

Let us now see how such a matrix equation may be solved by the methods of matrix algebra.

We know that the inverse A^{-1} of the matrix A exists, if A is non-singular, i.e. $|A| \neq 0$. Multiplying both sides of the matrix equation

$$AX = C$$

by A^{-1} from the left, we get

$$A^{-1}(AX) = A^{-1}C$$

or, $$(A^{-1}A)X = A^{-1}C$$

or $\quad IX = D$, (say) with $A^{-1}C = D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

or $\quad X = D$

or $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$

Hence, $x = d_1$ and $y = d_2$.

Worked Out Examples

Example 1

Solve $\quad 4x + 5y = 2$

$\quad\quad 2x + 3y = 0$

Solution :

Writing the system of linear equations in the matrix form, we have,

$$AX = C,$$

where, $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Here $|A| = \begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 12 - 10 = 2 \neq 0$

A_{11} = cofactor of $a_{11} = 3$ $\quad\quad$ A_{12} = cofactor of $a_{12} = -2$

A_{21} = cofactor of $a_{21} = -5$ $\quad\quad$ A_{22} = cofactor of $a_{22} = 4$

$$\text{Matrix of cofactors} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$$

Now, $\quad A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}$

Since $\quad X = A^{-1}C$, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Hence $x = 3$ and $y = -2$

Example 2.

Solve $x + y - z = 1$
$y + z = 2$
$x - y = 0$

Solution :

Writing this system of linear equations in the matrix form, we have

$AX = C$ where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1 + 1 + 1 = 3 \neq 0$$

A_{11} = cofactor of $a_{11} = \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} = 1$ $\quad$ A_{12} = cofactor of $a_{12} = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1$

A_{13} = cofactor of $a_{13} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$ $\quad$ A_{21} = cofactor of $a_{21} = -\begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = 1$

A_{22} = cofactor of $a_{22} = \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} = 1$ $\quad$ A_{23} = cofactor of $a_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2$

A_{31} = cofactor of $a_{31} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$ $\quad$ A_{32} = cofactor of $a_{32} = -\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = -1$

A_{33} = cofactor of $a_{33} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$

$$\text{Matrix of cofactors} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\text{adj } A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$$

Since, $X = A^{-1}C = \dfrac{1}{|A|} \text{adj } (A).C,$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1+2 \\ 1+2 \\ -1+4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and hence $x = 1, y = 1$ and $z = 1$.

b) Another important technique of solving a system of linear equations is the application of *Cramer's rule*. This technique is also a consequence of the technique of elimination which we have discussed in the previous pages. But here we use the notion of a determinant associated with a square matrix instead of the matrix itself.

Let us now see how the idea of a determinant can be used in solving system of linear equations. For instance consider the system of two linear equations

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

where x and y are variables and the rest are all constants.

In order to facilitate computation and to save space, we present computations involving x and y side by side:

Multiply the first equation by b_2, and the second by $-b_1$ to find $a_1b_2x + b_1b_2y = c_1b_2$ $-a_2b_1x - b_1b_2y = -c_2b_1$ Add the two equations to get $(a_1b_2 - a_2b_1)\,x = (c_1b_2 - c_2b_1)$,	Multiply the first equation by $-a_2$ and the second by a_1 to find $-a_1a_2x - a_2b_1y = -a_2c_1$ $a_1a_2x + a_1b_2y = a_1c_2$ Add the two equations to get $(a_1b_2 - a_2b_1)\,y = (a_1c_2 - a_2c_1)$,

Using the determinant notations, we have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Alternative method

The system of linear equations is

$$\left.\begin{aligned} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{aligned}\right\} \quad \ldots.(1)$$

We have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1x + b_1y & b_1 \\ a_2x + b_2y & b_2 \end{vmatrix} \quad \text{(by changing } C_1 \to C_1 + C_2y\text{)}$$

$$= \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \text{because of (1)}$$

Also we have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & b_1y \\ a_2 & b_2y \end{vmatrix}.$$

$$= \begin{vmatrix} a_1 & a_1x + b_1y \\ a_2 & a_2x + b_2y \end{vmatrix} \quad \text{(by applying } C_2 \to C_2 + C_1x\text{)}$$

$$= \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad \text{because of equation (1)]}$$

Let us now see how the determinants determine the nature of the solution of the system, i.e. what values can be assigned to x and y.

Case I.

Suppose $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$, then we can divide in each case by this quantity (number) to obtain

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This pair of values gives unique solution. In this case the system is *consistent* and *independent*

Case II.

Suppose $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ and $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = 0.$

Then we have

$$0.x = 0 \quad \text{and} \quad 0.y = 0$$

Clearly, these two are true whatever may be the values of x and y. In other words, given a value to x, we may find the corresponding value of y from one of the equations such that the pair of values satisfies both equations. Since we can assign an infinite number of values to x, we can find an infinite number of values of y. Thus we have an infinite number of solutions of the system. In such a case, we say that system is *consistent* and *dependent*.

Case III.

Suppose $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$ and either $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \neq 0$ or $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \neq 0,$

Then we have

$0.x \neq 0$ or $0.y \neq 0$, which is obviously false.

In such a case, we say that the system is *inconsistent* and *independent*. In other words, the system has no solution.

In conclusion, we say that the system is consistent and has exactly one solution when determinant of the coefficients, i.e.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0.$$

In what follows, we always assume that determinant of the coefficient matrix is NEVER zero, and in this case we have the following rules for the determinations in the values of x and y:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where the numerator determinants in the values of x and y may be obtained by replacing the first and second columns of the denominator determinant by $\begin{matrix} c_1 \\ c_2 \end{matrix}$ in turn.

The above rule of finding the solution of a system of linear equations is known as Cramer's Rule.

Cramer's rule for a system of linear equations in two variables can be extended to a system of linear equations in three or more variables. The procedure of obtaining formulae for the variables is the same, i.e. gradual elimination of variables till we get an equation involving a single variable.

We can use the alternative method also.

Consider the system of linear equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \text{.........(ii)}$$

We have $x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} \quad \text{(by changing } C_1 \to C_{1+} C_2y + C_3z)$$

$$= \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \quad \text{because of (ii)}$$

Also we have $y \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1y & c_1 \\ a_2 & b_2y & c_2 \\ a_3 & b_3y & c_3 \end{vmatrix}$

$$= \begin{vmatrix} a_1 & a_1x + b_1y + c_1z & c_1 \\ a_2 & a_2x + b_2y + c_2z & c_2 \\ a_3 & a_3x + b_3y + c_3z & c_3 \end{vmatrix} \quad \text{(by applying } C_2 \to C_1x + C_2 + C_3z)$$

$$= \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \text{because of (ii)}$$

Similarly

$$z \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Hence, the formulae for x, y and z are

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_1}{D} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_2}{D} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_3}{D}$$

where the quantities in the numerator and denominator of each formula is a determinant of third order (or three-by-three determinant). It is defined by the formula of the form

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b2 & c2 \\ b3 & c3 \end{vmatrix} - a_2\begin{vmatrix} b1 & c1 \\ b3 & c3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_1 - b_2c_1)$$

Note: the numerator determinants D_1, D_2, D_3 of the above formulae for x, y and z can be obtained by replacing the first, second and third columns of the denominator determinant D by the column $\begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix}$ in turn.

Worked Out Examples

Example 1

To illustrate the use of Cramer's rule, consider the system

$x - 2y = -7$

$3x + 7y = 5.$

Using Cramer's Rule, we have

$$x = \frac{\begin{vmatrix} -7 & -2 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 7 \end{vmatrix}} = \frac{(-7).7 - 5.(-2)}{1.7 - 3.(-2)} = \frac{-39}{13} = -3$$

$$y = \frac{\begin{vmatrix} 1 & -7 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & 7 \end{vmatrix}} = \frac{1.5 - 3.(-7)}{1.7 - 3.(-2)} = \frac{26}{13} = 2$$

Hence the solution is (–3, 2).

Example 2

Solve the following equations using Cramer's rule.

$$\frac{3}{x} + \frac{2}{y} = \frac{19}{20} \quad \text{and} \quad \frac{4}{x} + \frac{10}{y} = 2$$

Solution :

$$\left.\begin{aligned} \frac{3}{x} + \frac{2}{y} &= \frac{19}{20} \\ \frac{4}{x} + \frac{10}{y} &= 2 \end{aligned}\right\}$$

$$D = \begin{vmatrix} 3 & 2 \\ 4 & 10 \end{vmatrix} = 30 - 8 = 22 \qquad D_1 = \begin{vmatrix} 19/20 & 2 \\ 2 & 10 \end{vmatrix} = \frac{19}{2} - 4 = \frac{11}{2}$$

$$D_2 = \begin{vmatrix} 3 & 19/20 \\ 4 & 2 \end{vmatrix} = 6 - \frac{19}{5} = \frac{11}{5}$$

Using Cramer's rule,

$\frac{1}{x} = \frac{D_1}{D} = \frac{11/2}{22} = \frac{1}{4}$ and $\frac{1}{y} = \frac{D_2}{D} = \frac{11/5}{22} = \frac{1}{10}$

$\therefore \quad x = 4$ $\qquad \therefore \quad y = 10$

$\therefore$ the solution is (4, 10)

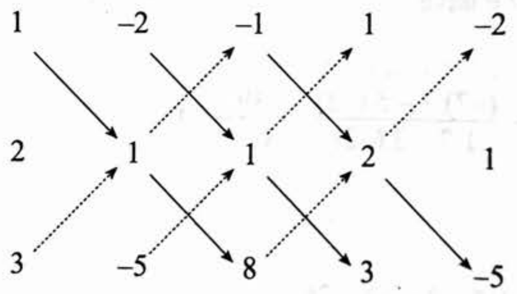
Example 3

Solve by using Cramer's rule, the system

$$x - 2y - z = -7$$
$$2x + y + z = 0$$
$$3x - 5y + 8z = 13$$

Solution

Rule of Sarrus used to find the value of D :



$$D = 8 + (-6) + 10 - (-3) - (-5) - (-32) = 52$$

Hence $\quad D = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix} = 52$

Similarly,

$$D_1 = \begin{vmatrix} -7 & -2 & -1 \\ 0 & 1 & 1 \\ 13 & -5 & 8 \end{vmatrix} = -104, \quad D_2 = \begin{vmatrix} 1 & -7 & -1 \\ 2 & 0 & 1 \\ 3 & 13 & 8 \end{vmatrix} = 52$$

and $D_3 = \begin{vmatrix} 1 & -2 & -7 \\ 2 & 1 & 0 \\ 3 & -5 & 13 \end{vmatrix} = 156$

Hence $\quad x = \frac{D_1}{D} = -\frac{104}{52} = -2 \qquad y = \frac{D_2}{D} = \frac{52}{52} = 1 \qquad z = \frac{D_3}{D} = \frac{154}{52} = 3.$

Hence the solution is (−2,1,3).

Note: In practice, it is actually not necessary to evaluate D_3, since if we know the values of x and y, we can use them in one of the equations to find z. We know $x = -2$ and $y = 1$ in the above case, so by using the equation $-2-2.1-z = -7$, we find $z = 3$.

EXERCISE

1. Apply matrix method or Cramer's rule to solve the following systems:

(a) $2x - y = 5$, $x - 2y = 1$

(b) $2x + 5y = 17$, $5x - 2y = -1$

(c) $3x + 4y = -2$, $5x - 7y = 24$

(d) $-2x + 4y = 3$, $3x - 7y = 1$

(e) $5x - 4y = -3$, $7x + 2y = 49$

(f) $2x + 5y = 24$, $2x + 3y = 12$

(g) $\frac{2x}{3} + y = 16$, $x + \frac{y}{4} = 14$

(h) $3x + \frac{4}{y} = 10$, $-2x + \frac{3}{y} = -1$

2. Solve the following system by using matrix method or Cramer's Rule:

(a) $x - 3y - 7z = 6$, $2x + 3y + z = 9$, $4x + y = 7$

(b) $2x - 3y - z = 4$, $x - 2y - z = 1$, $x - y + 2z = 9$

(c) $3x - y - 2z = 1$, $x - y + 2z = 3$, $-2x + 3y + z = 8$

(d) $3x + 5y = 2$, $2x - 3z = -7$, $4y + 2z = 2$

(e) $6y + 6z = -1$, $8x + 6z = -1$, $4x + 9y = 8$

(f) $x + 2y - z = -5$, $2x - y + z = 6$, $x - y - 3z = -3$

A. Form two simultaneous linear equations of two variables showing three different types of characteristics such as dependent, independent, consistent, inconsistent etc. Clarify what do each pair of linear equations represent?

If possible, solve each pair using any of the methods.

Answers

1. (a) (3,1) (b) (1, 3) (c) (2, –2) (d) (–25/2, –11/2) (e) (5, 7) (f) (–3, 6) (g) (12, 8) (h) (2, 1)

2. (a) (1,3–2) (b) (2, –1,3) (c) (3, 4, 2) (d) (4, –2, 5) (e) (1/2,2/3,–5/6) (f) (1, –2, 2)

Chapter 7

Inverse Circular Functions and Trigonometric Equation

Introduction

Trigonometric functions defined by

$y = \sin x$, $\quad y = \cos x$, $\quad y = \tan x$. etc.,

are known as the sine, cosine, tangent, etc. functions respectively. Here each x corresponds to a unique y. Expressed as a set of ordered pairs (x, y) of real numbers, each of them would look like

$f = \{ (x, y): y = \sin x\}$, $\quad g = \{ (x, y): y = \cos x\}$,

$h = \{ (x, y): y = \tan x\}$, etc.

On interchanging the roles of x and y, they will look like

$F = \{ (x, y): x = \sin y \}$, $\quad G = \{ (x, y): x = \cos y \}$,

$H = \{ (x, y): x = \tan y \}$, etc.

In each of these cases, each x corresponds to more than one y. For instance, if $x = \frac{1}{2}$, then y may be 30°, 150°, 360° + 30°, etc. so, none of them represents a function. But, on suitably restricting the values of y in these cases (or the values of x in the former cases), we can get only one value of y corresponding to each value of x. We then get well-defined functions. In the next section, we make this point more clear with special reference to the sine function.

Inverse Circular Functions

The circular function defined by $y = \sin x$, i.e.,

$$f = \{ (x, y): y = \sin x \}$$

is the sine function. Here the domain of the function f is the set of real numbers, and its range is the set of real numbers between –1 and 1 inclusive. We thus have

function $= f$

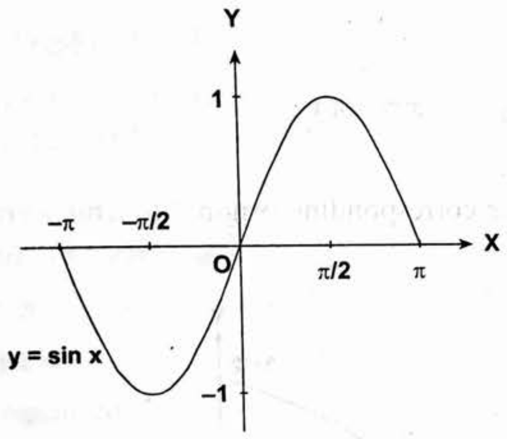
domain of f $= \{x: x \in R\}$

range of f $= \{y: -1 \le y \le 1 \}$.

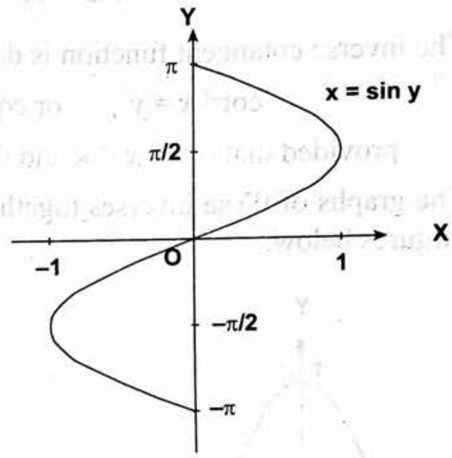
Interchanging the roles of x and y, we have

$$F = \{ (x, y): x = \sin y \}.$$

In this case, the domain of F is the set of real numbers between -1 and 1 inclusive; and its range is the set of all real numbers, several of which may correspond to one x. This can be seen more clearly from the graphs of $y = \sin x$ and $x = \sin y$. The graphs of $y = \sin x$ winds along the x-axis (Fig. a) and that of $x = \sin y$ (is of the same form but winds along the y-axis (Fig. b). In fact, one is the mirror image of the other (i.e. reflection) on the line $y = x$. Note that a horizontal line lying between $y = -1$ and $y = 1$ intersect the graph of $y = \sin x$ at several points. This shows that there are many values of x which correspond to a certain value of y. But one can also see that the length of the longest interval on the x-axis for which a horizontal line can intersect the curve $y = \sin x$ in at most one point in π. For convenience, this interval (in fact, the domain of the function) is chosen to be an interval $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$ (the heavy part of the graph). Moreover, this choice gives a full range of values (i.e. $-1 \le y \le 1$) of the function defined by $y = \sin x$.



(a)



(b)

On other hand, a vertical line lying between $x = -1$ and $x = 1$ intersects the graph of $x = \sin y$ at more than one point. This shows that $x = \sin y$, as it stands, does not define a function. But, on restricting the value of y so that $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$ (i.e. , restricting to the solid part of the curve only), each value of x between -1 and 1 inclusive corresponds to exactly one value of y. To indicate this restriction in $x = \sin y$, it is customary to write (capitalise s in sin)

$$x = \sin y$$

and the value of y so obtained is called the principal value (p.v.).

Other commonly used notations for this are

$$y = \sin^{-1} x, \text{ (read 'inverse sine of } x\text{')}$$

and $y = \text{Arc} \sin x$, (from the Latin phrase *arcs cuisus sinus x est* –the arc whose sine is x)

Note that $x = \sin y$, $y = \text{arc} \sin x$ and $y = \sin^{-1} x$ are three equivalent notations for the same thing. Also, it should be remembered that

$$\sin^{-1}x \quad \text{NEVER MEANS} \quad \frac{1}{\sin x}.$$

In short, we have the following working definitions of the inverse sine, the inverse cosine, the inverse tangent, the inverse cotangent, the inverse secant and the inverse cosecant being omitted as they are of less common use.

(a) The *inverse sine function* is defined by

$\sin^{-1} x = y$ or equivalently $x = \sin y$

provided that $-1 \le x \le 1$ and $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$.

(b) The *inverse cosine function* is defined by

$\cos^{-1} x = y$ or equivalently $x = \cos y$

provided that $-1 \le x \le 1$ and $0 \le y \le \pi$.

(c) The *inverse tangent function* is defined by

$\tan^{-1} x = y$ or equivalently $x = \tan y$

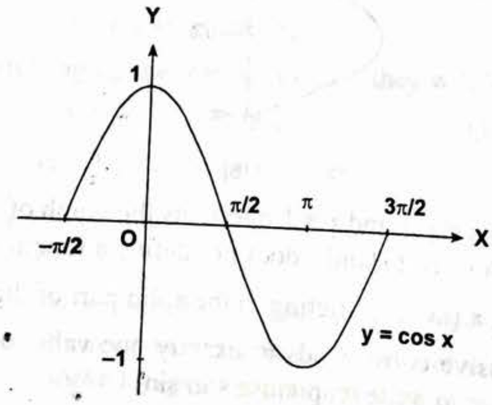
provided that $-\times < x < \times$ and $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

(d) The inverse cotangent function is defined by

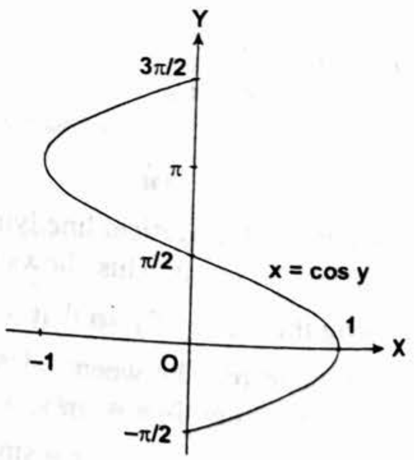
$\cot^{-1} x = y$ or equivalently $x = \cot y$

provided that $-\times < x < \times$ and $0 < y < \pi$.

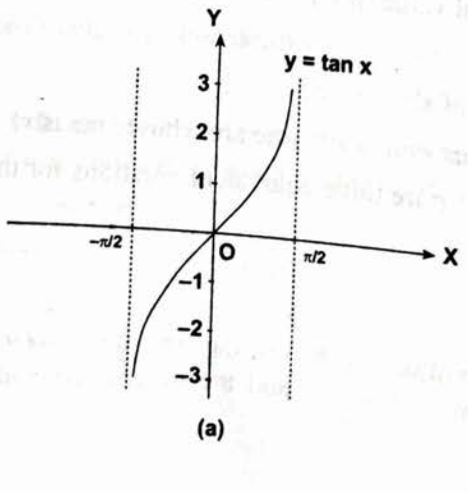
The graphs of these inverses together with the corresponding original functions are shown in the figures below.



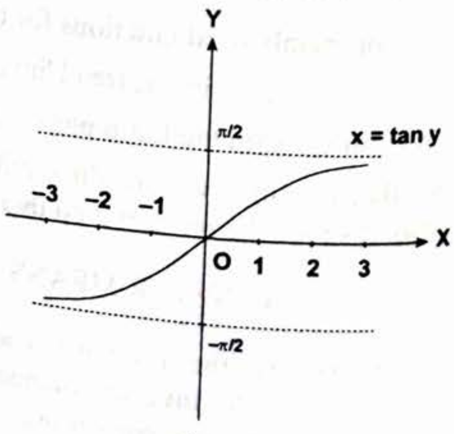
(a)



(b)



(a)



(b)

For emphasis and convenience, we list here the definitions of principal inverse trigonometric functions, together with corresponding domains and ranges.

Function	Notation for Principal Inverse Function	Defining Equation	Domain	Range
sine	$\sin^{-1}$	$y = \sin^{-1}x$	$-1 \le x \le 1$	$-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$
cosine	$\cos^{-1}$	$y = \cos^{-1}x$	$-1 \le x \le 1$	$0 \le y \le \pi$
tangent	$\tan^{-1}$	$y = \tan^{-1}x$	$-\infty < x < \infty$	$-\frac{1}{2}\pi < y < \frac{1}{2}\pi$
cotangent	$\cot^{-1}$	$y = \cot^{-1}x$	$-\infty < x < \infty$	$0 < y < \pi$

Some Useful Results

We shall now establish some elementary but useful results involving trigonometric and inverse trigonometric functions.

(a) For a given angle θ,

(i) $\theta = \sin^{-1} \sin \theta$ (ii) $\theta = \sin \sin^{-1} \theta$, etc.

Let $\sin \theta = x$, then, by definition, $\theta = \sin^{-1}x$.

Hence $\theta = \sin^{-1} \sin \theta$ (since $x = \sin \theta$)

Again, let $y = \sin^{-1} \theta$, then $\theta = \sin y$, and

$\theta = \sin \sin^{-1} \theta$ (since $y = \sin^{-1} \theta$)

The rest follows similarly.

(b) For a given numerical value x,

(i) $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$ (ii) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$

(iii) $\sin^{-1} x = \operatorname{cosec}^{-1} \frac{1}{x}$, etc.

We shall prove the first one only and the rest will be left as exercises;

Let $\theta = \operatorname{cosec}^{-1}x$, then, by definition, $\operatorname{cosec} \theta = x$ and $\sin \theta = \frac{1}{x}$,

and hence $\theta = \sin^{-1} \frac{1}{x}$.

Thus $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$.

(c) Expressions of a given inverse trigonometric function in terms of the remaining inverse trigonometric functions.

A few simple cases of the results that we are going to discuss here has already been given in (b) above. We shall consider a few similar but a bit more general relations.

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$ (ii) $\sin^{-1}x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

(iii) $\sin^{-1}x = \cot^{-1}\dfrac{\sqrt{1-x^2}}{x}$ (iv) $\sin^{-1}x = \sec^{-1}\dfrac{1}{\sqrt{1-x^2}}$.

To prove these, we put $\theta = \sin^{-1}x$ then $\sin\theta = x$, and

(i) $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$, i.e. $\theta = \cos^{-1}\sqrt{1-x^2}$.

(ii) $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{x}{\sqrt{1-x^2}}$, i.e. $\theta = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

The remaining two also follow similarly, and are left as exercises. Combining the above results, we find that

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$
$$= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}.$$

(d) For a given numerical value x

(i) $\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{1}{2}\pi$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{1}{2}\pi$.

Suppose $\sin^{-1}x = \theta$, then $x = \sin\theta$, and so $\cos(\frac{1}{2}\pi - \theta) = x$

This gives $\cos^{-1}x = \frac{\pi}{2} - \theta$; and hence

$$\sin^{-1}x + \cos^{-1}x = \frac{1}{2}\pi.$$

The other results also follow similarly.

(e) For a given numerical value x,

i) $\sin^{-1}(-x) = -\sin^{-1}x$

ii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

iii) $\tan^{-1}(-x) = -\tan^{-1}x$

iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$

Let $\sin^{-1}x = \theta$, then $x = \sin\theta$

So, $-x = -\sin\theta$ or, $-x = \sin(-\theta)$

$\therefore \sin^{-1}(-x) = -\sin^{-1}x$ or, $\sin^{-1}(-x) = -\theta$

Again, if $\cos^{-1}x = \theta$, then $x = \cos\theta$

So, $-x = -\cos\theta$ or, $-x = \cos(\pi - \theta)$

$\therefore \cos^{-1}(-x) = \pi - \cos^{-1}x$. or, $\cos^{-1}(-x) = \pi - \theta$

The other results follow similarly.

We shall now illustrate the procedure of handling with problems involving inverse trigonometric functions.

Worked Out Examples

Example 1

Use Trigonometric table if necessary to evaluate

(a) $\sin^{-1}(-1)$ (b) $\sin(\cos^{-1}\frac{2}{3})$ (c) Arc tan (tan 60°)

Solution.

Using trigonometric tables, we have

(a) $\sin^{-1}(-1) = -\frac{1}{2}\pi$ and

(b) Let $\cos^{-1}\frac{2}{3} = x$, then

$\cos x = \frac{2}{3}$ and $\sin(\cos^{-1}\frac{2}{3}) = \sin x = \sqrt{1-\cos^2 x}$

$$= \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$$

(c) Arc tan (tan 60°) = 60°.

Example 2

Find the value of $\cos\left(\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{1}{2}\right)$

Solution.

Let $x = \sin^{-1}\frac{1}{4}$ and $y = \cos^{-1}\frac{1}{2}$, then

$\sin x = \frac{1}{4}$ and $\cos y = \frac{1}{2}$, and hence

$\cos x = \sqrt{1-\frac{1}{16}} = \sqrt{\frac{15}{16}}$ and $\sin y = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}}$.

Therefore, $\cos\left(\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{1}{2}\right) = \cos(x+y)$

$$= \cos x \cos y - \sin x \sin y$$

$$= \frac{\sqrt{15}}{4}\cdot\frac{1}{2} - \frac{1}{4}\cdot\frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{15}-\sqrt{3}}{8}$$

Example 3

Prove that $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$.

Solution.

Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$, then

$x = \tan A$ and $y = \tan B$

Now, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy}$

Hence $A + B = \tan^{-1} \frac{x+y}{1-xy}$.

i.e. $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

Example 4

Show that $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} \{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$

Solution.

Let $\sin^{-1} x = A$ and $\sin^{-1} y = B$, then

$\sin A = x$ and $\sin B = y$ and

$\cos A = \sqrt{1-x^2}$ and $\cos B = \sqrt{1-y^2}$

Now, since $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$= x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$.

$A + B = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

i.e. $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}$

The second case follows similarly.

Example 5

Prove that $2\tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$.

Solution.

Let $x = \tan\theta$, then $2\tan^{-1} x = 2\tan^{-1} \tan\theta = 2\theta$,

$\sin^{-1} \frac{2x}{1+x^2} = \sin^{-1} \frac{2\tan\theta}{1+\tan^2\theta} = \sin^{-1}(\sin 2\theta) = 2\theta$,

$\cos^{-1} \frac{1-x^2}{1+x^2} = \cos^{-1} \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos^{-1}(\cos 2\theta) = 2\theta$

$\tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{2\tan\theta}{1-\tan^2\theta} = \tan^{-1}(\tan 2\theta) = 2\theta$.

Combining the above results, we get the required result.

Example 6

Solve $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$

Solution.

Let $a=\tan\theta \quad \therefore \quad \theta=\tan^{-1}a$

Then, $\sin^{-1}\frac{2a}{1+a^2}=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$=\sin^{-1}(\sin 2\theta)$

$=2\theta=2\tan^{-1}a$

$\therefore \quad \sin^{-1}\frac{2a}{1+a^2}=2\tan^{-1}a,$

similarly $\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}b$

$\therefore$ the given equation becomes

$2\tan^{-1}a+2\tan^{-1}b=2\tan^{-1}x$

or $\quad \tan^{-1}\frac{a+b}{1-ab}=\tan^{-1}x.$

Hence $\quad x=\frac{a+b}{1-ab}.$

Example 7

Find the value of $\cos\tan^{-1}\sin\cot^{-1}x$.

Solution.

Let $\cot^{-1}x=\theta$, then $\cot\theta=x$ and $\sin\theta=\frac{1}{\sqrt{1+x^2}}$

Again, suppose

$\tan^{-1}\sin\cot^{-1}x=\tan^{-1}\sin\theta=\varnothing$

then $\tan\varnothing=\sin\cot^{-1}x=\sin\theta=\frac{1}{\sqrt{1+x^2}}.$

Hence $\cos\tan^{-1}\sin\cot^{-1}x=\cos\varnothing$

$$=\frac{1}{\sqrt{1+\tan^2\varnothing}}=\frac{1}{\sqrt{1+\frac{1}{1+x^2}}}$$

$$=\sqrt{\frac{1+x^2}{2+x^2}}$$

Example 8

Prove that $\cos(3\cos^{-1}x)=4x^3-3x$

Solution:

Let $\cos^{-1}x = \theta$ then $x = \cos\theta$

Now, $\cos(3\cos^{-1}x) = \cos 3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$= 4x^3 - 3x$

Example 9

Prove that $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \frac{\pi}{2} - \frac{x}{2}$

Solution:

$$\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right) = \tan^{-1}\left(\frac{2\cos^2 x/2}{2\sin x/2 \cos x/2}\right)$$

$= \tan^{-1}(\cot x/2)$

$= \tan^{-1}(\tan(\pi/2 - x/2))$

$= \frac{\pi}{2} - \frac{x}{2}$

Example 10

Prove that $\cot^{-1}x - \cot^{-1}z = \cot^{-1}\frac{xy+1}{y-x} + \cot^{-1}\frac{yz+1}{z-y}$

Solution:

$\cot^{-1}x - \cot^{-1}z = \cot^{-1}x - \cot^{-1}y + \cot^{-1}y - \cot^{-1}z$

$= \cot^{-1}\frac{xy+1}{y-x} + \cot^{-1}\frac{yz+1}{z-y}$

Example 11

Prove that $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

Solution:

Since, $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$, so

$$\sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4/5}{\sqrt{1-\left(\frac{4}{5}\right)^2}} = \tan^{-1}\frac{4}{3}$$

Again since, $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, so

$$2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2\times 1/3}{1-\left(\frac{1}{3}\right)^2} = \tan^{-1}\frac{3}{4}$$

$$\text{Now, } \sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\frac{4}{3} + \cot^{-1}\frac{4}{3} \qquad \left(\because \tan^{-1}x = \cot^{-1}\frac{1}{x}\right)$$

$$= \frac{\pi}{2} \qquad \left(\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$$

Example 12

If $\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}$, prove that $x^2 + y^2 = 1$

Solution:

$$\cos^{-1}x + \cos^{-1}y = \frac{\pi}{2}$$

or, $\cos^{-1}x = \frac{\pi}{2} - \cos^{-1}y$

or, $\cos^{-1}x = \sin^{-1}y$

or, $\cos^{-1}x = \cos^{-1}\sqrt{1-y^2}$

or, $x = \sqrt{1-y^2}$

$x^2 + y^2 = 1$

EXERCISE

1. Evaluate the following without using tables:
   (a) $\sin^{-1} 1$ (b) $\sin^{-1}(-\frac{1}{2})$. (c) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
   (d) $\tan^{-1} 1$ (e) Arccot (-1) (f) Arctan $\left(-\frac{1}{\sqrt{3}}\right)$
2. Express each of the following in terms of x:
   (a) $\cos\tan^{-1}x$ (b) $\sin(\cot^{-1}x)$ (c) $\tan(\text{Arccot } x)$
   (d) $\cos\sin^{-1}x$ (e) $\tan(2\tan^{-1}x)$ (f) $\sin(2\tan^{-1}x)$
   (g) $\cos(2\cot^{-1}x)$ (h) $\cot(2\text{ Arccot } x)$
3. Evaluate each of the following, using tables if necessary;
   (a) $\sin\left(\cos^{-1}\frac{3}{5}\right)$ (b) $\cos\left(\text{Arccos}\frac{2}{3}\right)$ (c) Arctan $\left(\tan\frac{\pi}{6}\right)$
   (d) $\sin\left(\tan^{-1}\frac{3}{4}\right)$ (e) $\sin\left(2\cos^{-1}\frac{1}{2}\right)$ (f) $\sin^{-1}\left(2\cos\frac{\pi}{3}\right)$
4. Prove each of the following:
   (a) $2\cos^{-1}x = \cos^{-1}(2x^2 - 1)$ (b) $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$

(c) $3\tan^{-1}x = \tan^{-1}\dfrac{3x - x^3}{1 - 3x^2}$

(d) $\sin(\text{Arccos } t) = \cos(\text{Arcsin } t)$

(e) $\cos(2\text{ Arccos } t) = 2t^2 - 1$

(f) $\sin(2\sin^{-1}x) = 2x\sqrt{1 - x^2}$

(g) $\cos(\sin^{-1}u + \cos^{-1}v) = v\sqrt{1 - u^2} - u\sqrt{1 - v^2}$

(h) $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{6}{17}$

(i) $\tan^{-1}a - \tan^{-1}c = \tan^{-1}\dfrac{a - b}{1 + ab} + \tan^{-1}\dfrac{b - c}{1 + bc}$

(j) $\tan(\text{Arctan } u - \text{Arctan } v) = \dfrac{u - v}{1 + uv}$

(k) $\tan(2\tan^{-1}x) = 2\tan(\tan^{-1}x + \tan^{-1}x^3)$

(l) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{x + y + z - xyz}{1 - yz - zx - xy}$

(m) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{1}{2}\pi$

(n) Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\dfrac{1 - x}{1 + x}\right) = \frac{1}{2}\sin^{-1}\left(\dfrac{2\sqrt{x}}{1 + x}\right)$

5. Find the value of each of the following:

(a) $\cos\left(\sin^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right)$

(b) $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$

(c) $\tan\left(\text{Arccos}\frac{4}{5} - \text{Arcsin}\frac{12}{13}\right)$

(d) $\tan(\tan^{-1}x - \tan^{-1}2y)$

(e) $\tan^{-1}3 + \tan^{-1}\frac{1}{3}$

(f) $\text{Arcsin } t - \text{Arccos}(-t)$

6. Solve each of the following equation:

(a) $\cos^{-1}x - \sin^{-1}x = 0$

(b) $\tan^{-1}x - \cot^{-1}x = 0$

(c) $\sin^{-1}\frac{1}{2}x = \cos^{-1}x$

(d) $\cos^{-1}x = \cos^{-1}\dfrac{1}{2x}$

(e) $\tan^{-1}2x = 2\tan^{-1}x$

(f) $\cos^{-1}x + \cos^{-1}2x = \frac{1}{2}\pi$

(g) $\sin^{-1}x + \cos^{-1}(1 - x) = \frac{1}{2}\pi$

(h) $\sin^{-1}\dfrac{2a}{1 + a^2} - \cos^{-1}\dfrac{1 - b^2}{1 + b^2} = 2\tan^{-1}x$

(i) $\tan^{-1}\dfrac{x - 1}{x - 2} + \tan^{-1}\dfrac{x + 1}{x + 2} = \tan^{-1}1$

(j) $\tan^{-1}2x + \tan^{-1}3x = \frac{1}{4}\pi$

(k) $3\tan^{-1}\dfrac{1}{2 + \sqrt{3}} - \tan^{-1}\dfrac{1}{x} = \tan^{-1}\dfrac{1}{3}$

7. Prove that

(a) $x + y + z = xyz$, if $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

(b) $xy + yz + zx = 1$, if $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$

8. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, show that $x^2 + y^2 + z^2 + 2xyz = 1$.

9. Prove that:

(i) $\tan^{-1}\frac{3}{5} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{27}{11}$

(ii) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} = \frac{1}{4}\pi$

(iii) $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \frac{1}{4}\pi$

(iv) $4(\cot^{-1}3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$

(v) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi = 2\left(\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right)$

10. Prove that:

(i) $\sin^{-1}\sqrt{\frac{x-b}{a-b}} = \cos^{-1}\sqrt{\frac{a-x}{a-b}} = \tan^{-1}\sqrt{\frac{x-b}{a-x}}$

(ii) $\cos^{-1} x = 2\sin^{-1}\sqrt{\frac{1-x}{2}} = 2\cos^{-1}\sqrt{\frac{1+x}{2}}$

11. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, prove that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

12. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{\pi}{2}$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

A. What are the domains and the ranges of $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$ in order that they represent functions?

B. Draw a rough sketch of $\sin^{-1}x$, $-1 \le x \le 1$.

Answers

1. (a) $\frac{1}{2}\pi$ (b) $-\pi/6$ (c) $\frac{5\pi}{6}$ (d) $\frac{1}{4}\pi$ (e) $\frac{3\pi}{4}$ (f) $-\frac{\pi}{6}$

2. (a) $1/\sqrt{1+x^2}$ (b) $1/\sqrt{1+x^2}$ (c) $1/x$ (d) $\sqrt{1-x^2}$ (e) $2x/(1-x^2)$ (f) $2x/(1+x^2)$ (g) $\frac{x^2-1}{x^2+1}$ (h) $(x^2-1)/2x$

3. (a) 4/5 (b) 2/3 (c) $\pi/6$ (d) 3/5 (e) $\sqrt{3}/2$ (f) $\frac{1}{2}\pi$

5. (a) 16/65 (b) $\frac{4\sqrt{3}+3}{10}$ (c) $-\frac{33}{56}$ (d) $\frac{x-2y}{1+2xy}$ (e) $\frac{1}{2}\pi$ (f) $-\frac{\pi}{2}$

6. (a) $x = 1/\sqrt{2}$ (b) $x = 1$ (c) $x = 2/\sqrt{5}$ (d) $x = \pm 1/\sqrt{2}$ (e) $x = 0$ (f) $x = \pm\frac{1}{\sqrt{5}}$ (g) $x = \frac{1}{2}$ (h) $x = \frac{a-b}{1+ab}$ (i) $x = \pm 1/\sqrt{2}$ (j) $x = -1$ or $1/6$ (k) $x = 2$

Additional Questions

1. Prove that: $\tan^{-1}\frac{a-b}{1+ab} + \tan^{-1}\frac{b-c}{1+bc} + \tan^{-1}\frac{c-a}{1+ca}$
$= \tan^{-1}\frac{a^2-b^2}{1+a^2b^2} + \tan^{-1}\frac{b^2-c^2}{1+b^2c^2} + \tan^{-1}\frac{c^2-a^2}{1+c^2a^2}$

2. Prove that:
   (i) $\cot^{-1}\frac{ab+1}{a-b} + \cot^{-1}\frac{bc+1}{b-c} + \cot^{-1}\frac{ca+1}{c-a} = 0$
   (ii) $\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3 = \pi/2$
   (iii) $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3) = 15$
   (iv) $\tan\left(\frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} + \frac{1}{2}\cos^{-1}\frac{1-y^2}{1+y^2}\right) = \frac{x+y}{1-xy}$

3. Prove that
   (i) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \frac{\pi}{4} - x$
   (ii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}$
   (iii) $\tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$

4. (i) If $\sec^{-1}x = \text{cosec}^{-1}y$, show that $\frac{1}{x^2} + \frac{1}{y^2} = 1$
   (ii) If $\cos^{-1}x + \cos^{-1}y = \theta$, show that $x^2 - 2xy\cos\theta + y^2 = \sin^2\theta$
   (iii) If $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \alpha$, prove that $x^2 = \sin 2\alpha$.

Introduction of Trigonometric Equations

A **trigonometric equation** is an equation involving one or more trigonometric functions of a variable. The equation may be true for one or more values, but not every value of the variable.

By solving a trigonometric equation, we mean to find the angle or angles which satisfy the equation. The values of the variable (angle) are known as the **root(s)** of the equation.

The simplest type of trigonometric equation is that in which a trigonometric function of a variable (angle) is equal to a constant. The equations $\sin x = \frac{1}{2}$ is such an example. It is satisfied by $x = 30°$, $x = 150°$ and all angles which differ from these by any integral multiple of 360°, that is, for all integers, the solutions are

$x = 30° + n.\,360°$ and $x = 150° + n.\,360°$.

thus there exists an infinite number of roots of the equation $\sin x = \frac{1}{2}$.

The set of all possible solutions of a trigonometric equation form the **general solution** of the equation.

General Solution of the Equations sin x = k, cos x = k and tan x = k.

Any angle x and each of the angles formed by adding or subtracting any integral multiple of 360° to or from the angle x, have the same initial and terminal arms, i.e. are *coterminal*. Therefore, any trigonometric function of an angle x has the same value as the same trigonometric function of every angle coterminal with the angle. In particular

$$\sin 30° = \sin(\pm 360° + 30°) = \frac{1}{2} = \sin(n\, 360° + 30°) \qquad \text{for } n = 0, \pm1, \pm2, \pm3 \ldots, \text{etc.}$$

(a) The General Solution of sin x = k, (–1 ≤ k ≤ 1).

Let θ be the angle (preferably the smallest) whose sine is k. Then, $\pi - \theta$ is the other angle having the same sine. Since the value of the trigonometric function is unaltered by adding or subtracting an integral multiple of 360° to or from θ, we have

$$x = 2m\pi + \theta = 2m\pi + (-1)^{2m}\theta \qquad (1)$$

and $$x = 2m\pi + \pi - \theta = (2m+1)\pi + (-1)^{2m+1}\theta \qquad (2)$$

where m is an integer, as the solution of the equation $\sin x = k$.

Combining (1) and (2), we have $x = n\pi + (-1)^n\theta$, n = 0, ±1, ±2 …

But $x = n\pi + (-1)^n(\pm\theta)$ i.e. $x = n\pi \pm (-1)^n\theta$

is same as to write $x = n\pi \pm \theta$.

Because if n is even, $(-1)^n = 1$, so

$$x = n\pi \pm (-1)^n\theta = n\pi \pm \theta$$

and if n is odd, $(-1)^n = -1$, so

$$x = n\pi \pm (-1)^n\theta = n\pi \pm (-1)\theta = n\pi \pm \theta$$

$\therefore x = n\pi \pm (-1)^n\theta = n\pi \pm \theta$, for all integral values of n.

Cor. 1 If $k = 0$, $\sin x = \sin n\pi$, or $x = n\pi$ for all integers n.

Cor. 2 If $\operatorname{cosec} x = k$, $x = n\pi + (-1)^n\theta$, for all integral values of n.

Cor. 3 If $\sin x = 1$, $x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$ for all integers n.

Cor. 4 If $\sin x = -1$, $x = 2n\pi - \frac{\pi}{2} = (4n-1)\frac{\pi}{2}$ for all integers n.

(b) The General Solution of cos x = k, (–1 ≤ k ≤ 1)

Let θ be a particular angle such that $\cos\theta = k$. Then $-\theta$ is another value of x whose cosine is the same costant k. Since all coterminal angles of θ and $-\theta$ are given by $2n\pi + \theta$ and $2n\pi - \theta$,

$$\cos x = \cos(2n\pi \pm \theta).$$

Hence $x = 2n\pi \pm \theta$, for any integer n.

Cor. 1 If $k = 1$, $\cos x = \cos 2n\pi$ and $x = 2n\pi$, for any integer n.

Cor. 2 If $\sec x = k$, then $x = 2n\pi \pm \theta$, for any integer n.

Cor. 3 If $k = 0$, $\cos x = \cos\frac{\pi}{2}$ and $x = (2n+1)\frac{\pi}{2}$

Cor. 4 If $k = -1$, $\cos x = \cos \pi$ and $x = (2n+1)\pi$

(c) The General Solution of $\tan \theta = k$

Let θ be a particular angle so that $\tan \theta = k$. The other value of x for which $\tan x = k$, is $\pi + \theta$. Thus the coterminal angles are given by

$$2m\pi + \theta \qquad \text{and} \qquad 2m\pi + \pi + \theta = (2m+1)\pi + \theta.$$

Also $\tan x = \tan(n\pi + \theta)$, for any integer n.

Hence the required general solution is $x = n\pi + \theta$, for any integer n.

Cor. 1 If $k = 0$, $x = n\pi$, for any integer n.

Cor. 2 If $\cot x = k$, then $x = n\pi + \theta$, for any integer n.

Trigonometric Equations in Other Forms

If a given trigonometric equation is not in the simplest form discussed so far, the usual procedure is to derive two or more simple equations which yield all the solutions of the given equation. In order to obtain simple equations from a given equation, the most useful tools are the algebraic operations and trigonometric identities. We give below a few hints that will be of adequate help in solving a trigonometric equation.

(a) Express, if possible, all trigonometric functions in terms of a single trigonometric function of same angle.

(b) Transfer every term to the left.

(c) If quadratic in certain trigonometric function is obtained, use the formula for the solution of a quadratic equation.

(d) Or factorise the left side, and equate each factor to zero

(e) Use the general values of the last section.

Remark. In the process of solving trigonometric equations, algebraic operations such as squaring, cubing, etc. give rise to some additional equations and consequently some additional roots. It is therefore advisable to check which of the roots thus obtained do not satisfy the given equation. Such roots must be discarded. We illustrate this with an example.

Suppose we have to solve the equation

$$\cos x - \sin x = 1 \text{ for } 0° \leq x \leq 360°.$$

It is usual to write the above equation in the form

$$\cos x = 1 + \sin x\,.$$

Squaring both sides, we have

$$\cos^2 x = \sin^2 x + 2\sin x + 1$$

Using $\qquad \cos^2 x = 1 - \sin^2 x \qquad$ and simplifying, we have

$2\sin^2 x + 2\sin x = 0$

or, $\sin x(\sin x + 1) = 0.$

Hence, either $\sin x = 0$ or $\sin x = -1$.

That is, $x = 0°$ and $180°$ in the given range if $\sin x = 0$;

and $x = 270°$ in the same range if $\sin x = -1$.

It is easy to see that the roots $x = 0°$ and $x = 270°$ satisfy the given equation where as the value $x = 180°$ does not. Hence $x = 180°$ should be discarded. (Why ?)

Worked Out Examples

Example 1.

Solve $2\cos^2 x - 5\cos x + 2 = 0$ for $0° \leq x \leq 360°$.

Solution:

Factoring the left hand side, we get

$$(\cos x - 2)(2\cos x - 1) = 0.$$

Thus $\cos x - 2 = 0$ and $2\cos x - 1 = 0$

or $\cos x = 2$ and $\cos x = \frac{1}{2}$

Since $\cos x$ can never be greater than (numerically) 1, $\cos x = 2$ has no solution.

From $\cos x = \frac{1}{2}$, we have $x = 60°$ and $x = 300°$ as the required solution in the given range. (Checking our results is quite simple.)

Example 2.

Find all values of x in the interval $0 \leq x \leq 2\pi$ which satisfies the equation

$$6\cos^2 x + 4\sin^2 x = 5.$$

Solution:

The given equation may be written as

$$6\cos^2 x + 4(1 - \cos^2 x) = 5$$

or, $2\cos^2 x = 1$

or, $\cos^2 x = \frac{1}{2}$

Hence $\cos x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

Clearly, the roots of the simple equation $\cos x = \sqrt{\frac{1}{2}}$ in the given range are $x = 45°$ and $x = 315°$; and those of $\cos x = -\sqrt{\frac{1}{2}}$ in the same range are $x = 135°$ and $x = 225°$. All these roots satisfy the given equation (Checking is left as exercise).

Example 3.

Solve $\cot x + \tan x = 2.$

Solution

The given equation is same as

$$\frac{1}{\tan x} + \tan x = 2$$

or, $1 + \tan^2 x = 2 \tan x$

or, $\tan^2 x - 2 \tan x + 1 = 0$

or, $(\tan x - 1)^2 = 0$

$\therefore \tan x = 1,$ (repeated)

$\tan x = \tan \frac{\pi}{4}$

Hence $x = n\pi + \frac{1}{4}\pi$, for any integral values of n.

Checking is quite simple.

Example 4

Solve: $a \cos x + b \sin x = c$, where a, b and c are constants.

Solution:

If $a = 0$ or $b = 0$, the equation reduces to simple equation in each case, and the solution can be written down directly. However, we shall assume that a and b are not simultaneously (which is obviously the case) equal to zero. For definiteness, let us assume that $a \neq 0$.

We then put

$$\tan \theta = \frac{\text{coefficient of the second term (i.e. } \sin x)}{\text{coefficient of the first term (i.e. } \cos x)} = \frac{b}{a}.$$

Dividing both sides of the given equation by a and using $\tan \theta = \frac{b}{a}$, we have

$$\cos x + \tan \theta \sin x = \frac{c}{a}.$$

or, $\cos x \cos \theta + \sin x \sin \theta = \frac{c}{a} \cos \theta$

or, $\cos (x - \theta) = \frac{c}{\sqrt{a^2 + b^2}}$ since $\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$

If $c^2 > a^2 + b^2$, no real solution of the equation exists (Why ?)

If $c^2 \leq a^2 + b^2$, we can find an angle ø such that

$$\cos ø = \frac{c}{\sqrt{a^2 + b^2}}$$

Therefore $\cos(x - \theta) = \cos \phi$

This gives the following general solutions of the given equation

$$x - \theta = 2n\pi \pm \phi$$

i.e. $$x = 2n\pi + \theta \pm \phi$$

Example 5.

Solve $\sqrt{3}\sin x - \cos x = \sqrt{3}$ for $0 \le x \le 2\pi$.

Solution:

Divide both sides of the equation by $\sqrt{(\text{coefficient of } \sin x)^2 + (\text{coefficient of } \cos x)^2}$

$$= \sqrt{3+1} = 2.$$

Then, we have

$$\sin x . \frac{\sqrt{3}}{2} - \cos x . \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$\sin x \cos 30° - \cos x \sin 30° = \frac{\sqrt{3}}{2} \quad \left(\text{using the smallest positive angle}\right)$$

$$\sin(x - 30°) = \sin 60°.$$

Hence $x - 30° = n\pi + (-1)^n \frac{\pi}{3}$.

Putting $n = 0$ and $n = 1$,

we get $x = 90°$ and $x = 150°$.

Example 6.

Solve the equation $2\sin 3x - 2\sin x + 5\cos 2x = 0$.

Solution:

Since $\sin 3x - \sin x = 2\cos 2x \sin x$, the given equation reduces to

$$4\cos 2x \sin x + 5\cos 2x = 0$$

$$\cos 2x(4\sin x + 5) = 0$$

Thus $\cos 2x = 0$

or $4\sin x + 5 = 0$.

Since $\sin x = -\frac{5}{4}$ does not have any solution, the required solution

$$2x = 90°, 270°, 450°, 630°, \text{etc.}$$

or $x = 45°, 135°, 225°, 315°$, etc.

Example 7.

Solve $\tan ax = \cot bx$.

Solution:

Here $\tan ax = \cot bx = \tan\left(\frac{1}{2}\pi - bx\right)$

Hence $ax = n\pi + \frac{1}{2}\pi - bx$

or $x = \frac{2n+1}{a+b} \cdot \frac{\pi}{2}$.

Example 8

Solve $\cos 3x + \cos x = \cos 2x$

Solution:

$\cos 3x + \cos x = \cos 2x$

or, $2 \cos 2x \cos x - \cos 2x = 0$

or, $\cos 2x (2 \cos x - 1) = 0$

Either $\cos 2x = 0$ or, $2 \cos x - 1 = 0$

$2x = (2n + 1)\frac{\pi}{2}$ $\cos x = \frac{1}{2}$

$x = (2n + 1)\frac{\pi}{4}$ $x = 2n\pi \pm \frac{\pi}{3}$

$\therefore$ $x = (2n + 1)\frac{\pi}{4}, 2n\pi \pm \frac{\pi}{3}$

Example 9

Solve $\sin^2\theta - 2 \cos \theta + \frac{1}{4} = 0$

Solution:

$\sin^2\theta - 2 \cos \theta + \frac{1}{4} = 0$

or, $4 \sin^2\theta - 8 \cos \theta + 1 = 0$

or, $4 - 4 \cos^2\theta - 8 \cos \theta + 1 = 0$

or, $-4 \cos^2\theta - 8 \cos \theta + 5 = 0$

or, $4 \cos^2\theta + 8\cos \theta - 5 = 0$

or, $(2 \cos \theta + 5) (2 \cos \theta - 1) = 0$

Either $2 \cos \theta + 5 = 0$

$\cos \theta = -\frac{5}{2}$

Since $\cos \theta \nless -1$, so

$\cos \theta = -\frac{5}{2}$ is not possible.

or, $2 \cos \theta - 1 = 0$

$\cos \theta = \frac{1}{2}$

or, $\cos\theta = \cos\frac{\pi}{3}$

$\therefore$ $\theta = 2n\pi \pm \frac{\pi}{3}$

Example 10

Solve $2\tan 3x\cos 2x + 1 = \tan 3x + 2\cos 2x$

Solution:

$2\tan 3x\cos 2x + 1 = \tan 3x + 2\cos 2x$

or, $2\tan 3x\cos 2x + 1 - \tan 3x - 2\cos 2x = 0$

or, $\tan 3x(2\cos 2x - 1) - 1(2\cos 2x - 1) = 0$

or, $(2\cos 2x - 1)(\tan 3x - 1) = 0$

Either $2\cos 2x - 1 = 0$

$\cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$

$2x = 2n\pi \pm \frac{\pi}{3}$

$\therefore$ $x = n\pi \pm \frac{\pi}{6}$

or, $\tan 3x - 1 = 0$

$\tan 3x = 1 = \tan\frac{\pi}{4}$

$3x = n\pi + \frac{\pi}{4} = (4n + 1)\frac{\pi}{4}$

$\therefore$ $x = (4n + 1)\frac{\pi}{12}$

$\therefore x = n\pi \pm \frac{\pi}{6}, (4n + 1)\frac{\pi}{12}$

Example 11

Solve $\sin\theta - \sqrt{3}\cos\theta = 2\ (-2\pi < \theta < 2\pi)$

Solution:

$\sin\theta - \sqrt{3}\cos\theta = 2$

Dividing each term by 2,

$\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = 1$

or, $\sin\theta\cos\frac{\pi}{3} - \cos\theta\sin\frac{\pi}{3} = 1$

or, $\sin\left(\theta - \frac{\pi}{3}\right) = 1$

$\therefore \quad \theta - \frac{\pi}{3} = (4n + 1)\frac{\pi}{2}$

$\therefore \quad \theta = (4n + 1)\frac{\pi}{2} + \frac{\pi}{3}$

When n = 0, $\theta = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

n = 1, $\theta = \frac{5\pi}{2} + \frac{\pi}{3} > 2\pi$

n = –1, $\theta = -\frac{3\pi}{2} + \frac{\pi}{3} = -\frac{7\pi}{6}$

n = –2, $\theta = -\frac{7\pi}{2} + \frac{\pi}{3} = -\frac{19\pi}{6} < -2\pi$

$\therefore$ the values of θ lying between -2π and 2π are $-\frac{7\pi}{6}, \frac{5\pi}{6}$.

Example 12

Solve $\tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 4$

Solution:

$\tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 4$

or, $\frac{\tan\theta + \tan \pi/3}{1 - \tan\theta \tan \pi/3} + \frac{\tan\theta + \tan 2\pi/3}{1 - \tan\theta \tan 2\pi/3} = 4$

or, $\frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} = 4$

or, $\frac{(\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) + (\tan\theta - \sqrt{3})(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta} = 4$

or, $\frac{8\tan\theta}{1 - 3\tan^2\theta} = 4$

or, $2\tan\theta = 1 - 3\tan^2\theta$

or, $3\tan^2\theta + 2\tan\theta - 1 = 0$

or, $(\tan\theta + 1)(3\tan\theta - 1) = 0$

Either $\tan\theta + 1 = 0$

$\tan\theta = -1 = \tan\left(-\frac{\pi}{4}\right)$

$\therefore \quad \theta = n\pi - \frac{\pi}{4}$

or, $3\tan\theta - 1 = 0$

$\tan\theta = \frac{1}{3}$

$\tan\theta = \tan\alpha$ where $\tan\alpha = \frac{1}{3}$

$\therefore\quad \theta = n\pi + \alpha$

$\therefore\quad \theta = n\pi - \frac{\pi}{4},\ n\pi + \alpha$ where $\alpha = \tan^{-1}\frac{1}{3}$

EXERCISE

Solve the following equations (Exs. 1 to 10):

1. (a) $4\cos^2 x = 1$ (b) $\cos 2x - \sin x = 0$
   (c) $2\sin 2x = \sin x$ (d) $\sin 2x + \sin x = 0$
2. (a) $\sin^2 x - \cos x = 1$ (b) $\cos^2 x - \sin x + 5 = 0$
   (c) $4\cos x + \sec x - 4 = 0$ (d) $2\sin x + \cot x - \operatorname{cosec} x = 0$.
3. (a) $2\cos^2 x + 4\sin^2 x = 3$ (b) $7\sin^2 x + 3\cos^2 x = 4$.
   (c) $\tan x + \cot x = 2\operatorname{cosec} x$ (d) $\tan^2 x = \sec x + 1$
4. (a) $\sin x + \sqrt{3}\cos x = \sqrt{2}$
   (b) $\sqrt{3}\sin x - \cos x = \sqrt{2}\ (0 \le x \le \pi)$
   (c) $\cos x + \sqrt{3}\sin x = \sqrt{2}$
   (d) $\sin x + \cos x = \sqrt{2}\ (-2\pi \le x \le 2\pi)$
5. (a) $\sin 3x + \sin x = 0$ (b) $\sin 3x + \sin x = \sin 2x$
   (c) $\cos 3x - \cos x = 0$ (d) $\tan 2x + \tan x = 0\ (-\pi/2 \le x \le \pi/2)$
6. (a) $2\cos^2 x + \sin x\cos x - \sin^2 x = 0$
   (b) $\sin 2x - 4\sin x - \cos x + 2 = 0$
7. (a) $\sin 9\theta = \sin\theta$ (b) $\tan 5\theta = \cot 2\theta$
   (c) $\tan 2x - \cot x = 0$ (d) $\tan m\theta + \cot n\theta = 0$
8. (a) $\cos\theta + \cos 2\theta + \cos 3\theta = 0$
   (b) $\cos\theta - \sin 3\theta = \cos 2\theta$
   (c) $\tan\theta + \tan 2\theta = \tan 3\theta$
   (d) $2\sin x\sin 3x = 1$
9. (a) $\cot^2 x - \operatorname{cosec} x - 1 = 0$ (b) $2\cos x + 1 = \sin x$
   (c) $\sec x\tan x = \sqrt{2}$ (d) $4\sin^4 x - \cos^2 2x = 0$
10. (a) $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$
    (b) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$.

11. Find all the solutions of $\tan\theta - 3\cot\theta = 2\tan 3\theta$ that lie between $0°$ and $360°$.

12. Find the solution of the equations (general solution not required)

$$\tan x + \tan y = 2$$
$$2\cos x \cos y = 1$$

13. (i) $\tan\theta + \tan 2\theta + \tan 3\theta = \tan\theta \tan 2\theta \tan 3\theta$

    (ii) $\tan\theta + \tan 2\theta + \tan\theta.\tan 2\theta = 1$

    (iii) $\tan\left(\frac{\pi}{4}+\theta\right) + \tan\left(\frac{\pi}{4}-\theta\right) = 4$

14. (i) $\sin 2x.\tan x + 1 = \sin 2x + \tan x$

    (ii) $2\sin x.\tan x + 1 = \tan x + 2\sin x$

15. (i) $\cos x + \sin x = \cos 2x + \sin 2x$

    (ii) $\cos x - \sin x = \cos\alpha + \sin\alpha$

A. Find the general values of x satisfying the following pair of equations:

a) $\sin x = \frac{1}{2}$, $\cos x = -\frac{\sqrt{3}}{2}$ b) $\sin x = -\frac{1}{\sqrt{2}}$, $\tan x = 1$

c) $\sin x = -\frac{\sqrt{3}}{2}$, $\cos x = \frac{1}{2}$ d) $\operatorname{cosec} x = -\frac{2}{\sqrt{3}}$, $\tan x = -\sqrt{3}$

Answers

1. (a) $n\pi \pm \frac{\pi}{3}$ (b) $(4n+1)\frac{\pi}{6}, (4n-1)\frac{\pi}{2}$ (c) $n\pi, 2n\pi \pm (\cos^{-1}\frac{1}{4})$(d) $n\pi, (6n\pm 2)\frac{\pi}{3}$
2. (a) $(4n\pm 1)\frac{\pi}{2}, (2n\pm 1)\pi$ (b) No solution (c) $(6n\pm 1)\frac{\pi}{3}$ (d) $(6n\pm 2)\frac{\pi}{3}$
3. (a) $n\pi \pm \frac{\pi}{4}$ (b) $n\pi \pm \frac{\pi}{6}$ (c) $2n\pi \pm \frac{\pi}{3}$ (d) $(2n+1)\pi, 2n\pi \pm \frac{\pi}{3}$
4. (a) $2n\pi + \frac{\pi}{6} \pm \frac{\pi}{4}$ (b) $\frac{5\pi}{12}, \frac{11\pi}{12}$ (c) $2n\pi + \frac{\pi}{3} \pm \frac{\pi}{4}$ (d) $-\frac{7\pi}{4}, \frac{\pi}{4}$
5. (a) $n\pi, (2n\pm 1)\frac{\pi}{2}$ $\left[\text{or } \frac{n\pi}{2}, (4n\pm 1)\frac{\pi}{2}\right]$ (b) $\frac{n\pi}{2}, (6n\pm 1)\frac{\pi}{3}$ (c) $n\pi, \frac{n\pi}{2}$
   (d) $-\frac{\pi}{3}, 0, \frac{\pi}{3}$
6. (a) $n\pi - \frac{\pi}{4}, n\pi + \theta$ where $\theta = \tan^{-1} 2$. (b) $n\pi + (-1)^n\frac{\pi}{6}$
7. (a) $\frac{n\pi}{4}, (4n\pm 1)\frac{\pi}{10}$ (b) $(4n\pm 1)\frac{\pi}{14}$ or $(2n+1)\frac{\pi}{14}$
   (c) $(4n\pm 1)\frac{\pi}{6}$ or $(2n+1)\frac{\pi}{6}$ (d) $(4n\pm 1)\frac{\pi}{2(m-n)}$ or $\frac{(2n+1)\pi}{2(m-n)}$
8. (a) $(4n\pm 1)\frac{\pi}{4}, (6n\pm 2)\frac{\pi}{3}$ (b) $\frac{2n\pi}{3}, (4n+1)\frac{\pi}{4}, (4n-1)\frac{\pi}{2}$
   (c) $n\pi, \frac{n\pi}{2}, \frac{n\pi}{3}$ (d) $(4n\pm 1)\frac{\pi}{4}, (6n\pm 1)\frac{\pi}{6}$
9. (a) $n\pi + (-1)^n\frac{\pi}{6}, (4n-1)\frac{\pi}{2}$ (b) $(4n+1)\frac{\pi}{2}, 2n\pi + \theta, \theta = \cos^{-1}\left(-\frac{4}{5}\right)$

(c) $n\pi + (-1)^n . \frac{\pi}{4}$ (d) $(6n \pm 1)\frac{\pi}{6}$

10. (a) $(4n \pm 1)\frac{\pi}{2}, (4n \pm 1)\frac{\pi}{4}, (4n \pm 1)\frac{\pi}{8}$ (b) $\frac{n\pi}{3}, n\pi \pm \theta, \theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

11. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

12. $\frac{\pi}{4}, \frac{\pi}{4}$

13. (i) $\frac{n\pi}{3}$ (ii) $(4n + 1)\frac{\pi}{12}$ (iii) $n\pi \pm \frac{\pi}{6}$

14. (i) $n\pi + \frac{\pi}{4}$ (ii) $n\pi + \frac{\pi}{4}, n\pi + (-1)^n\frac{\pi}{6}$

15. (i) $2n\pi, (4n + 1)\frac{\pi}{6}$ (ii) $2n\pi - \alpha, 2n\pi - \frac{\pi}{2} + \alpha$

Additional Questions

1. What does the trigonometrical equation mean ? What is its solution ? What does the general value of the trigonometrical equation mean ?
2. Solve the following equations:
   (i) $\sin 2x + \sin 4x + \sin 6x = 0$
   (ii) $2\sin^2 x + \sin^2 2x = 2$
   (iii) $2\sin^2 x + 3\cos x = 0 \ (0 < x < 2\pi)$
   (iv) $\sin^2\theta - \cos\theta = 1/4 \ (0 \le \theta \le 2\pi)$
   (v) $2\tan\theta - \cot\theta = -1$
   (vi) $\tan^2\theta + (1 - \sqrt{3})\tan\theta - \sqrt{3} = 0$
3. Solve the following equations:
   (i) $\sin\theta + \cos\theta = \frac{1}{\sqrt{2}} \ (0 \le \theta \le 2\pi)$
   (ii) $\cos\theta - \sin\theta = \frac{1}{\sqrt{2}} \ (-2\pi \le \theta \le 2\pi)$
   (iii) $\tan\theta + \sec\theta = \sqrt{3}$
   (iv) $\sqrt{2}\sec\theta + \tan\theta = 1$
   (v) $\sqrt{3}\cos\theta + \sin\theta = 1 \ (-2\pi < \theta < 2\pi)$
   (vi) $\cos\theta + \sqrt{3}\sin\theta = 2 \ (-2\pi \le \theta \le 2\pi)$

Answers

2. (i) $\frac{1}{4}n\pi, n\pi \pm \pi/3$ (ii) $n\pi + \pi/2, n\pi \pm \pi/4$ (iii) $\frac{2\pi}{3}, \frac{4\pi}{3}$
   (iv) $\frac{\pi}{3}, \frac{5\pi}{3}$ (v) $n\pi - \frac{\pi}{4}, n\pi + \alpha, \alpha = \tan^{-1}\frac{1}{2}$ (vi) $n\pi + \frac{\pi}{3}, n\pi - \frac{\pi}{4}$
3. (i) $\frac{7\pi}{12}, \frac{23\pi}{12}$ (ii) $-\frac{23\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{17\pi}{12}$ (iii) $2n\pi \pm \frac{\pi}{3} - \frac{\pi}{6}$
   (iv) $2n\pi - \frac{\pi}{4}$ (v) $-\frac{3\pi}{2}, -\frac{\pi}{6}, \frac{\pi}{2}, \frac{11\pi}{6}$ (vi) $-\frac{5\pi}{3}, \frac{\pi}{3}$

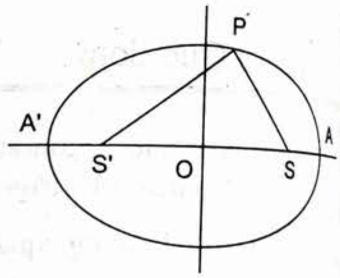
●

Chapter 8
Conic Section

Ellipse

An *ellipse* is the locus of a point in a plane such that the sum of the distances of the point from two fixed points is constant.

The two fixed points S and S′ are called *foci* (*singular focus*) and the point midway between them is the centre of the ellipse. We call the line through the two foci the *major axis* of the ellipse; the line through the centre and perpendicular to the major axis is its *minor axis*. The intersection of the ellipse with the major axis determines the two points A and A′ which are called *vertices*.



Alternative definition of an ellipse

An *ellipse* is the locus of the point in a plane such that its distance from a fixed point (called the *focus*) bears a constant ratio (called *eccenticity*) to its distance from a fixed straight line (called the *directrix*). e, the eccentricity is any number between 0 and 1.

a) The Standard Equation to an Ellipse

Place the ellipse in the rectangular coordinate plane with the centre at the origin and the major axis along the x–axis. We call this the ellipse in the *standard position*. Let the two foci S and S′ be located at $(c, 0)$ and $(-c, 0)$ where c is a positive constant.

If $P(x, y)$ be any point on the ellipse, we have

$$PS + PS' = \text{constant, say } 2a.$$

Obviously $2a > 2c$.

Using distance formula,

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

or, $(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$

or $4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$

or $(cx - a^2)^2 = a^2[(x-c)^2 + y^2]$

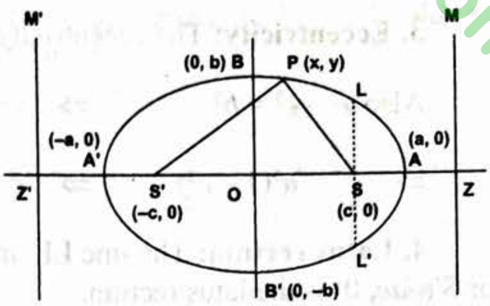
or $c^2x^2 - 2a^2cx + a^4 = a^2\,[x^2 - 2cx + c^2 + y^2]$

or $a^2(a^2 - c^2) = (a^2 - c^2)\,x^2 + a^2y^2$

or $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 - c^2} = 1$

$a^2 - c^2$ being a positive constant we may let $b^2 = a^2 - c^2$, so that the equation to the ellipse is,

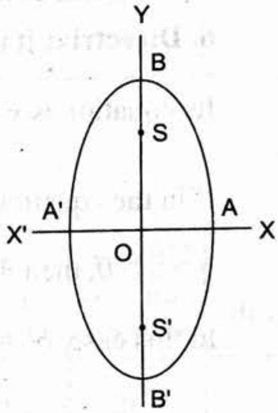
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



The centre O of the ellipse is taken as the origin and the line joining S and S′ known as the axis is taken as the x-axis.

The vertices of the ellipse are A(a, 0) and A′($-a$, 0). The distance between them, $2a$, is called the *length* of the major axis. The ellipse intersects the y-axis at B(0, b) and B′(0, $-b$). The distance between them, $2b$, is called the *length* of the minor axis. Note that $a^2 = b^2 + c^2$.

The eccentricity e, of the ellipse is defined as



$$e = \frac{c}{a}$$

Also, $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}}$

[$\because$ AS = e.AZ and A′S = e.A′Z

Then, A′S – AS = e(A′Z – AZ)

$\Rightarrow$ (A′O + OS) – (OA – OS) = eAA′

$\Rightarrow$ 2.OS = e.2a

$\therefore$ OS = ae i.e. $c = ae$

Also, AS + A′S = e(AZ + A′Z) $\Rightarrow$ $2a = e.2$ OZ

$\therefore$ OZ = $\frac{a}{e}$]

If $a > b$, the major axis is along the x-axis.

Since $c < a$, the eccentricity is always between 0 and 1. The coordinates of the foci are ($\pm ae$, 0).

Some of the major terms of the ellipse

1. Major axis: The distance between the vertices A(a, 0) and A′($-a$, 0) is known as the major axis. Length of the major axis is AA′ = $2a$.

2. Minor axis: The ellipse cuts the y-axis at the points B and B′ where x = 0.

So, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{y^2}{b^2} = 1 \quad (\because x = 0)$

$\therefore \quad y = \pm b$

$\therefore$ the coordinates of B and B′ are (0, b) and (0, $-b$) respectively.

The length of BB′ is known as the minor axis.

Length of the minor axis = $2b$

3. Eccentricity: The eccentricity of the ellipse $e = \frac{c}{a}$

Also $a^2 - c^2 = b^2$ $\Rightarrow$ $a^2 - a^2e^2 = b^2$

$\Rightarrow$ $b^2 = a^2(1 - e^2)$ $\Rightarrow$ $e = \sqrt{1 - \frac{b^2}{a^2}}$

4. Latus rectum: The line LL′ intercepted by the ellipse passing through the focus S(ae, 0) or S′(–ae, 0) is the latus rectum.

Length of the latus rectum = $LL' = \frac{2b^2}{a}$ $\left(\because \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } x = ae \Rightarrow y = \frac{b^2}{a}\right)$

5. Axis: The line joining the two foci of the ellipse is known as the axis of the ellipse. In above case x-axis is the axis of the ellipse.

6. Directrix: It is the fixed line perpendicular to the axis of the ellipse.

Its equation is $x = \pm \frac{a}{e}$

If in the equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$b > a > 0$, then the major axis will be along the y-axis and minor axis along the x-axis.

In this case, $b^2 = a^2 + c^2$ and ecentricity $(e) = \frac{c}{b}$

So, $e = \frac{c}{b} = \frac{\sqrt{b^2 - a^2}}{b} = \sqrt{1 - \frac{a^2}{b^2}}$

Major axis: $2b$ Minor axis: $2a$

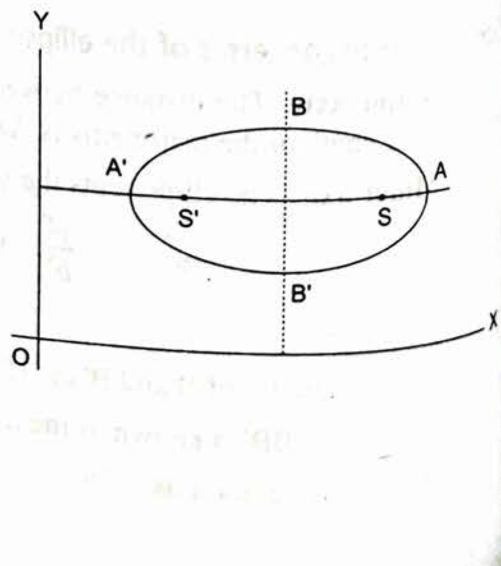
Vertices: B(0, b) and B′(0, $-b$) Foci: S(0, be) and S′(0, $-be$)

Length of latus rectum = $\frac{2a^2}{b}$ Directrix: Equation of the directrix are $y = \pm \frac{b}{e}$

b) Centre not at the origin

The equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b > 0$

also represents an ellipse of the same size and shape as the one represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ but its centre will be shifted to the point (h, k), the major and minor axes will be parallel to the x-axis and y-axis respectively. This will be clear if we recapitulate that $x^2 + y^2 = r^2$ and $(x - h)^2 + (y - k)^2 = r^2$ represent circles of the same size with the centre at (0, 0) in the first and at (h, k) in the second.



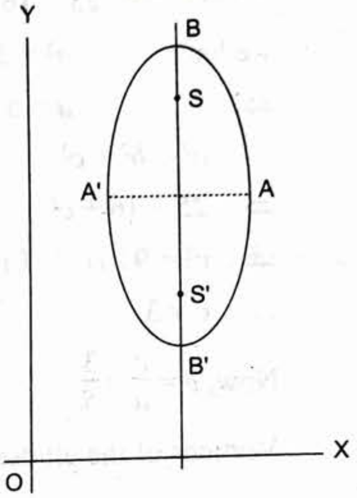
The equation $y^2 = 4ax$ represents a parabola with vertex at the origin, but $(y - k)^2 = 4a(x - h)$ represents the same sized parabola with the vertex shifted to (h, k).

The equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ changes to

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

if we let $x - h = X$ and $y - k = Y$.

In the above equation of an ellipse, if $b > a > 0$, then the major axis and minor axis will be respectively parallel to y-axis and x-axis.



For easy memory the important results of an ellipse are tabulated below:

Eq. of an ellipse	Centre	Vertex	Focus	Major axis	Minor axis	Eccentricities	Length of Latus rectum	Eq. of directrix
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b > 0$	(0, 0)	$(\pm a, 0)$	$(\pm ae, 0)$	$2a$	$2b$	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$\frac{2b^2}{a}$	$x = \pm\frac{a}{e}$
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $b > a > 0$	(0, 0)	$(0, \pm b)$	$(0, \pm be)$	$2b$	$2a$	$e = \sqrt{1 - \frac{a^2}{b^2}}$	$\frac{2a^2}{b}$	$y = \pm\frac{b}{e}$
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $a > b > 0$	(h, k)	$(h \pm a, k)$	$(h \pm ae, k)$	$2a$	$2b$	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$\frac{2b^2}{a}$	$x = h \pm \frac{a}{e}$
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, $b > a > 0$	(h, k)	$(h, k \pm b)$	$(h, k \pm be)$	$2b$	$2a$	$e = \sqrt{1 - \frac{a^2}{b^2}}$	$\frac{2a^2}{b}$	$y = k \pm \frac{b}{e}$

Worked Out Examples

Example 1

Find the eccentricity, coordinates of the vertices and foci and also the lengths of the major axis, the minor axis and the latus rectum of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

Solution:

Comparing $\frac{x^2}{25}+\frac{y^2}{16}=1$ with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

we have, $a^2 = 25$ and $b^2 = 16$

$\Rightarrow$ $a = 5$ and $b = 4$

$a^2 = b^2 + c^2$

$\Rightarrow$ $25 = 16 + c^2$

$\Rightarrow$ $c^2 = 9$

$\therefore$ $c = 3$

Now, $e = \frac{c}{a} = \frac{3}{5}$

Vertices of the ellipse = $(\pm a, 0) = (\pm 5, 0)$

Foci of the ellipse = $(\pm ae, 0) = (\pm 5 \times \frac{3}{5}, 0) = (\pm 3, 0)$

Length of the major axis = $2a = 2 \times 5 = 10$

Length of the minor axis = $2b = 2 \times 4 = 8$

Length of latus rectum = $\frac{2b^2}{a} = 2 \times \frac{16}{5} = \frac{32}{5}$

Example 2

Find the coordinates of the vertices, eccentricity, coordinates of the foci, length of the major axis and the length of the minor axis and the latus rectum of the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$

Solution:

Comparing the given equation with the standard form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

We have $a^2 = 9$ and $b^2 = 25$

$\Rightarrow$ $a = 3$ and $b = 5$

Since $b > a$, so the major axis is along the y-axis.

$b^2 = a^2 + c^2$

$\Rightarrow$ $25 = 9 + c^2$

$\Rightarrow$ $c^2 = 16$

$\therefore$ $c = 4$

Now, $e = \frac{c}{b} = \frac{4}{5}$

Vertices of the ellipse = $(0, \pm b) = (0, \pm 5)$

Foci of the ellipse = $(0, \pm be) = (0, \pm 5 \times \frac{4}{5}) = (0, \pm 4)$

Length of the major axis = $2b = 2 \times 5 = 10$

Length of the minor axis = $2a = 2 \times 3 = 6$

Length of latus rectum = $\frac{2a^2}{b} = \frac{2 \times 9}{5} = \frac{18}{5}$

Example 3

Find the equation to the locus of the point whose distance from (3, 0) is 1/3 of its distance from the line $x = 27$.

Solution :

Let $P(x, y)$ be the point.

so $\sqrt{(x-3)^2 + y^2} = \frac{1}{3}(27 - x)$

or $x^2 - 6x + 9 + y^2 = 81 - 6x + \frac{x^2}{9}$

or $8x^2 + 9y^2 = 648$

or $\frac{x^2}{(9)^2} + \frac{y^2}{(6\sqrt{2})^2} = 1$ which is an ellipse.

Example 4

Find the eccentricity and the coordinates of the vertices and the foci of the ellipse

$$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{4} = 1$$

Solution :

Comparing the given equation with the standard form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

We have $h = 1, k = 2, a^2 = 16$ and $b^2 = 4$

so that $a = 4$ and $b = 2$.

Since $a > b$, so the major axis is parallel to the x-axis.

$a^2 = b^2 + c^2$

$\Rightarrow$ $16 = 4 + c^2$

$\Rightarrow$ $c^2 = 12$

$\therefore$ $c = 2\sqrt{3}$

Now $e = \frac{c}{a} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

Coordinates of the vertices of the ellipse = $(h \pm a, k)$

$= (1 \pm 4, 2) = (5, 2)$ and $(-3, 2)$

Coordinates of the foci of the ellipse = $(h\pm ae, k)$

$$= (1\pm 4\times\frac{\sqrt{3}}{2}, 2) = (1\pm 2\sqrt{3}, 2)$$

Example 5

Show that $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ represents the equation of an ellipse. Find its centre, vertex, focus, eccentricity and the equation of directrix of ellipse.

Solution:

$$9x^2 + 4y^2 - 18x - 16y - 11 = 0$$

or, $9(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 36$

or, $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$

Comparing this equation with

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

we have, $h = 1$, $k = 2$, $a^2 = 4$ and $b^2 = 9$

Since $b > a$, so the major axis is parallel to the y-axis.

$$e^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{4}{9} \qquad \therefore e = \frac{\sqrt{5}}{3}$$

Centre = $(h, k) = (1, 2)$

Vertex = $(h, k \pm b) = (1, 2 \pm 3) = (1, 5)$ and $(1, -1)$

Focus = $(h, k \pm be) = (1, 2 \pm 3.\frac{\sqrt{5}}{3}) = (1, 2 \pm \sqrt{5})$

Equation of the directrix is

$$y = k \pm \frac{b}{e} \quad \Rightarrow \quad y = 2 \pm \frac{3}{\frac{\sqrt{5}}{3}}$$

$$\Rightarrow \quad y = 2 \pm \frac{9}{\sqrt{5}}$$

Example 6

Find the equation of the ellipse in standard form with its length of the major axis = 8 and eccentricity = $\frac{3}{4}$.

Solution:

Here $2a = 8$ $\qquad \therefore a = 4$

Again $\quad e = \frac{3}{4}$

Also, $\quad e = 1 - \dfrac{b^2}{a^2}$

$\Rightarrow \dfrac{9}{16} = 1 - \dfrac{b^2}{16}$

$\Rightarrow \dfrac{b^2}{16} = 1 - \dfrac{9}{16}$

$\Rightarrow \dfrac{b^2}{16} = \dfrac{7}{16} \qquad \therefore b^2 = 7$

Now, the required equation of the ellipse in the standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$

$\Rightarrow 7x^2 + 16y^2 = 112$

Example 7

Find the equation of the ellipse whose foci are $(\pm 2, 0)$ and the length of the latus rectum is 6.

Solution:

By given, $ae = 2 \quad \ldots\ldots\text{(i)}$

and $\quad \dfrac{2b^2}{a} = 6 \Rightarrow b^2 = 3a \quad \ldots\ldots\text{(ii)}$

Again, $a^2 = b^2 + c^2$

$\Rightarrow a^2 = 3a + 4 \qquad (\because c = ae)$

$\Rightarrow a^2 - 3a - 4 = 0$

$\Rightarrow (a - 4)(a + 1) = 0$

$\therefore a = 4, -1$

The possible value of $a = 4$

$b^2 = 3a = 3 \times 4 = 12$

Now, the equation of the parabola is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or, $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

EXERCISE

1. Find the coordinates of the vertices, the eccentricity, the coordinates of the foci, the lengths of the major and minor axes of the following ellipses
   i) $\frac{x^2}{16}+\frac{y^2}{4}=1$ ii) $\frac{x^2}{10}+\frac{y^2}{5}=1$ iii) $\frac{x^2}{9}+\frac{y^2}{16}=1$
   iv) $25x^2+4y^2=100$ v) $3x^2+4y^2=36$
2. Deduce the equation of the ellipse in the standard position with the following data
   a) A focus at (–2, 0) and a vertex at (5, 0).
   b) A vertex at (±9, 0) and eccentricity 2/3.
   c) A focus at (0, –5) and eccentricity 1/3.
   d) A focus at (0, 3) and length of minor axis 8.
   e) A vertex at (0, 8) and passing through (3, 32/5).
   f) Passing through the points (1, 4) and (–3, 2).
3. Find the eccentricity, the coordinates of the vertices, centre and the foci of the following ellipses.
   a) $\frac{(x+2)^2}{16}+\frac{(y-5)^2}{9}=1$ b) $\frac{(x+6)^2}{4}+\frac{y^2}{36}=1$
   c) $\frac{x^2}{8}+\frac{(y-2)^2}{12}=1$ d) $9x^2+5y^2-30y=0$
   e) $x^2+4y^2-4x+24y+24=0$ f) $x^2+2(y+1)^2=8$
4. Find the equation of the ellipse whose
   a) major axis is twice its minor axis and which passes through the point (0, 1).
   b) latus rectum is equal to half its major axis and which passes through the point $(\sqrt{6}, 1)$.
   c) latus rectum is 3 and eccentricity is $\frac{1}{\sqrt{2}}$.
   d) distance between the two foci is 8 and the semi-latus rectum is 6
   e) latus rectum is half the major axis and focus is at (3, 0)

A. In an ellipse
   a) S and S′ are the foci
   b) A and A′ are the vertices
   c) ZM and Z′M′ are the directrices
   d) P is any point on the ellipse such that PS + P′S = 12 and AZ = 3

   Draw a rough sketch of the ellipse and then find the vertices, centre, eccentricity, foci and the equation of the directrices if O, the centre of the ellipse be taken as the origin and AA′ as the x-axis.

Answers

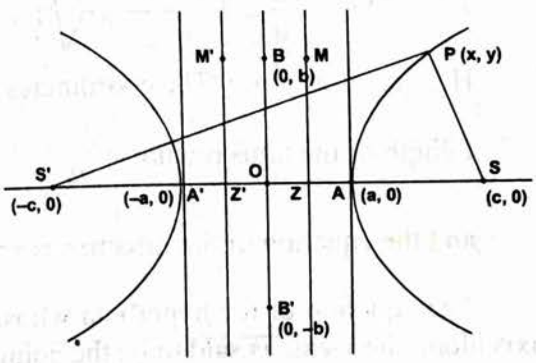
1. i) $(\pm 4, 0), \frac{\sqrt{3}}{2}, (\pm 2\sqrt{3}, 0), 8, 4$ ii) $(\pm\sqrt{10}, 0), \frac{\sqrt{2}}{2}, (\pm\sqrt{5}, 0), 2\sqrt{10}, 2\sqrt{5}$
   iii) $(0, \pm 4), \frac{\sqrt{7}}{4}, (0, \pm\sqrt{7}), 8, 6$ iv) $(0, \pm 5), \frac{\sqrt{21}}{5}, (0, \pm\sqrt{21}), 10, 4$
   v) $(\pm 2\sqrt{3}, 0), \frac{1}{2}, (\pm\sqrt{3}, 0), 4\sqrt{3}, 6$
2. a) $\frac{x^2}{25} + \frac{y^2}{21} = 1$ b) $5x^2 + 9y^2 = 405$ c) $225x^2 + 200y^2 = 45000$
   d) $\frac{x^2}{16} + \frac{y^2}{25} = 1$ e) $\frac{x^2}{25} + \frac{y^2}{64} = 1$ f) $3x^2 + 2y^2 = 35$
3. a) $\frac{\sqrt{7}}{4}$, (–6, 5) and (2, 5), (–2, 5), $(-2 \pm \sqrt{7}, 5)$ b) $\frac{2\sqrt{2}}{3}$, (–6, ±6), (–6, 0), $(-6, \pm 4\sqrt{2})$
   c) $\frac{\sqrt{3}}{3}$, $(0, 2\pm2\sqrt{3})$, (0, 2), (0, 0) & (0, 4) d) $\frac{2}{3}$, (0, 5) and (0, 1), (0,3); (0,5) and (0,1)
   e) $\frac{\sqrt{3}}{2}$, (6, –3) and (–2, –3), (2, –3), $(2 \pm 2\sqrt{3}, -3)$ f) $\frac{1}{\sqrt{2}}$, $(\pm 2\sqrt{2}, -1)$, (0 –1); (–2, –1) and (2, –1)
4. a) $x^2 + 4y^2 = 4$ b) $x^2 + 2y^2 = 8$ c) $x^2 + 2y^2 = 9$
   d) $3x^2 + 4y^2 = 192$ e) $x^2 + 2y^2 = 18$

Hyperbola

A *hyperbola* is the locus of a point in a plane such that the difference of the distances from two fixed points is constant.

Alternatively, a conic section with eccentricity greater than 1 is said to be a hyperbola.

a) The Standard Equation to a Hyperbola



Taking the two points, called foci, at S (*c*, 0) and S′ (–*c*, 0) and the constant equal to 2*a*. O, the middle point of SS′ is taken as the origin and the line joining S and S′ as the x-axis. If a point P(*x*, *y*) lies on the hyperbola then the difference of PS and PS′ is 2a. So,

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a, \text{ if } PS > PS'$$

or, $$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a, \text{ if } PS < PS'$$

Transposing and squaring

$$(x+c)^2 + y^2 = 4a^2 + (x-c)^2 + y^2 + 4a\sqrt{(x-c)^2 + y^2}$$

which simplifies to

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)$$

or, $$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1,$$

Since the difference of two sides of a triangle is less than the third,

$2a < 2c$

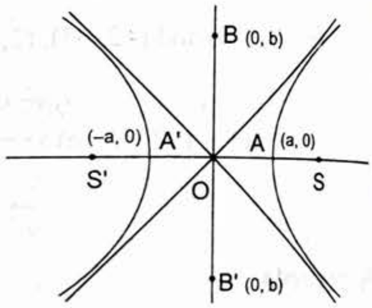
$\therefore$ $a^2 - c^2$ is negative

$\therefore$ the equation is $\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$

or, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

or, $b^2 = c^2 - a^2$.

This is the equation of hyperbola in the standard form. Its graph, as shown in the figure, consists of two branches that go to infinity. The two straight lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are asymptotes to the hyperbola. That is, the distance of a point on the hyperbola from the line tends to zero as the point moves to infinity along the curve.



The vertices of the hyperbola are A(a, 0) and A′ ($-a$, 0). The line AA′ joining the vertices A and A′ is called the transverse axis. Its length is $2a$.

$\therefore$ length of transverse axis = $2a$

The line BB′ through O perpendicular to the AA′ such that OB = OB′ = b is called the conjugate axis.

Length of the conjugate axis = $2b$

Here the hyperbola does not intersect the y-axis at B and B′. The eccentricity e, is defined as

$$e = \frac{c}{a} \qquad \text{where} \qquad c = \sqrt{a^2 + b^2}$$

so $$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \sqrt{1 + \frac{b^2}{a^2}}.$$

Here $e > 1$ as $c > a$. The coordinates of the foci are $(ae, 0)$ and $(-ae, 0)$

Length of the latus rectum = $\frac{2b^2}{a}$

and the equation of the directrix is $x = \pm\frac{a}{e}$

The equation of the hyperbola whose transverse axis is along the y-axis and the conjugate axis along the x-axis is said to be the cojugate hyperbola.

Its equation is, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

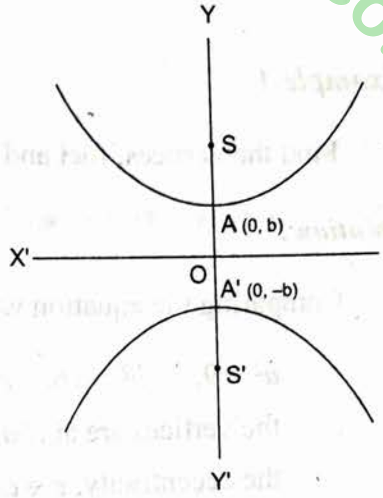
Its vertices are A (0, b) and A′ (0, $-b$). The eccentricity e is $\frac{c}{b}$. ($c > a$). The co-ordinates of the foci are (0, $\pm$ be)

If the foci of a hyperbola be at S (a, a) and S′$(-a, -a)$ and if P(x, y) be any point on the hyperbola, then

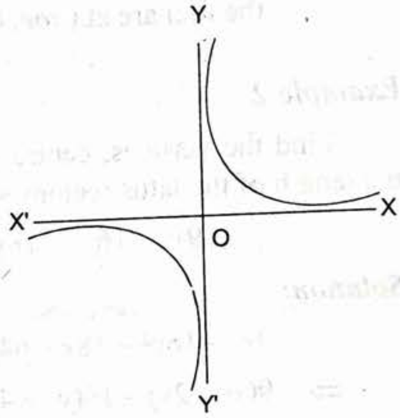
$$\sqrt{(x+a)^2+(y+a)^2}-\sqrt{(x-a)^2+(y-a)^2} = 2a$$

which reduces to $2xy = a^2$

or $\quad xy = c^2 \quad$ where $\quad c^2 = \dfrac{a^2}{2}$



Its asymptotes are the axes of coordinates and its graph is as shown in figure aside.



For easy memory, the important results of a hyperbola are tabulated below

Eq. of a hyperbola	Vertex	Focus	Transverse axis	Conjugate axis	Eccentricities	Eq. of directrix	Length of latus rectum
$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$	$(\pm a, 0)$	$(\pm ae, 0)$	$2a$	$2b$	$e=\sqrt{1+\dfrac{b^2}{a^2}}$	$x=\pm\dfrac{a}{e}$	$\dfrac{2b^2}{a}$
$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=-1$	$(0, \pm b)$	$(0, \pm be)$	$2b$	$2a$	$e=\sqrt{1+\dfrac{a^2}{b^2}}$	$y=\pm\dfrac{b}{e}$	$\dfrac{2a^2}{b}$
$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$	$(h\pm a, k)$	$(h\pm ae, k)$	$2a$	$2b$	$e=\sqrt{1+\dfrac{b^2}{a^2}}$	$x=h\pm\dfrac{a}{e}$	$\dfrac{2b^2}{a}$
$\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=-1$	$(h, k\pm b)$	$(h, k\pm be)$	$2b$	$2a$	$e=\sqrt{1+\dfrac{a^2}{b^2}}$	$x=k\pm\dfrac{b}{e}$	$\dfrac{2a^2}{b}$

Worked Out Examples

Example 1

Find the vertices, foci and eccentricity of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

Solution :

Comparing the equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

$a^2 = 9$, $b^2 = 16$, $c^2 = a^2 + b^2 = 25$ $\quad \therefore c = 5$

$\therefore$ the vertices are at $(\pm a, 0) = (\pm 3, 0)$

the eccentricity, $e = c/a = 5/3$

the foci are at $(\pm ae, 0) = (\pm 3 \times \frac{5}{3}, 0) = (\pm 5, 0)$.

Example 2

Find the vertices, centre, eccentricity, foci, lengths of transverse axis, conjugate axis and the length of the latus rectum of the hyperbola

$$9x^2 - 16y^2 - 18x - 64y - 199 = 0$$

Solution:

$9x^2 - 16y^2 - 18x - 64y - 199 = 0$

$\Rightarrow 9(x^2 - 2x) - 16(y^2 + 4y) = 199$

$\Rightarrow 9(x^2 - 2x + 1) - 16(y^2 + 4y + 4) = 199 - 55$

$\Rightarrow 9(x - 1)^2 - 16(y + 2)^2 = 144$

$\Rightarrow \frac{(x-1)^2}{16} - \frac{(y+2)^2}{9} = 1$

Comparing this equation with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

We have, $h = 1$, $k = -2$, $a^2 = 16$, $b^2 = 9$

$\Rightarrow a = 4, b = 3$

Coordinates of the vertices = $(h \pm a, k) = (1 \pm 4, -2)$ i.e. $(5, -2)$ and $(-3, -2)$

Coordinates of the centre = $\left(\frac{5-3}{2}, \frac{-2-2}{2}\right) = (1, -2)$

Eccentricity = $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$ $\quad \left(\because e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} = \sqrt{1 + \frac{b^2}{a^2}}\right)$

Coordinates of the foci = $(h \pm ae, k)$

$= (1 \pm 4 \times \frac{5}{4}, -2) = (1 \pm 5, -2)$ i.e. $(6, -2)$ and $(-4, -2)$

Length of transverse axis = $2a = 2 \times 4 = 8$

Length of the conjugate axis = $2b = 2 \times 3 = 6$

Length of the latus rectum = $\frac{2b^2}{a} = \frac{2 \times 9}{4} = \frac{9}{2}$

Example 3

Determine the equation of the hyperbola with a focus at (6, 0) and a vertex at (4, 0)

Solution :

Here $c = a\,e = 6$ and $a = 4$.

$\therefore \; e = 6/4 = 3/2 = 1.5$

$c^2 = a^2 + b^2$ gives $36 = 16 + b^2$,

$b^2 = 20$

so $b = 2\sqrt{5}$

The equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{20} = 1$

EXERCISE

1. Find the coordinates of the vertices, the eccentricity and the coordinates of the foci of the hyperbola.

   a) $\frac{x^2}{16} - \frac{y^2}{4} = 1$
   b) $\frac{x^2}{9} - \frac{y^2}{16} = -1$
   c) $3x^2 - 4y^2 = 36$
   d) $5x^2 - 20y^2 - 20x = 0$
   e) $9x^2 - 16y^2 + 36x + 32y - 124 = 0$
   f) $16x^2 - 9y^2 - 64x - 54y + 127 = 0$

2. Deduce the equation to the hyperbola in the Standard form

   a) with a focus at (–5, 0) and a vertex at (2, 0).
   b) with a focus at (0, 5) and a vertex at (0, –3).
   c) with a focus at (–7, 0) and eccentricity 7/4.
   d) with a vertex at (0, 8) and passing through (4, 8 √2).
   e) with length of the transverse axis = 8 and eccentricity = 2

A. In a hyperbola,

   a) S and S′ are the foci
   b) A and A′ are the vertices
   c) ZM and Z′M′ are the directrices
   d) P is any point on the hyperbola such that PS′ ~ PS = 12 and AZ = 2

Draw a rough sketch of the hyperbola and then find the vertices, centre, length of the transverse axis, length of the conjugate axis, eccentricity, foci, length of the latus rectum and the equation of the directrices supposing that O, the centre of the origin and AA′ as the x-axis.

Answers

1. a) $(\pm 4, 0), \frac{\sqrt{5}}{2}, (\pm 2\sqrt{5}, 0)$ b) $(0, \pm 4), \frac{5}{4}, (0, \pm 5)$

   c) $(\pm 2\sqrt{3}, 0), \frac{\sqrt{7}}{2}, (\pm\sqrt{21}, 0)$ d) $(4, 0), (0, 0); \frac{\sqrt{5}}{2}, (2 \pm \sqrt{5}, 0)$.

   e) $(-6, 1), (2, 1); \frac{5}{4}; (-7, 1), (3, 1)$ f) $(2, 1)$ and $(2, -7); \frac{5}{4}; (2, 2)$ and $(2, -8)$

2. a) $21x^2 - 4y^2 = 84$ b) $16y^2 - 9x^2 = 144$ c) $\frac{x^2}{16} - \frac{y^2}{33} = 1$

   d) $\frac{y^2}{64} - \frac{x^2}{16} = 1$ e) $3x^2 - y^2 = 48$

Chapter 9

Coordinates in Space

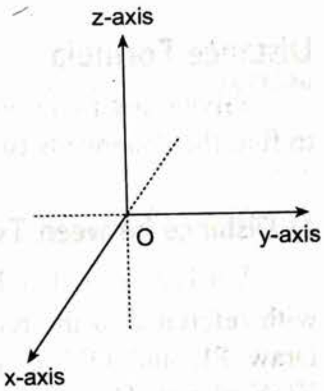
Introduction

Presence of material objects in relation to one another or their movement makes us feel about what is ordinarily known as **space**. In other words, the notion of space is closely associated with material objects in relative rest and/or motion. We often regard a *point* as an idealized model of a small material object (or particle). Small material objects or particles, when taken together, form bigger objects. Just the same way, we agree that points, when taken together, give rise to various forms or shapes. A space may therefore be visualized as the set of all points that give rise to various forms or shapes. A systematic study of such set of points or shapes can be done by associating each of such points with an ordered set of real numbers. Such numbers are said to be coordinates of a given point in the space under consideration.

Coordinates in Space

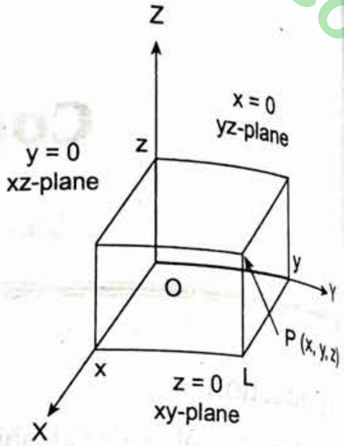
We began our study of coordinate geometry by establishing a one-to-one correspondence between the points on a line and the real numbers. This was followed by associating each point in a plane with an *ordered pair* of real numbers. Now we intend to explain how each point in the ordinary space (or three-dimensional space) can be associated with an *ordered triple* of real numbers.

To start with, we consider three mutually perpendicular lines intersecting at a fixed point. We call this point the **origin** and the three lines the **coordinate axes**. They are usually labeled as the x-axis, the y-axis and the z-axis. On each of the three lines we select appropriate scale and positive direction as indicated below:



The three lines, the origin and the directions form a system or coordinate system. In practice, we denote the origin by O and the x-axis by XOX′, the y-axis by YOY′ and the z-axis by ZOZ′. The direction of the positive x-axis, the direction of the positive y-axis and the direction of the positive z-axis are chosen along the *index finger*, the *middle finger* and the *thumb* respectively. In other words, we follow the **right-hand-thumb rule** in deciding the positive direction.

If we start from the origin O and move a certain distance (x units) along the x-axis, we arrive at a fixed point A on the x-axis. From A, if we move a certain distance (y units) along a line parallel to the y-axis, we arrive at a point L on the xy-plane. The point L is denoted by the ordered pair (x, y). From L, if we again move a distance (z units) along a line parallel to the z-axis (or perpendicular to the xy-plane), we arrive at a unique point P in the ordinary space or three-dimensional space. This point is denoted by the ordered triple (x, y, z). We may, however, proceed in the reversed order, we can start with a point and unique set of three numbers.



XOY is the plane formed with the help of OX and OY. In the plane, z-coordinate of every point will be zero. So, this plane is known as xy-plane or z = 0 plane. Similarly YOZ and ZOX are the planes known as yz-plane, zx-plane or x = 0, y = 0 planes respectively.

In a plane the coordinate axes divide the plane into four quadrants. In space the coordinate planes divide the whole space into eight parts called **octants**. The signs of the coordinates of any point can be found by applying the right-hand thumb rule.

The following table gives a summarized picture of the signs:

Octants / Coordinates	oxyz	ox'yz	oxy'z	oxyz'	ox'y'z	oxy'z'	ox'yz'	ox'y'z'
x	+	–	+	+	–	+	–	–
y	+	+	–	+	–	–	+	–
z	+	+	+	–	+	–	–	–

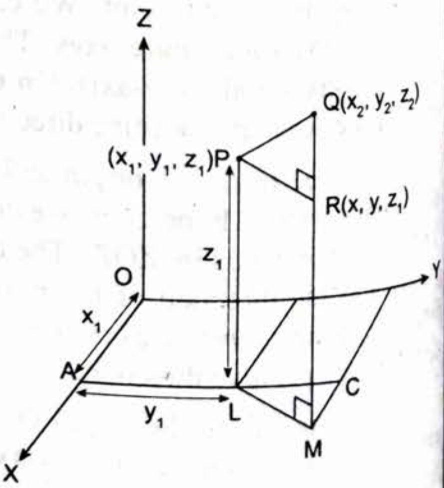
Distance Formula

Given any two points in space, it is, as in the case of plane coordinate geometry, possible to find the distance between them in terms of the coordinates of the points.

a) Distance between Two Points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points with reference to the rectangular axes OX, OY and OZ. Draw PL and QM perpendiculars to the xy-plane (or XOY-plane). Then, the coordinates of the points L and M in the xy-plane are respectively $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$. Clearly, the distance between two points L $(x_1, y_1, 0)$ and M $(x_2, y_2, 0)$ [i.e. L (x_1, y_1) and M (x_2, y_2)] is given by

$$LM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$



Now, from P draw PR perpendicular to MQ. Then PR is parallel and equal to LM. So, from the right angled triangle PRQ,

$$PQ^2 = PR^2 + RQ^2$$
$$= LM^2 + (MQ - MR)^2$$

$$= [(x_2 - x_1)^2 + (y_2 - y_1)^2] + (z_2 - z_1)^2.$$

Hence, $\quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Cor. If one point say $Q(x_2, y_2, z_2)$ be at the origin then

$$OP = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

Locus and Equations

Points in space may or may not be scattered or spread uniformly. They may or may not satisfy certain condition or conditions. Since points are associated with coordinates in coordinate geometry, conditions satisfied by given point or points can be considered as conditions placed on the coordinates of the points. These conditions can be expressed as equations or inequalities. *The set of those points and only those points that satisfy certain conditions is called a* ***locus***. The corresponding equations or inequalities are called the **equations** or **inequalities of the locus**. Moreover, each such equation or inequality defines a relation in such a way that there is a one-to-one correspondence between the ordered triple of numbers and the points of the locus. In short, *a locus is a graph of a relation.*

Section Formulae

Given some points of a locus, we often have to find the coordinates of certain point or points. One such case of frequent occurrence is the determination of the coordinates of a point which divides the join of two points in a given ratio. Obviously, there are two cases:

a) Internal division and b) External division.

The formulae giving the coordinates of such points of division are called the **section formulae.**

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points in the three-dimensional space. Suppose a point $R(x, y, z)$ divides the join of P and Q in the ratio $m : n$ internally in Fig. (a) and externally in Fig. (b).

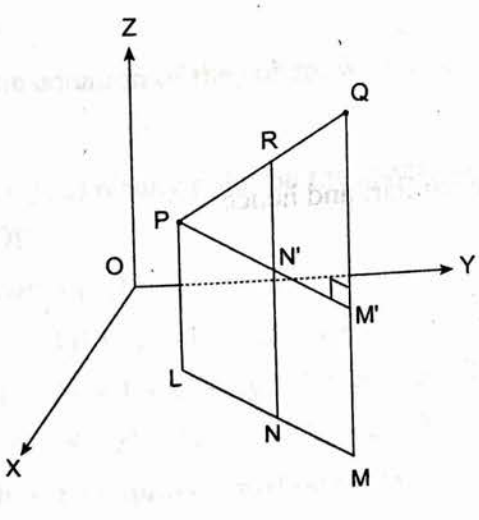


Fig. (a)

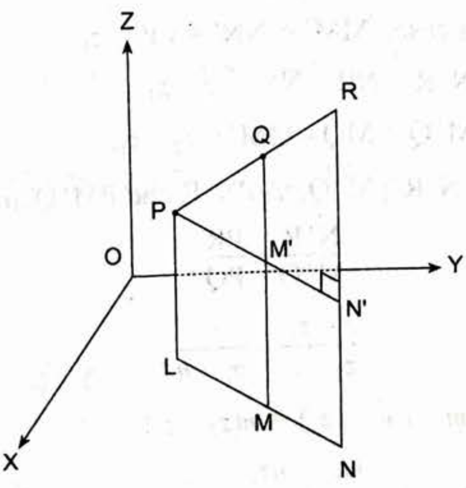


Fig. (b)

To find the coordinates of R in terms of the known coordinates of P and Q, we drop perpendiculars PL from P on the xy-plane or XOY-plane; and then draw parallel lines QM and RN cutting the xy-plane at M and N respectively. Clearly, L, N and M lie in the same plane and in the same line LM. Also QM and RN are perpendicular to the plane. A line PN′ M′ drawn parallel to LMN cuts RN and QM perpendicularly at N′ and M′ respectively.

a) Internal Division

In this case,

$MM' = NN' = LP = z_1,$

$N'R = NR - NN' = z - z_1$

and $M'Q = MQ - MM' = z_2 - z_1.$

Since N′R // M′Q, ˢΔ PN′R and PM′Q are similar; and hence

$$\frac{N'R}{M'Q} = \frac{PR}{PQ},$$

or, $\dfrac{z - z_1}{z_2 - z_1} = \dfrac{m}{m + n},$

or, $(m + n)(z - z_1) = m(z_2 - z_1).$

Hence, $z = \dfrac{mz_2 + nz_1}{m + n}.$

Proceeding similarly, we can show that

$$y = \frac{my_2 + ny_1}{m + n} \quad \text{and} \quad x = \frac{mx_2 + nx_1}{m + n}.$$

Therefore, the coordinates of R are $\left(\dfrac{mx_2 + nx_1}{m + n}, \dfrac{my_2 + ny_1}{m + n}, \dfrac{mz_2 + nz_1}{m + n}\right).$

b) External Division

In this case, $MM' = NN' = LP = z_1,$

$N'R = NR - NN' = z - z_1$

and $M'Q = MQ - MM' = z_2 - z_1.$

Since N′R || M′Q, ˢΔ PN′R and PM′Q are similar; and hence

$$\frac{N'R}{M'Q} = \frac{PR}{PQ},$$

or, $\dfrac{z - z_1}{z_2 - z_1} = \dfrac{m}{m - n},$

or, $(m - n)(z - z_1) = m(z_2 - z_1).$

Hence, $z = \dfrac{mz_2 - nz_1}{m - n}.$

Proceeding similarly, we can show that

$$y = \frac{my_2 - ny_1}{m - n} \quad \text{and} \quad x = \frac{mx_2 - nx_1}{m - n}.$$

Therefore, the coordinates of R are $\left(\frac{mx_2 - nx_1}{m - n}, \frac{my_2 - ny_1}{m - n}, \frac{mz_2 - nz_1}{m - n}\right)$.

Cor.1 The coordinates of the middle point of the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Cor.2 The coordinates of the point that divides the line segment joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $\lambda : 1$ are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right)$.

Worked Out Examples

Example 1

Find the distance between the points (4, 3, –6) and (–2, 1, 3).

Solution:

Using the distance formula,

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(-2 - 4)^2 + (1 - 3)^2 + (3 + 6)^2} \\ &= \sqrt{36 + 4 + 81} \\ &= \sqrt{121} = 11 \end{aligned}$$

Example 2

Find the equation of the sphere whose centre is at (1, –1, 1) and radius is 2.

Solution:

Let $P(x, y, z)$ be any point on the sphere with centre at O(1, –1, 1). Then,

$$OP = 2$$

$$\Rightarrow \quad OP^2 = (2)^2$$

$$\Rightarrow \quad (x - 1)^2 + (y + 1)^2 + (z - 1)^2 = 4$$

$$\Rightarrow \quad x^2 - 2x + 1 + y^2 + 2y + 1 + z^2 - 2z + 1 - 4 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 2x + 2y - 2z - 1 = 0$$

which is the required equation of the sphere.

Example 3

Find the locus of a point P(x, y, z) satisfying the condition $PA^2 + PB^2 = 6$ where A(–1, 2, –1) and B(0, 3, –2) are two fixed points.

Solution:

Here, $PA^2 = (x + 1)^2 + (y - 2)^2 + (z + 1)^2$

$= x^2 + y^2 + z^2 + 2x - 4y + 2z + 6$

and $PB^2 = x^2 + (y - 3)^2 + (z + 2)^2$

$= x^2 + y^2 + z^2 - 6y + 4z + 13$

Since $PA^2 + PB^2 = 6$

So, we have

$$x^2 + y^2 + z^2 + 2x - 4y + 2z + 6 + x^2 + y^2 + z^2 - 6y + 4z + 13 = 6$$

$$\Rightarrow \quad 2x^2 + 2y^2 + 2z^2 + 2x - 10y + 6z + 13 = 0$$

which is the required equation of the locus of a point.

Example 4

Using distance formula, show that A(4, 3, 1), B(3, 1, 2) and C(2, –1, 3) are collinear.

Solution:

Using distance formula

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2,$$

we have

$AB = \sqrt{(3-4)^2 + (1-3)^2 + (2-1)^2} = \sqrt{1+4+1} = \sqrt{6}$

$BC = \sqrt{(2-3)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{1+4+1} = \sqrt{6}$

$AC = \sqrt{(2-4)^2 + (-1-3)^2 + (3-1)^2} = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

Since $AB + BC = \sqrt{6} + \sqrt{6} = 2\sqrt{6} = AC$

$\therefore$ the points A, B and C are collinear.

Example 5

Show that the triangle whose vertices are (2, –1, 1), (1, –3, –5) and (3, –4, –4) is a right angled.

Solution:

Let A(2, –1, 1), B(1, –3, –5) and C(3, –4, –4) be the vertices of a triangle.

Using distance formula,

$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$, we have

$AB^2 = (1-2)^2 + (-3+1)^2 + (-5-1)^2 = 1 + 4 + 36 = 41$

$BC^2 = (3-1)^2 + (-4+3)^2 + (-4+5)^2 = 4 + 1 + 1 = 6$

and $CA^2 = (2-3)^2 + (-1+4)^2 + (1+4)^2 = 1 + 9 + 25 = 35$

$\because BC^2 + CA^2 = 6 + 35 = 41 = AB^2$

$\therefore$ the triangle is a right angled.

Example 6

Show that the centre of the sphere which passes through the points (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c) is $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$.

Solution:

Let P(x, y, z) be the centre of the sphere passing through the points O(0, 0, 0), A(a, 0, 0), B(0, b, 0) and C(0, 0, c)

Using $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$, we have

$OP^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2$ ......(i)

$PA^2 = (a - x)^2 + (0 - y)^2 + (0 - z)^2 = (a - x)^2 + y^2 + z^2$ ......(ii)

$PB^2 = (0 - x)^2 + (b - y)^2 + (0 - z)^2 = x^2 + (b - y)^2 + z^2$ ......(iii)

$PC^2 = (0 - x)^2 + (0 - y)^2 + (c - z)^2 = x^2 + y^2 + (c - z)^2$ ......(iv)

$\because$ OP = PA

$\Rightarrow OP^2 = PA^2$

$\Rightarrow (a - x)^2 + y^2 + z^2 = x^2 + y^2 + z^2$

$\Rightarrow 2ax - x^2 = 0$

$\therefore x = \frac{1}{2}a$

Similarly making (i) equal to (iii) and (i) equal to (iv).

we have, $y = \frac{1}{2}a$ and $z = \frac{1}{2}a$

$\therefore$ the centre of the sphere = $(x, y, z) = (\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a)$

Example 7

Find the coordinates of the point which divides the line segment joining the points (1, –4, 1) and (2, 3, –5) a) internally b) externally in the ratio 1:2.

Solution:

a) **Internal division:** Let (x, y, z) be the coordinates of a point of division.

Then using section formula for internal division,

$$(x, y, z) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

$$= \left(\frac{1.2 + 2.1}{1+2}, \frac{1.3 + 2.(-4)}{1+2}, \frac{1.(-5) + 2.1}{1+2}\right)$$

$$= \left(\frac{4}{3}, \frac{-5}{3}, -1\right)$$

b) External division: Let (x, y, z) be the coordinates of a point of division. Then using section formula for external division

$$(x, y, z) = \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$$

$$= \left(\frac{1.2 - 2.1}{1-2}, \frac{1.3 - 2.(-4)}{1-2}, \frac{1.(-5) - 2.1}{1-2}\right)$$

$$= (0, -11, 7)$$

Example 8

Find the coordinates of the point where the line through the points A(5, 6, 1) and B(5, 1, 6) crosses the xy-plane.

Solution:

Let the line AB cross the xy-plane at P.

The z-coordinate of any point on the xy-plane is 0. So, P, a point on xy-plane, has the coordinates of the form $(x, y, 0)$. If P divides the join of A(5, 6, 1) and B(5, 1, 6) in the ratio λ:1, then

$$z = \frac{\lambda z_2 + z_1}{\lambda + 1}$$

$$0 = \frac{\lambda.6 + 1}{\lambda + 1} \quad \Rightarrow \quad 6\lambda + 1 = 0$$

$$\therefore \quad \lambda = -\frac{1}{6}$$

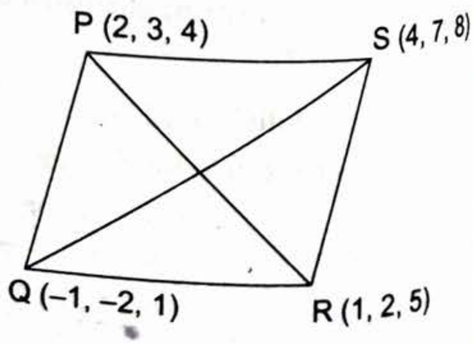
$\therefore$ the point P divides the join of two given points externally in the ratio of 1:6.

The coordinates of P are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, 0\right)$

$$= \left(\frac{-1/6\,.\,5 + 5}{-1/6 + 1}, \frac{-1/6\,.\,1 + 6}{-1/6 + 1}, 0\right) = (5, 7, 0)$$

Example 9

Show that the points (2, 3, 4), (–1, –2, 1), (1, 2, 5) and (4, 7, 8) taken in order form a parallelogram.



Solution:

Let P(2, 3, 4), Q(–1, –2, 1), R(1, 2, 5) and S(4, 7, 8) be the vertices of a quadrilateral PQRS.

Mid-point of the diagonal PR

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

$$= \left(\frac{2+1}{2}, \frac{3+2}{2}, \frac{4+5}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right)$$

Mid-point of the diagonal QS

$$= \left(\frac{-1+4}{2}, \frac{-2+7}{2}, \frac{1+8}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}, \frac{9}{2}\right)$$

Since the coordinates of the mid-points of the diagonals PR and QS are the same, so diagonals PR and QS bisect each other.

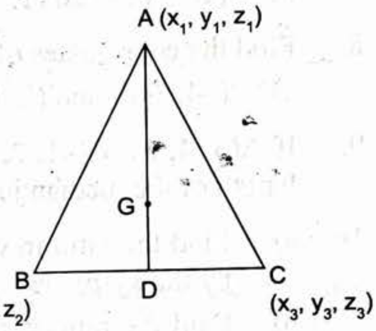
$\therefore$ PQRS is a parallelogram.

Example 10

Show that the coordinates of the centroid of the triangle ABC whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

Solution:

Let AD be the median so that D is the mid point of BC. Then, the coordinates of the mid-point D of BC are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$



If G (x, y, z) be the centroid of the triangle ABC, then G divides AD in the ratio 2 : 1.

Then the coordinates of G are

$$\left(\frac{1.x_1 + 2.\frac{x_2+x_3}{2}}{1+2}, \frac{1.y_1 + 2.\frac{y_2+y_3}{2}}{1+2}, \frac{1.z_1 + 2.\frac{z_2+z_3}{2}}{1+2}\right)$$

$$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

EXERCISE

1. Find the distance between the points
   a) (–1, 4, 3) and (2, 2, –3)
   b) (4, –1, 5) and (–4, 3, 6)
2. a) Show that the point (2, 0, –4), (4, 2, 4) and (10, 2, –2) are the vertices of an equilateral triangle.
   b) Show that the point (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) are the vertices of a right angled isosceles triangle.
3. a) Show that the points (1, 2, 3), (–1, –2, –1), (2, 3, 2) and (4, 7, 6) are the vertices of a parallelogram.

b) Show that the points (1, 1, 1), (–2, 4, 1), (–1, 5, 5) and (2, 2, 5) are the vertices of a square.

4. Show that the following points are collinear
   a) (1, 2, 3), (–2, 3, 4) and (7, 0, 1)
   b) (–2, 3, 5), (1, 2, 3) and (7, 0, –1)

5. a) Find the locus of a point which moves such that its distance from the fixed point (1, 2, –2) is always 5.
   b) Find the locus of a point which moves such that it is equidistant from two fixed point (–1, 2, 3) and (4, –1, 5).

6. Find the coordinates of the point which is equidistant from the four points O, A, B and C where O is the origin and A, B and C are the points on the x-, y- and z-axis respectively at distances a, b and c from the origin.

7. Find the coordinates of the point which divides the line segment joining each of the following pair of points internally in the ratio 1:2 and externally in the ratio 3:2
   a) (1, –2, 3) and (1, 2, 3)
   b) (2, 0, 1) and (4, –2, 5)

8. Find the coordinates of the mid-points of the join of each of the following pair of points
   a) (–4, 3, 6) and (2, 1, –3)
   b) (2, 5, –8) and (4, –1, 6)

9. If A(3, 4, 5), B(–1, 2, 0) and C(–3, 4, –2) are the vertices of the triangle ABC, find the length of the median joining the vertex A(3, 4, 5) and middle point of its opposite side.

10. a) Find the ratio in which the line joining the points (–2, 4, 7) and (3, –5, –8) is divided by the xy-plane.
    b) Find the ratio in which the yz-plane divides the line joining the points (4, 6, 7) and (–1, 2, 5). Find also the coordinates of the points on the yz-plane.
    c) Given three collinear points A(3, 2, –4), B(5, 4, –6) and C(9, 8, –10), find the ratio in which B divides AC.

11. a) Find the point where the line through the points (1, 2, 3) and (4, –4, 9) meets the zx-plane.
    b) Find the point where the line joining the points (2, –3, 1) and (3, –4, –5) cuts the plane $2x + y + z = 7$.

12. Show that the following points represent the vertices of a parallelogram
    a) (3, 0, 1), (2, 2, 2), (–1, 3, 3) and (0, 1, 2)
    b) (1, 3, 4), (–1, 6, 10), (–7, 4, 7) and (–5, 1, 1)

13. Two vertices of a triangle ABC are A(2, –4, 3) and B(3, –1, –2) and its centroid is (1, 0, 3). Find the third vertex C.

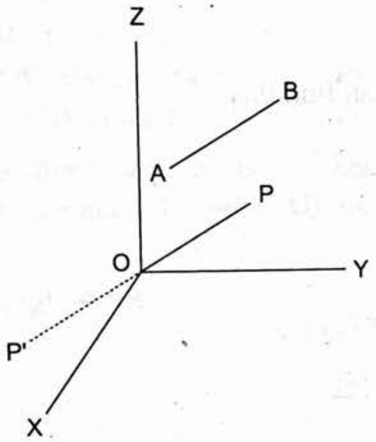
14. Three vertices of a parallelogram ABCD are A(–5, 5, 2), B(–9, –1, 2) and C(–3, –3, 0). Find the coordinates of the fourth vertex.

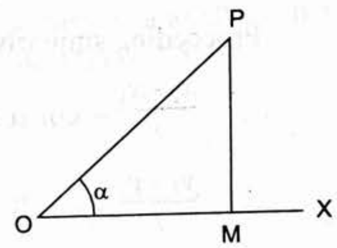
Answers

1. a) 7 b) 9
5. a) $x^2 + y^2 + z^2 - 2x - 4y + 4z - 16 = 0$ b) $5x - 3y + 2z - 14 = 0$
6. $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$
7. a) $(1, -\frac{2}{3}, 3)$ and $(1, 10, 3)$ b) $(\frac{8}{3}, -\frac{2}{3}, \frac{7}{3})$ and $(8, -6, 13)$
8. a) $(-1, 2, \frac{3}{2})$ b) $(3, 2, -1)$ c) 1:2
9. $\sqrt{62}$
10. a) 7:8 b) 4:1, $(0, \frac{14}{5}, \frac{27}{5})$
11. a) (2, 0, 5) b) (1, –2, 7)
13. (–2, 5, 8)
14. (1, 3, 0)

Direction cosines of a line

If a line AB makes angles α, β, γ with the directions x-, y-, z-axis respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called the direction cosines (d.c.s) of the line AB and α, β, γ the direction angles. The direction cosines are generally denoted by l, m, n.





Note that the direction cosines of the line BA are $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$ and $\cos(\pi - \gamma)$ as the direction angles of the line OP′ through O parallel to BA are $\pi - \alpha$, $\pi - \beta$ and $\pi - \gamma$. Hence if l, m, n are the direction cosines of the line AB, then $-l, -m, -n$ are the direction cosines of BA.

Direction cosines of the coordinate axes

Since the x-axis makes angles $0, \frac{\pi}{2}$ and $\frac{\pi}{2}$ with x-axis, y-axis and z-axis respectively, so the direction cosines of the x-axis are $\cos 0, \cos\frac{\pi}{2}, \cos\frac{\pi}{2}$, i.e 1, 0, 0. Similarly, the direction cosines of y- and z-axis are 0, 1, 0 and 0, 0, 1 respectively.

To prove that $l^2 + m^2 + n^2 = 1$ where l, m, n are the direction cosines of a line AB.

In the above figure, OP $(= r)$ makes angles α, β, γ with the coordinate axes so that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ i.e. l, m, n are the direction cosines of the line AB. If (x, y, z) be the coordinates of P and PM is drawn perpendicular to x-axis, then from the right angled triangle POM

$$\cos\alpha = \frac{OM}{OP} \Rightarrow l = \frac{x}{r} \quad \therefore x = lr$$

Similarly, $y = mr$ and $z = nr$

Also, $OP^2 = x^2 + y^2 + z^2$

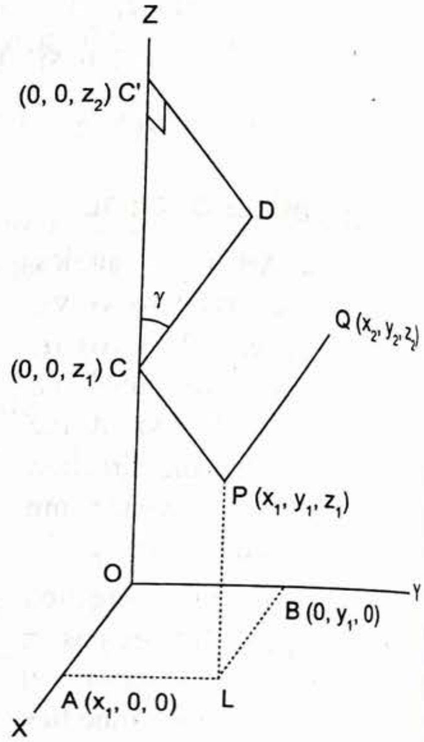
$\Rightarrow r^2 = (lr)^2 + (mr)^2 + (nr)^2$

$\Rightarrow 1 = l^2 + m^2 + n^2$

$\therefore l^2 + m^2 + n^2 = 1$

Direction cosines of the Line through Two Given Points

Suppose a straight line PQ passes through two given points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. In order to find the direction cosines of PQ, draw PC perpendicular to the z-axis. From C draw CD equal and parallel to PQ = r.



Then CD and PQ will be inclined to the z-axis at the same angle γ, say. Also draw DC' perpendicular to the z axis. Then, clearly

$$\frac{CC'}{CD} = \cos\gamma$$

or, $$\frac{z_2 - z_1}{r} = \cos\gamma = n$$

Proceeding similarly, we can find that

$$\frac{x_2 - x_1}{r} = \cos\alpha = l,$$

$$\frac{y_2 - y_1}{r} = \cos\beta = m.$$

Hence, direction cosines of PQ are

$$\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}$$

Also, $$\frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma} = r,$$

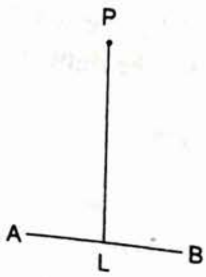
where $$r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

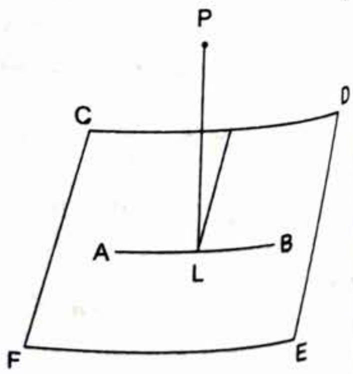
Furthermore, the quantities $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ are the direction ratios of the line PQ.

Projection

a) Projection of a point on a line or a plane

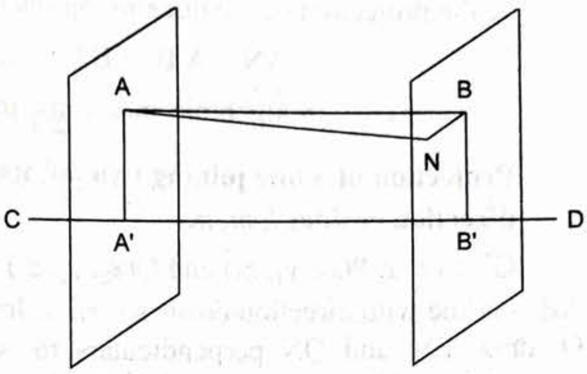
The projection of a point P on a line AB or a plane CDEF is the foot L of the perpendicular from the point P on the line AB or plane CDEF. In the first case, it is also interpreted as the point L of intersection of the plane through the given point P and perpendicular to the given line.





b) Projection of a line segment on a line

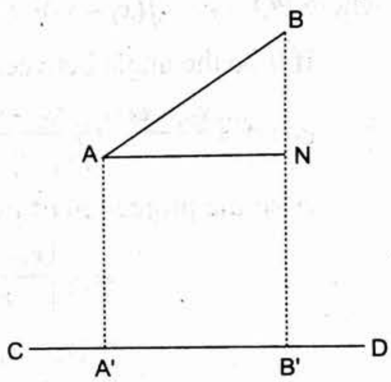
The projection of the line segment AB of a given line on another line CD is the segment A'B' of CD where A' and B' are the projections of A and B on CD. Here we may also say that A' and B' are the points in which planes through A and B perpendicular to CD meet the line CD.



To find the length of the projection of a given line on another line

Let AB be a given line and CD be the line on which the projection of AB is to be determined. From A and B, draw AA′ and BB′ perpendiculars to CD so that A′ and B′ are the projections of A and B on CD. Then A'B' is the length of the projection of AB on CD.

Again from A, draw AN perpendicular to BB′. Then AN = A'B′. Let ∠NAB = θ i.e. θ is the angle between AB and CD.



Then from the right angled triangle ANB

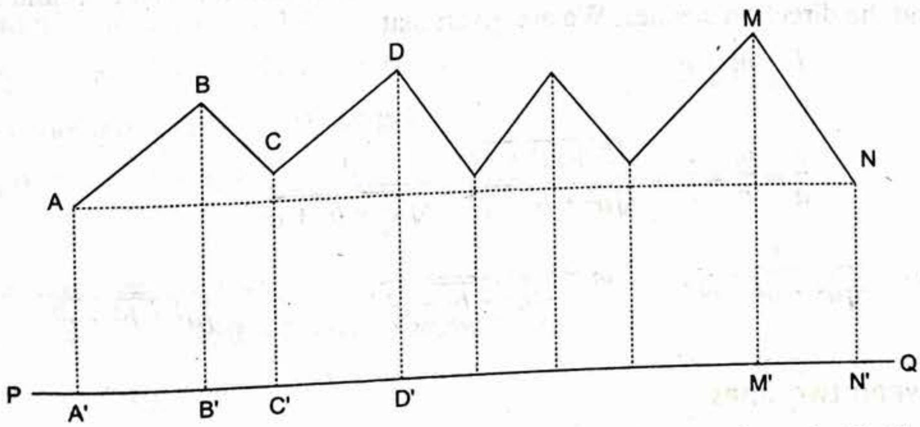
$$\cos\theta = \frac{AN}{AB}$$

$\Rightarrow$ AN = AB cos θ

$\Rightarrow$ A′B′ = AB cos θ

Projection of a broken line on a given line

If A, B, C, D, ……, M, N be any number of points in space then the sum of the projections of AB, BC, …, MN on any given line PQ is equal to the projection of the straight line AN on PQ.



Let A′, B′, C′, D′, …, M′, N′ be the feet of the perpendiculars from A, B, C, D, …, M, N on the line PQ. Then the projections of AB, BC, CD, …, MN on the line PQ are A′B′, B′C′, C′D′, …, M′N′ respectively and

$$A'B' + B'C' + C'D' + \ldots\ldots + M'N' = A'N'$$

Again the projection of AN on PQ = A′N′

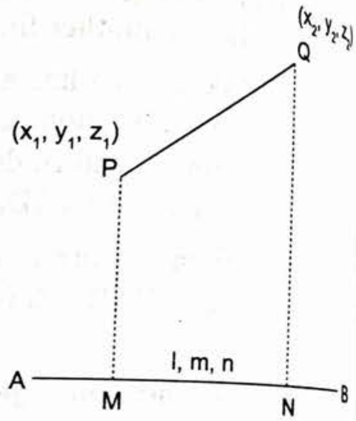
∴ the projection of the line joining the end points A and N

= AN = A′B′ + B′C′ + … + M′N′

= algebraic sum of the projections of the broken lines on PQ.

Projection of a line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line with direction cosines l, m, n.

Given that $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points and AB is a line with direction cosines l, m, n. Join PQ. From P and Q, draw PM and QN perpendiculars to AB meeting at the points M and N respectively. Then the direction cosines of PQ are $\frac{x_2-x_1}{r}, \frac{y_2-y_1}{r}, \frac{z_2-z_1}{r}$



where $PQ = r = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

If θ be the angle between PQ and AB, then

$$\cos\theta = \frac{x_2-x_1}{r}.l + \frac{y_2-y_1}{r}.m + \frac{z_2-z_1}{r}.n$$

Also the projection of PQ on AB = MN = PQ cos θ

$$= r.\left\{\frac{x_2-x_1}{r}.l + \frac{y_2-y_1}{r}.m + \frac{z_2-z_1}{r}.n\right\}$$

$$= (x_2-x_1).l + (y_2-y_1).m + (z_2-z_1).n$$

Direction Ratios

If a set of three numbers a, b and c are proportional to the directions cosines l, m and n of a line, they are called its **direction ratios** (or d.r.'s). If the direction ratios a, b and c are given, it is easy to find the direction cosines. We are given that

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}.$$

Then, $$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{a^2+b^2+c^2}}.$$

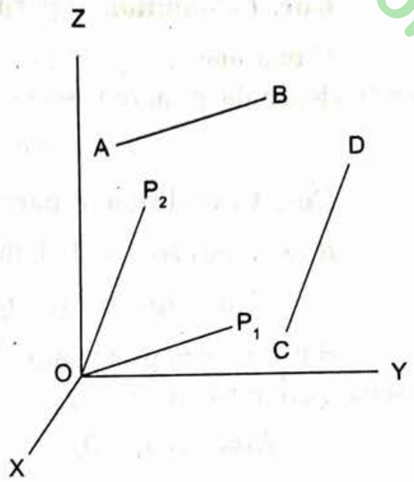
Hence, $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$, $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$, $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$

Angle between two lines

To find the angle between two lines with direction cosine (l_1, m_1, n_1) and (l_2, m_2, n_2)

Let (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosines of two given lines AB and CD respectively. Let OP_1 and OP_2 be the lines thorugh the origin O parallel to the lines AB and CD respectively so that the angle between them is the same as the angle between the lines AB and

CD. Let this angle be θ. Also the direction cosines of OP_1 and OP_2 are respectively (l_1, m_1, n_1) and (l_2, m_2, n_2). Let the coordinates of P_1 and P_2 be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.



If $OP_2 = r_2$, the projection of OP_2 on OP_1 is $r_2 \cos\theta$.

Also the projection of OP_2 on OP_1 is equal to $l_1x_2 + m_1y_2 + n_1z_2$. Therefore we have

$$r_2 \cos\theta = l_1x_2 + m_1y_2 + n_1z_2$$

or, $$\cos\theta = l_1\frac{x_2}{r_2} + m_1\frac{y_2}{r_2} + n_1\frac{z_2}{r_2}$$

$$= l_1l_2 + m_1m_2 + n_1n_2$$

which gives the angle θ.

Alternative method:

Let (l_1, m_1, n_1) and (l_2, m_2, n_2) be the direction cosines of OP_1 and OP_2 respectively. Let $OP_1 = r_1$ and $OP_2 = r_2$. Let the coordinates of P_1 and P_2 be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then

$$x_1^2 + y_1^2 + z_1^2 = r_1^2 \quad \text{and} \quad x_2^2 + y_2^2 + z_2^2 = r_2^2$$

Also $x_1 = l_1r_1,\quad y_1 = m_1r_1,\quad z_1 = n_1r_1,$

and $x_2 = l_2r_2,\quad y_2 = m_2r_2,\quad z_2 = n_2r_2$

Now $$P_1P_2^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$
$$= (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) - 2(x_1x_2 + y_1y_2 + z_1z_2)$$
$$= r_1^2 + r_2^2 - 2r_1r_2(l_1l_2 + m_1m_2 + n_1n_2) \quad \text{......(i)}$$

Also from trigonometry, we have

$$P_1P_2^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\theta \quad \text{......(ii)}$$

Hence by comparing (i) and (ii), we get

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

Cor. 1. Expression for sin θ

We have $\sin^2\theta = 1 - \cos^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$

$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$$

$$(\because l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2)$$

$$= (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 \quad \text{by Lagrange's identity}$$

$$= \Sigma(l_1m_2 - l_2m_1)^2$$

Therefore $\sin\theta = \pm\sqrt{\Sigma(l_1m_2 - l_2m_1)^2}$

Cor. 2 Condition of perpendicularity of two lines

If two lines are perpendicular to each other, then angle between them, $\theta = 90°$ i.e. $\cos 90° = 0$. Hence the required condition is

$$l_1l_2 + m_1m_2 + n_1n_2 = \cos 90° = 0.$$

Cor. 3 Condition of parallelism of two lines

If two lines are parallel, then $\theta = 0$, i.e. $\sin\theta = 0$. So from the above result we have

$$(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 = 0 \text{ (by cor. 1)}$$

But this, being the sum of square of three quantities, can be zero only if each of them is separately zero, i.e. if

$$l_1m_2 - l_2m_1 = 0, \quad m_1n_2 - m_2n_1 = 0, \quad n_1l_2 - n_2l_1 = 0$$

which gives $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

$$\Rightarrow \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\therefore \quad l_1 = l_2, \quad m_1 = m_2 \quad \text{and} \quad n_1 = n_2$$

which is the required condition of parallelism of two straight lines.

Angle between two straight lines whose direction ratios are given

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of two lines whose corresponding direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2. Then

$$l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \quad n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$$

and $$l_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \quad m_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \quad n_2 = \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If θ be the angle between the two lines, then

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(Substituting the values of l_1, m_1, n_1 and l_2, m_2, n_2)

The two lines will be perpendicular to each other if $\theta = 90°$

i.e. $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The two lines will be parallel if $\theta = 0°$ or 180, so

$$1 = \frac{(a_1a_2 + b_1b_2 + c_1c_2)^2}{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}$$

$$\Rightarrow \quad (a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2 = 0$$

$$\Rightarrow \quad (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$$

This relation is possible only when

$a_1b_2 - a_2b_1 = 0 \quad \Rightarrow \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$, $\quad b_1c_2 - b_2c_1 = 0 \quad \Rightarrow \quad \frac{b_1}{b_2} = \frac{c_1}{c_2}$

and $c_1a_2 - c_2a_1 = 0 \quad \Rightarrow \quad \frac{c_1}{c_2} = \frac{a_1}{a_2}$

i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence the two lines will be parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Worked Out Examples

Example 1

If a straight line makes an angle $\alpha = \frac{\pi}{3}$ with the positive x-axis, an angle $\beta = \frac{\pi}{3}$ with the positive y-axis and an acute angle γ with the positive z-axis, find γ.

Solution:

Since $l^2 + m^2 + n^2 = 1$, we have

$$\cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{3} + \cos^2\gamma = 1$$

$$\Rightarrow \frac{1}{4} + \frac{1}{4} + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\gamma = \frac{1}{2}$$

$$\therefore \cos\gamma = \pm\frac{1}{\sqrt{2}}$$

Neglecting the negative value as we need only the acute angle, we have

$$\cos\gamma = \frac{1}{\sqrt{2}}$$

$$\therefore \gamma = \frac{\pi}{4}$$

Example 2

If $-1, 2, 2$ are the direction ratios of a line, find its direction cosines.

Solution:

If a, b, c be the direction ratios of a line, then by given $a = -1, b = 2, c = 2$.

$$a^2 + b^2 + c^2 = (-1)^2 + (2)^2 + (2)^2 = 9$$

If l, m, n be the direction cosines of the same line, then

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{-1}{\sqrt{9}} = -\frac{1}{3}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

which are the direction cosines of the line.

Example 3

Find the direction cosines of the line passing through the points A(−1, 2, 5) and B(−2, 4, 3).

Solution:

If r be the distance between two given points A and B then using distance formula,

$$r = \sqrt{(-2+1)^2+(4-2)^2+(3-5)^2} = \sqrt{1+4+4} = 3$$

The direction cosines of AB are

$$\frac{x_2-x_1}{r}, \frac{y_2-y_1}{r}, \frac{z_2-z_1}{r}$$

$$\frac{-2+1}{3}, \frac{4-2}{3}, \frac{3-5}{3}$$

i.e. $-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$

Example 4

Find the angle between the two lines whose direction cosines are proportional to 1, 2, 3 and 3, 4, 5.

Solution:

If a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of the two lines then by given,

$a_1 = 1, \quad b_1 = 2, \quad c_1 = 3 \quad$ and $\quad a_2 = 3, \quad b_2 = 4, \quad c_2 = 5$

Now using the formula,

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

where θ is the angle between the two lines.

$$= \frac{1.3+2.4+3.5}{\sqrt{1^2+2^2+3^2}\sqrt{3^2+4^2+5^2}} = \frac{26}{\sqrt{14}\sqrt{50}} = \frac{13}{5\sqrt{7}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{13}{5\sqrt{7}}\right)$$

Example 5

Show that the line AB is perpendicular to CD if A, B, C, D are the points (2, 3, 4), (5, 4, −1), (3, 6, 2) and (1, 2, 0) respectively.

Solution:

For AB: If a_1, b_1, c_1 be the direction cosines of AB, then

$$a_1 = 5 - 2 = 3, \quad b_1 = 4 - 3 = 1, \quad c_1 = -1 - 4 = -5$$

Again if a_2, b_2, c_2 be the direction cosines of CD, then

$$a_2 = 1 - 3 = -2, \quad b_2 = 2 - 6 = -4, \quad c_2 = 0 - 2 = -2$$

Now, $a_1a_2 + b_1b_2 + c_1c_2 = 3(-2) + 1(-4) + (-5)(-2) = 0$

Hence AB is perpendicular to CD.

Example 6

Find the direction cosines of two lines which satisfy the relations $2l + 2m - n = 0$ and $lm + mn + nl = 0$. Also find the angle between the two lines.

Solution:

Given relations are

$$2l + 2m - n = 0 \quad \ldots\ldots(i)$$

and

$$lm + mn + nl = 0 \quad \ldots\ldots(ii)$$

Eliminating n between (i) and (ii),

$$lm + m(2l + 2m) + (2l + 2m)l = 0$$

$$\Rightarrow \quad 2l^2 + 5lm + 2m^2 = 0$$

$$\Rightarrow \quad (l + 2m)(2l + m) = 0$$

Either $\quad l + 2m = 0 \quad \ldots\ldots(iii)$

and $\quad 2l + m = 0 \quad \ldots\ldots(iv)$

From (i) and (iii),

$$2l + 2m - n = 0$$

$$l + 2m + 0.n = 0$$

By the rule of cross multiplication,

$$\frac{l}{0+2} = \frac{m}{-1-0} = \frac{n}{4-2}$$

$$\Rightarrow \quad \frac{l}{2} = \frac{m}{-1} = \frac{n}{2} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{4+1+4}} = \frac{1}{3}$$

$\therefore l = \frac{2}{3}, \; m = -\frac{1}{3}, \; n = \frac{2}{3}$ which gives the direction cosines of the first line.

Again from (i) and (iv), we have

$$2l + 2m - n = 0$$

$$2l + m + 0.n = 0$$

By the rule of cross-multiplication

$$\frac{l}{0+1}=\frac{m}{-2-0}=\frac{n}{2-4} \quad \Rightarrow \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{-2}=\frac{1}{3}$$

$\therefore l=\frac{1}{3}, \quad m=-\frac{2}{3}, \quad n=-\frac{2}{3}$ which gives the direction cosines of second line.

Now, $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$$= \frac{2}{9}+\frac{2}{9}-\frac{4}{9}=0$$

$$\therefore \quad \theta = \frac{\pi}{2}$$

Example 7

If l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of two lines then show that the direction cosines of a line perpendicular to both are proportional to $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, $l_1m_2 - l_2m_1$.

Solution:

Let l, m, n be the direction cosines of a line perpendicular to the two given lines.

$\therefore \quad l.l_1 + m.m_1 + n.n_1 = 0$

$l.l_2 + m.m_2 + n.n_2 = 0$

By the rule of cross-multiplication

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

Hence the direction cosines of the required line are proportional to

$$m_1n_2 - m_2n_1, \quad n_1l_2 - n_2l_1, \quad l_1m_2 - l_2m_1$$

Example 8

Prove that the lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel, if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$ and paralle, if $\sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$

Solution:

Given relations are

$$al + bm + cn = 0 \quad \text{......(i)}$$

$$fmn + gnl + hlm = 0 \quad \text{......(ii)}$$

Eliminating n between (i) and (ii), we have

$$fm\left(-\frac{al+bm}{c}\right) + g\left(-\frac{al+bm}{c}\right)l + hlm = 0$$

$$\Rightarrow \quad agl^2 + (af + bg - ch)lm + bfm^2 = 0$$

$$\Rightarrow \quad ag\left(\frac{l}{m}\right)^2 + (af + bg - ch)\left(\frac{l}{m}\right) + bf = 0 \quad \text{......(iii)}$$

which is quadratic in $\frac{l}{m}$. Let the two roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

Then, $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bf}{ag}$

$$\Rightarrow \quad \frac{l_1 l_2}{bf} = \frac{m_1 m_2}{ag}$$

$$\Rightarrow \quad \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} \qquad \ldots\ldots(iv)$$

Similarly, if we eliminate l between (i) and (ii), we have

$$\frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} \qquad \ldots\ldots(v)$$

From (iv) and (v)

$$\frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c}$$

If each ratio be equal to k, then

$$l_1 l_2 = k\frac{f}{a}, \qquad m_1 m_2 = k\frac{g}{b} \quad \text{and} \quad n_1 n_2 = k\frac{h}{c}$$

The two lines will be perpendicular if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$k\frac{f}{a} + k\frac{g}{b} + k\frac{h}{c} = 0$$

i.e. $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

The two lines will be parallel if the discriminant of the equation (iii) is zero.

$$(af + bg - ch)^2 = 4abfg$$

$$\Rightarrow \quad af + bg - ch = \pm 2\sqrt{abfg}$$

$$\Rightarrow \quad af + bg \pm 2\sqrt{abfg} = ch$$

$$\Rightarrow \quad (\sqrt{af} \pm \sqrt{bg})^2 = ch$$

$$\Rightarrow \quad \sqrt{af} \pm \sqrt{bg} = \pm\sqrt{ch}$$

$$\Rightarrow \quad \sqrt{af} \pm \sqrt{bg} \pm \sqrt{ch} = 0$$

Example 9

If the coordinates of P, Q, R and S be (1, 2, 2), (2, 4, 0), (–3, 0, 1) and (–1, –2, 2) respectively, find the projection of RS on PQ.

Solution:

Using the distance formula $PQ = \sqrt{(2-1)^2 + (4-2)^2 + (0-2)^2} = \sqrt{9} = 3$

If the direction cosines of PQ are l, m, n, then

$$l = \frac{x_2 - x_1}{PQ} = \frac{2-1}{3} = \frac{1}{3} \quad m = \frac{y_2 - y_1}{PQ} = \frac{4-2}{3} = \frac{2}{3} \quad n = \frac{z_2 - z_1}{PQ} = \frac{0-2}{3} = -\frac{2}{3}$$

Now, the projection of RS on PQ

$$= (x_2 - x_1).l + (y_2 - y_1).m + (z_2 - z_1).n$$

$$= (-1+3)\frac{1}{3} + (-2-0)\frac{2}{3} + (2-1)\left(-\frac{2}{3}\right)$$

$$= \frac{2}{3} - \frac{4}{3} - \frac{2}{3} = -\frac{4}{3}$$

EXERCISE

1. If a line makes angles of $\frac{\pi}{3}$ and $\frac{\pi}{4}$ with the positive x-axis and z-axis respectively, find the acute angle made by the line with the positive y-axis.
2. Show that the direction cosines of a line equally inclined to the axes are

$$\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}.$$

3. a) If α, β and γ are the angles which a line makes with the coordinate axes, prove that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

   b) If α, β and γ are the direction angles of a line, prove that

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$$

4. Find the direction cosines of each of the lines whose direction ratios are
   a) $-1, 2, 2$
   b) $2, 3, 6$.
5. Find the direction cosines of the line passing through the points
   a) O(0, 0, 0) and P(2, 3, 4)
   b) P(2, 3, 4) and Q(1, 4, 6)
   c) M(−1, 2, −3) and N(4, −1, 1)
6. Find the angle between the two lines whose direction ratios are
   a) 1, 2, 4 and −2, 1, 5
   b) 2, 3, 4 and 1, −2, 1
   c) 1, 2, 2 and 2, 3, 6
7. Show that the line joining the points (1, 2, 3) and (−1, −2, −3) is
   a) parallel to the line joining the points (2, 3, 4) and (5, 9, 13)
   b) perpendicular to the line joining the points (−2, 1, 5) and (3, 3, 2)
8. a) For what value of k makes the line joining the points (1, 2, k) and (5, 7, 15) perpendicular to the line joining the points (4, 7, 1) and (3, 5, 3)?

b) For what value of k makes the line joining the points (1, 2, k) and (4, 5, 6) parallel to the line joining the points (–4, 3, –6) and (2, 9, 2)?

9. If O is the origin and P(2, 3, 4) and Q(1, –2, 1) be any two points, show that OP is perpendicular to OQ.

10. Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to 3, –1, 1 and –3, 2, 4.

11. Show that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

12. The projection of a line on the axes are 6, 2, 3. Find the length of the line and its direction cosines.

13. Find the projection of the join of the pair of points (3, –1, 2) and (5, –7, 4)

a) on the coordinate axes

b) on a line whose direction cosines are proportional to 1, –1, 2

c) on a line joining the points (0, 1, 0) and (1, 3, 7)

14. a) A, B, C and D are four points with coordinates (2, 3, 1), (3, 2, 5), (–1, 2, 4) and (–1, 5, 7) respectively. Prove that the projection of AB on CD is equal to the projection of CD on AB. Also, show that the angle between them is $\frac{\pi}{3}$.

b) A(1, 2, 3), B(–2, 2, 0) and C(3, 1, 1) are three points. Find the foot of the perpendicular drawn from A to the line BC.

15. Find the direction cosines l, m, n of two lines which satisfy the equations

a) $l + m + n = 0$ and $2lm - mn + 2nl = 0$

b) $4l + 3m - 2n = 0$ and $lm - mn + nl = 0$

c) $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$

Also find the angle between the lines.

16. a) A line makes α, β, γ, δ with the four diagonals of a cube, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

b) Prove that the lines whose direction cosines are given by the relations $pl + qm + rn = 0$ and $amn + bnl + clm = 0$ are perpendicular if $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0$ and parallel if $\sqrt{ap} \pm \sqrt{bq} \pm \sqrt{cr} = 0$.

A. Examine which of the following are the direction cosines of the line and which are not

a) $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}$

b) $-\frac{3}{5}, 0, \frac{4}{5}$

c) $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}$

d) (2, –3, 4)

B. If O(0, 0, 0) be the origin and P(3, 0, 0), Q(0, 4, 0) and R(0, 0, –5) be three points, what are the direction cosines of OP, OQ and OR?

C. Consider a cube or a cuboid. Take three adjacent edges as the axes of coordinates and their intersection as the origin. Next consider thre corners P, Q, R opposite to O of the three adjacent faces.

a) What are the direction cosines of the axes of coordinates?

b) Write the coordinates of the three corners P, Q and R. (You may guess it.)

c) Find the direction cosines of OP, OQ and OR.

Answers

1. $\frac{\pi}{3}$

4. a) $-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ b) $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

5. a) $\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$ b) $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ c) $\frac{1}{\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}$

6. a) $\cos^{-1}\left(\frac{20}{3\sqrt{70}}\right)$ b) $\frac{\pi}{2}$ c) $\cos^{-1}\frac{20}{21}$

8. a) 8 b) 2

10. $\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}$

12. $7, \frac{6}{7}, \frac{2}{7}, \frac{3}{7}$

13. a) 2, –6, 2 b) $2\sqrt{6}$ c) $\frac{4}{\sqrt{6}}$

14. a) $\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}$ b) $\left(\frac{4}{3}, \frac{4}{3}, \frac{2}{3}\right)$

15. a) $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}; -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$, 60° b) $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}; \frac{3}{13}, \frac{4}{13}, \frac{12}{13}; \cos^{-1}\left(\frac{10}{39}\right)$

c) $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$; 120° (or 60°)

Chapter 10
Plane

Definition

A surface such that the straight line joining any two point on it lies wholly on the surface is called a plane surface or simply a plane.

The general equation of the first degree in x, y and z represents a plane.

The general equation of the first degree in x, y and z is

$$Ax + By + Cz + D = 0 \quad \text{......(i)}$$

where A, B, C and D are constants and A, B, C are not all zero.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points lying on the locus represented by equation (i), then we have

$$Ax_1 + By_1 + Cz_1 + D = 0 \quad \text{......(ii)}$$

and $$Ax_2 + By_2 + Cz_2 + D = 0 \quad \text{......(iii)}$$

Multiplying equation (iii) by k and adding to equation (ii) we get

$$A(kx_2 + x_1) + B(ky_2 + y_1) + C(kz_2 + z_1) + D(k + 1) = 0$$

or $$A\left(\frac{kx_2 + x_1}{k+1}\right) + B\left(\frac{ky_2 + y_1}{k+1}\right) + C\left(\frac{kz_2 + z_1}{k+1}\right) + D = 0$$

This shows that the point

$$\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$$

lies on the locus represented by the equation (i). But this point lies on the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ and divides it in the ratio k:1. Giving different values to k except –1, we get all the points on the line PQ. Therefore, by definition of the plane the locus of the equation (i) is a plane.

Hence the general equation of the first degree in x, y and z represents a plane.

Equation of the plane

a) The intercept form

We have to find the equation of the plane which cuts off intercepts equal to a, b and c on the coordinate axes.

Let the equation of the plane be $Ax + By + Cx + D = 0$

It meets the x-axis where its y- and z- coordinates are zero i.e. if it meets at the point $(a, 0, 0)$ then we have

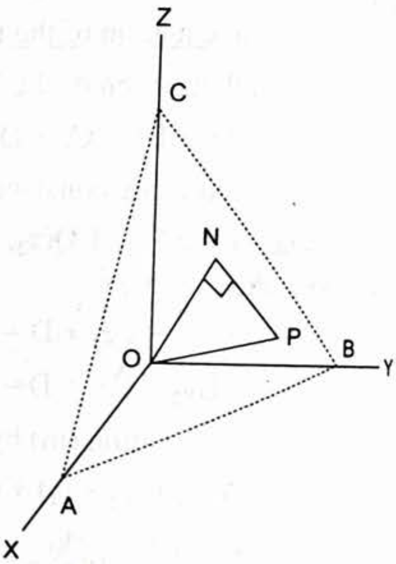
$$Aa + D = 0 \quad \text{or,} \quad A = \frac{-D}{a}$$

Similarly, if it meets the y- and z-axis at the points $(0, b, 0)$ and $(0, 0, c)$ respectively, we have

$$B = \frac{-D}{b} \quad \text{and} \quad C = \frac{-D}{c}$$

Hence the equation (i) takes the form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

b) The normal form



Let p be the length of the perpendicular ON from the origin O on the plane and let l, m, n be the direction cosines of ON, we have to find out the equation of the plane in terms of l, m, n and p.

Let $P(x, y, z)$ be any point on the plane ABC. Since ON is perpendicular to the plane, it must be perpendicular to every line lying in the plane.

Join NP, then $\angle ONP = 90^\circ$

Also ON = projection of OP on the line ON

= sum of the projections of the segments of OP on the line ON

$= lx + my + nz$

But ON $= p$, therefore $p = lx + my + nz$

Thus $lx + my + nz = p$ is the required equation of the plane.

Reduction of the general equation of the plane to the normal form

If $Ax + By + Cz + D = 0$ and $lx + my + nz = p$ represent the same plane, then

$$\frac{l}{A} = \frac{m}{B} = \frac{n}{C} = \frac{-p}{D} \quad \ldots\ldots(i)$$

Therefore the direction cosines of the normal to the plane

$$Ax + By + Cz + D = 0$$

are proportional to A, B, C. Since the axes being rectangular, each of the ratios in (i) is equal to $\pm\frac{1}{\sqrt{A^2 + B^2 + C^2}}$

But p is a positive number, therefore if D is positive we have

$$p = \frac{D}{\sqrt{A^2 + B^2 + C^2}}, \quad l = \frac{-A}{\sqrt{A^2 + B^2 + C^2}}, \quad m = \frac{-B}{\sqrt{A^2 + B^2 + C^2}}$$

and $n = \dfrac{-C}{\sqrt{A^2 + B^2 + C^2}}$

If D is negative, we must change the sign of $\sqrt{A^2 + B^2 + C^2}$.

Plane through three given points

Let the three non-collinear points be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Let the equation of the plane be

$$Ax + By + Cz + D = 0$$

Since it passes through the given points, so we have

$$Ax_1 + By_1 + Cz_1 + D = 0$$

$$Ax_2 + By_2 + Cz_2 + D = 0$$

$$Ax_3 + By_3 + Cz_3 + D = 0$$

Eliminating A, B, C, D from the above relations we get

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad \text{......(i)}$$

This gives the required equation of the plane.

Cor. The condition that the four points (x_r, y_r, z_r), r = 1, 2, 3, 4 be coplanar is that the point (x_4, y_4, z_4) satisfies the equation (1).

Hence the required condition is

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Plane through the intersection of two planes

Let $P = Ax + By + Cz + D = 0$ ......(i)

and $Q = A'x + B'y + C'z + D' = 0$ ......(ii)

be the equations of two given planes. Consider the equation

$$P + \lambda Q = 0 \quad \text{......(iii)}$$

Both P and Q being linear in x, y, z, $P + \lambda Q$ is also linear in x, y, z. Hence equation (iii) represents a plane. Again the coordinates of every point in the intersection of the plane (i) and (ii) will satisfy both the equations $P = 0$ and $Q = 0$ and hence will also satisfy $P + \lambda Q = 0$. Thus for all values of λ, (iii) represents some plane through the line of intersection of (i) and (ii), i.e. $P + \lambda Q = 0$ is the general equation of the plane through the intersection of $P = 0$ and $Q = 0$.

Angle between two planes

We know that the angle between the planes is the angle between their normals.

Let the equations of two planes be

$$A_1x + B_1y + C_1z + D_1 = 0 \quad \text{and} \quad A_2x + B_2y + C_2z + D_2 = 0$$

It is clear that the normals of the above two planes have their direction cosines proportional to A_1, B_1, C_1 and A_2, B_2, C_2 respectively.

If θ be the angle between two planes i.e. between their normals, then

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{\Sigma A_1^2}\sqrt{\Sigma A_2^2}}$$

Cor. 1. If the two planes be perpendicular to each other, then $\theta = \frac{\pi}{2}$, hence

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

Alternatively if the two planes be perpendicular, then their normals with direction cosines proportional to A_1, B_1, C_1 and A_2, B_2, C_2 are perpendicular so that

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

Cor. 2. If the two planes be parallel, then $\theta = 0$

and $$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{\Sigma A_1^2}\sqrt{\Sigma A_2^2}}$$

or, $$1 = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{\Sigma A_1^2}\sqrt{\Sigma A_2^2}}$$

or, $(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2) = (A_1A_2 + B_1B_2 + C_1C_2)^2$

or, $(A_1^2 + B_1^2 + C_1^2)(A_2^2 + B_2^2 + C_2^2) - (A_1A_2 + B_1B_2 + C_1C_2)^2 = 0$

or, $(B_1C_2 - B_2C_1)^2 + (C_1A_2 - C_2A_1)^2 + (A_1B_2 - A_2B_1)^2 = 0$ (Lagrange's identity)

This result is possible only when each term is zero.

Thus, $\frac{B_1}{B_2} = \frac{C_1}{C_2}$, $\frac{C_1}{C_2} = \frac{A_1}{A_2}$ and $\frac{A_1}{A_2} = \frac{B_1}{B_2}$

Therefore $$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

is the condition that given planes are parallel.

Alternatively if the two planes be parallel then their normals with direction cosines proportional to A_1, B_1, C_1 and A_2, B_2, C_2 are also parallel so that $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

Cor. 3. We have provided that the two planes

$$A_1x + B_1y + C_1z + D_1 = 0 \qquad \text{......(i)}$$

and $$A_2x + B_2y + C_2z + D_2 = 0 \qquad \text{......(ii)}$$

are parallel if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = k$ (say) and $k \neq 0$.

Then $A_1 = A_2k$, $B_1 = B_2k$ and $C_1 = C_2k$

Substituting the values of A_1, B_1, C_1 in equation (i) we get

$$kA_2x + kB_2y + kC_2z + D_1 = 0$$

or, $$A_2x + B_2y + C_2z + D = 0 \qquad \text{......(iii)}$$

where $D = \frac{D_1}{k}$.

Thus if plane (i) is parallel to the plane (ii) then plane (i) can be expressed in the form of (iii) which shows that we can write the equations of two parallel planes in such a way that the left hand sides of equations differ only by a constant.

Angle between a line and a plane

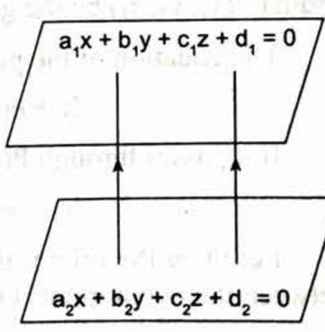
If θ be the angle between the line and the plane $A_1x + B_1y + C_1z + D_1 = 0$ then the angle between the line and the normal to the plane is $\frac{\pi}{2} - \theta$. Let the line have direction cosines proportional to A_2, B_2, C_2, then we have

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{\Sigma A_1^2}\sqrt{\Sigma A_2^2}}$$

Cor. If the line lies on the plane (or parallel to the plane) then clearly

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

While we find the equation of a plane under given condition, the following points are to be remembered.



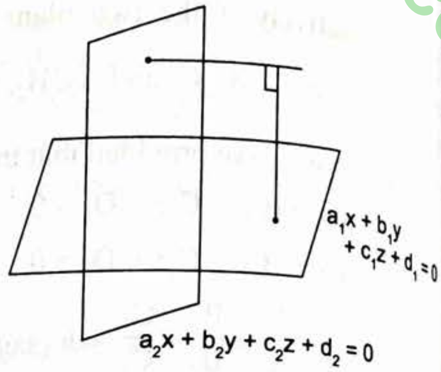
Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be the equations of two planes. Again let l, m, n be the direction cosines of a line.

a) If two planes are parallel, then the lines normal to the planes are also parallel.

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

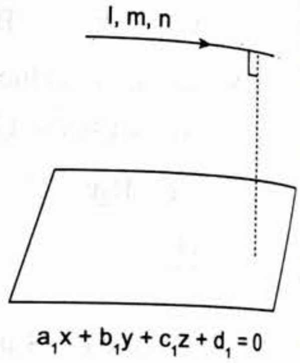
b) If two planes are perpendicular to each other, then the lines normal to the planes are also perpendicular to each other.

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 0$



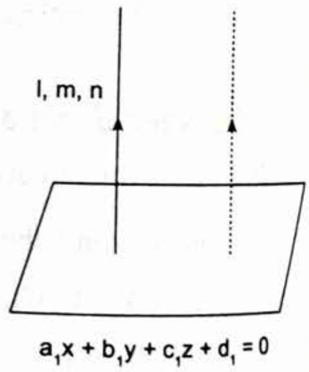
c) If a line is parallel to the plane, then the line and the line normal to the plane are perpendicular to each other.

$\therefore a_1l + b_1m + c_1n = 0$



d) If a line is perpendicular to the plane, then the line and the line normal to the plane are parallel

$\therefore \quad \frac{a_1}{l} = \frac{b_1}{m} = \frac{c_1}{n}$



Length of perpendicular from a given point on a given plane

Let the equation of the plane in the normal form be

$$lx + my + nz = p$$

and let $P(x_1, y_1, z_1)$ be the given point. ……(i)

The equation of the plane parallel to the given plane is

$$lx + my + nz = p_1$$

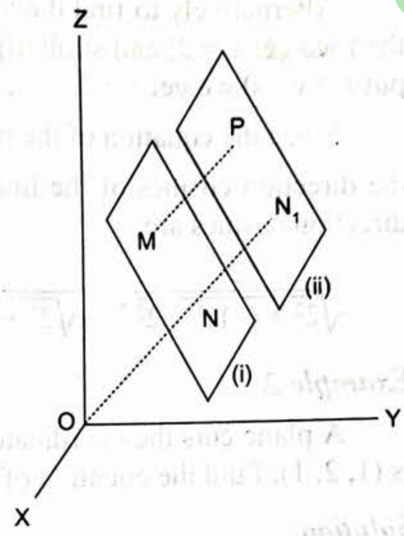
If it passes through $P(x_1, y_1, z_1)$, then

$$lx_1 + my_1 + nz_1 = p_1 \quad \text{……(ii)}$$

Let O be the origin, then the distance between the two planes (i) and (ii) is the difference between the two normals ON_1 and ON i.e. p_1 and p.

If PM be perpendicular from P on the given plane then

$$PM = NN_1 = ON_1 - ON$$
$$= p_1 - p$$
$$= lx_1 + my_1 + nz_1 - p$$



This is the required length of the perpendicular.

If the point P and the origin O lie on the same side of the given plane, it will be seen that

$$PM = N_1N = ON - ON_1$$
$$= p - p_1$$
$$= p - (lx_1 + my_1 + nz_1)$$

Thus $PM = lx_1 + my_1 + nz_1 - p$

or, $p - lx_1 - my_1 - nz_1$

according as the points P and O lie on the opposite side or same side of the plane.

Cor. If the equation of the given plane be $Ax + By + Cz + D = 0$ and the given point be $P(x_1, y_1, z_1)$, we change the given equation of the plane to the normal form which is

$$\frac{A}{\sqrt{\Sigma A^2}}x + \frac{B}{\sqrt{\Sigma A^2}}y + \frac{C}{\sqrt{\Sigma A^2}}z = \frac{-D}{\sqrt{\Sigma A^2}}$$

Hence the length of the perpendicular from the given point $P(x_1, y_1, z_1)$ will be

$$\pm\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{\Sigma A^2}}$$

This sign of radical is to be taken a positive or negative according as D is positive or negative.

Worked Out Examples

Example 1

Find the intercepts made on the coordinate axes by the plane $2x - y + 2z = 4$. Find also the direction cosines of the line normal to this plane.

Solution:

To reduce the equation of the plane $2x - y + 2z = 4$ to the intercept form, we divide each term by 4, then we get

$$\frac{2x}{4} - \frac{y}{4} + \frac{2z}{4} = 1 \quad \text{or,} \quad \frac{x}{2} - \frac{y}{4} + \frac{z}{2} = 1$$

Therefore the intercepts on the coordinate axes are 2, –4, 2.

Alternatively to find the intercepts on the x-axis, put $y = z = 0$ in the equation of the plane, then we get $x = 2$, and similarly for intercepts on y and z-axis, put $z = x = 0$, we get $y = -4$ and put $x = y = 0$ we get $z = 2$.

Since the equation of the plane is $2x - y + 2z = 4$

the direction cosines of the line normal to the plane are proportional to 2, −1, 2. Therefore its direction cosines are

$$\frac{2}{\sqrt{2^2+(-1)^2+2^2}}, \frac{-1}{\sqrt{2^2+(-1)^2+2^2}}, \frac{2}{\sqrt{2^2+(-1)^2+2^2}} \quad \text{i.e.} \quad \frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$$

Example 2

A plane cuts the coordinate axes at the points A, B, C and the centroid of the triangle ABC is (1, 2, 1). Find the equation of the plane.

Solution:

Let the coordinates of A, B, C be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ then by definition of the centroid, we have

$$\frac{a+0+0}{3} = 1, \quad \frac{0+b+0}{3} = 2, \quad \frac{0+0+c}{3} = 1$$

which give $a = 3, b = 6, c = 3$.

Hence the equation of the plane cutting off intercepts 3, 6, 3 on the coordinate axes is

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{3} = 1 \quad \text{or,} \quad 2x + y + 2z = 6$$

Example 3

Reduce the equation $x + 2y + 2z = 9$ to the normal form and also determine the direction cosines of the normal and the length of the perpendicular to it from the origin.

Solution:

The given equation of the plane is $x + 2y + 2z = 9$ ……(i)

To reduce it to the normal form, we divide both sides by

$\sqrt{1^2+2^2+2^2}$ i.e. 3 and thus the equation reduces to

$$\frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = \frac{9}{3}$$

$$\text{or,} \quad \frac{x}{3} + \frac{2y}{3} + \frac{2z}{3} = 3 \qquad \text{……(ii)}$$

Equation (ii) is the required equation of the normal form. On comparing equation (ii) with $lx + my + nz = p$ we have $l = \frac{1}{3}$, $m = \frac{2}{3}$, $n = \frac{2}{3}$ and $p = 3$.

∴ direction cosines of the normal to the plane $\frac{1}{2}, \frac{2}{3}, \frac{2}{3}$ and the length of the perpencicular from origin = 3.

Example 4

Find the equation of the plane passing through the points (0, 1, 1), (1, 1, 2), (–1, 2, –2).

Solution:

The required equation of the plane is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 2 & -2 & 1 \end{vmatrix} = 0$$

which when simplified reduces to

$$x - 2y - z + 3 = 0$$

Alternative method

Let the equation of a plane be

$$Ax + By + Cz + D = 0 \quad ......(i)$$

Since the plane passes through (0, 1, 1), so

$$A(x - 0) + B(y - 1) + C(z - 1) = 0$$

or, $Ax + B(y - 1) + C(z - 1) = 0 \quad ......(ii)$

Since it passes through (1, 1, 2) and (–1, 2, –2), so

$$A.1 + B(1 - 1) + C(2 - 1) = 0$$

and $A.(-1) + B(2 - 1) + C(-2 - 1) = 0$

i.e. $A + 0.B + C = 0$

$$-A + B - 3C = 0$$

Solving these equations we get

$$\frac{A}{0-1} = \frac{B}{-1+3} = \frac{C}{1-0} = k \text{ (say)}$$

which gives $A = -k$, $B = 2k$, $C = k$.

Substituting these values in equation (ii) we get

$$-kx + 2k(y - 1) + k(z - 1) = 0$$

or, $x - 2y - z + 3 = 0$

which is the required equation of the plane.

Example 5

Show that the four points (0, 1, 1) (1, 1, 2), (–1, 2, –2) and (2, 1, 3) are coplanar.

Solution:

As in above example we can show that the equation of the plane through (0, 1, 1), (1, 1, 2) and (–1, 2, –2) is $x - 2y - z + 3 = 0$

It is evident that the point (2, 1, 3) lies on this plane as the coordinates of this point satisfy the equation of the plane. Hence four points are coplanar.

Example 6

Find the equation of the plane through the intersection of the planes

$2x + 3y + 10z = 8$ and $2x - 3y + 7z = 2$

and perpendicular to the plane $3x - 2y + 4z = 5$.

Solution:

Any plane through the intersection of the plane $2x + 3y + 10z - 8 = 0$ and $2x - 3y + 7z - 2 = 0$ is

$$(2x + 3y + 10z - 8) + \lambda(2x - 3y + 7z - 2) = 0 \quad \ldots\ldots(i)$$

or, $(2 + 2\lambda)x + (3 - 3\lambda)y + (10 + 7\lambda)z - (8 + 2\lambda) = 0$

This plane is perpendicular to the plane $3x - 2y + 4z = 5$ if

$$3(2 + 2\lambda) - 2(3 - 3\lambda) + 4(10 + 7\lambda) = 0$$

i.e. if $\lambda = -1$

Substituting this value of λ in (i) we get the required equation as $2y + z = 2$

Example 7

Find the equation of the plane through (1, 2, 3) and parallel to the plane $3x - 4y + 5z = 0$

Solution:

Equation of any plane parallel to the given plane

$3x - 4y + 5z = 0$ is $3x - 4y + 5z + k = 0 \quad \ldots\ldots(i)$

It will pass through (1, 2, 3), if $3.1 - 4.2 + 5.3 + k = 0$

or, $k = -10$

Substituting the value of k in (i), we get $3x - 4y + 5z - 10 = 0$

which is the required equation of the plane.

Example 8

The foot of the perpendicular drawn from (1, 2, 3) to the plane is (2, –1, 1). Find the equation of the plane.

Solution:

The foot of the perpendicular (2, –1, 1) is a point on the plane.

The equation of the plane through (2, –1, 1) is

$$A(x - 2) + B(y + 1) + C(z - 1) = 0 \quad \ldots\ldots(i)$$

The dirs normal to the plane (i) are A, B, C

Again the dirs of the line joining the points (2, –1, 1) and (1, 2, 3) and $1 - 2, 2 + 1, 3 - 1$

i.e. –1, 3, 2.

Since the line normal to the plane is parallel to the line joining the points (2, –1, 1) and (1, 2, 3), so

$$\frac{A}{-1}=\frac{B}{3}=\frac{C}{2}$$

$$\frac{A}{-1}=\frac{B}{3}=\frac{C}{2}=k \text{ (suppose)}$$

$\therefore\quad A=-k,\quad B=3k,\quad C=2k$

Substituting the values of A, B and C in (i), we have

$$-k(x-2)+3k(y+1)+2k(z-1)=0$$

$$\Rightarrow\quad -x+2+3y+3+2z-2=0$$

$$\Rightarrow\quad x-3y-2z=3$$

Example 9

Find the equation of the plane through the point (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane $x+2y+2z=5$.

Solution:

Equation of the plane through the point (–1, 1, 1) is

$$A(x+1)+B(y-1)+C(z-1)=0 \qquad \text{......(i)}$$

It will pass through the point (1, –1, 1) if

$$A(1+1)+B(-1-1)+C(1-1)=0$$

$$2A-2B+0C=0 \qquad \text{......(ii)}$$

Again (i) will be perpendicular to the given plane

$$x+2y+2z=5$$

if $\quad A+2B+2C=0 \qquad$ ......(iii)

Solving (ii) and (iii), we get

$$\frac{A}{-4-0}=\frac{B}{0-4}=\frac{C}{4+2}$$

or, $\quad \dfrac{A}{2}=\dfrac{B}{2}=\dfrac{C}{-3}=k \text{ (say)}$

which gives A = 2k, B = 2k, C = –3k

Substituting the values of A, B and C in (i) we get

$$2(x+1)+2(y-1)-3(z-1)=0$$

or, $\quad 2x+2y-3z+3=0 \qquad$ which is the required equation of a plane.

Example 10

Find the equation of the plane through (–2, 3, 4) and perpendicular to the planes $2x+3y+4z=6$, $3x+2y+2z=8$.

Solution:

Equation of a plane through the given point (–2, 3, 4) is

$$A(x+2)+B(y-3)+C(z-4)=0 \quad \ldots\ldots(i)$$

If plane (i) is perpendicular to each of the given planes

$$2x+3y+4z=6 \quad \ldots\ldots(ii)$$

and $$3x+2y+2z=8 \quad \ldots\ldots(iii)$$

then $$2A+3B+4C=0 \quad \ldots\ldots(iv)$$

and $$3A+2B+2C=0 \quad \ldots\ldots(v)$$

Solving (iv) and (v) we get

$$\frac{A}{6-8}=\frac{B}{12-4}=\frac{C}{4-9}$$

or, $$\frac{A}{-2}=\frac{B}{8}=\frac{C}{-5}=k\text{ (say)}$$

which gives $A=-2k$, $B=8k$ and $C=-5k$

Substituting the values of A, B and C in (i) we get

$$-2(x+2)+8(y-3)-5(z-4)=0$$

or, $2x-8y+5z+8=0$

which is the required equation of the plane.

Example 11

Find the equation of the plane through (3, 2, 1) and perpendicular to the line joining (–5, 3, 7) and (2, –4, 5).

Solution:

Equation of a plane through (3, 2, 1) is

$$A(x-3)+B(y-2)+C(z-1)=0 \quad \ldots\ldots(i)$$

The direction cosines of the line joining the points (–5, 3, 7) and (2, –4, 5) are proportional to 7, –7, –2.

The plane (i) is perpendicular to this line, if any normal to the plane is parallel to this line. But direction cosines of the normal to the plane are proportional to A, B, C and therefore condition for parallelism gives

$$\frac{A}{7}=\frac{B}{-7}=\frac{C}{-2}=k\text{ (say)}$$

Then $A=7k$, $B=-7k$ and $C=-2k$

Substituting the values of A, B and C in (i) we get,

$$7(x-3)-7(y-2)-2(z-1)=0$$

or, $7x-7y-2z-5=0$

which is the required equation of the plane.

Example 12

Find the length of the perpendicular from the point (2, 3, 4) on the plane

$3x - 2y + 6z + 4 = 0$

Solution:

The equation of the given plane is $3x - 2y + 6z + 4 = 0$

The length of perpendicular from the given point (2, 3, 4) is then

$$\frac{3(2) - 2(3) + 6(4) + 4}{\sqrt{3^2 + 2^2 + 6^2}} = \frac{28}{7} = 4 \text{ unit}$$

(Note that the distance is always positive)

Example 13

Find the distance between the parallel planes $2x - 2y + z + 1 = 0$ and $4x - 4y + 2z + 3 = 0$.

Solution:

The given planes are

$2x - 2y + z + 1 = 0$ and $4x - 4y + 2z + 3 = 0$

They can be written in the normal form as

$$\frac{2x - 2y + z + 1}{\sqrt{2^2 + (-2)^2 + 1^2}} = 0 \quad \text{and} \quad \frac{4x - 4y + 2z + 3}{\sqrt{4^2 + (-4)^2 + 2^2}} = 0$$

Therefore the length of perpendicular from the origin to these planes are $\frac{1}{3}$ and $\frac{1}{2}$ respectively, and these planes are on the same side of the origin. Therefore, the required distance between them $= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

EXERCISE

1. a) Find the intercepts made by the plane $2x + 3y + 4z = 12$ on the coordinate axes.
   b) Reduce the equation $4x + 8y + 8z = 9$ to the normal form and also determine the direction cosines of the normal and the length of the perpendicular to it from the origin.
2. a) Find the equation of a plane which makes intercepts 2, 3, 4 on x-axis, y-axis and z-axis respectively.
   b) Find the equation of the plane which makes equal intercepts on the axes and passes through the point (2, 3, 4).
3. a) Find the equation of the plane passing through the points (1, 1, 0), (–2, 2, –1) and (1, 2, 1)

b) Show that the four points (0, –1, –1), (4, 5, 1), (3, 9, 4) and (–4, 4, 4) are coplanar.

4. a) Find the equation of the plane through the point (3, –4, 5) and parallel to the plane $3x - 4y + 5z = 7$.

b) Show that the equation of the plane through $(\alpha, -\beta, \gamma)$ and parallel to the plane $ax + by + cz = 0$ is $ax + by + cz = a\alpha + b\beta + c\gamma$.

5. Find the angle between the two planes

i) $x + 2y + z + 7 = 0$ and $2x + y - z + 13 = 0$

ii) $x + 2y + 3z = 6$ and $3x - 3y + z = 1$

6. Show that the plane $2x + 3y - 4z = 3$ is parallel to the plane $10x + 15y - 20z = 12$ and perpendicular to the plane $3x + 2y + 3z = 5$.

7. a) Find the equation of the plane through the point (1, 2, 3) and normal to the planes $x - y - z = 5$ and $2x - 5y - 3z = 7$.

b) Find the equation of the plane through the point (2, 1, 4) and perpendicular to each of the planes $9x - 7y + 6z + 48 = 0$ and $x + y + z = 0$.

8. a) Find the equation of the plane through P(a, b, c) and perpendicular to OP.

b) Find the equation of the plane through the point (2, –3, 1) and perpendicular to the line joining the two points (3, 4, –1) and (2, –1, 5).

c) The foot of the perpendicular drawn from the origin to a plane is (3, –4, 2). Find the equation of the plane.

9. a) Find the equation of the plane through the point (2, 2, 1) and (9, 3, 6) and normal to the plane $2x + 6y + 6z = 9$.

b) Find the equation of the plane through the points (–1, 1, –1) and (6, 2, 1) and perpendicular to the plane $2x + y + z = 5$.

10. a) Find the equation of the plane through the intersection of the planes $x - 2y + 3z + 10 = 0$ and $3x + 5y - 2z - 14 = 0$ and (i) passing through the point (–1, 4, 3) (ii) parallel to the x-axis.

b) Find the equation of the plane through the intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$.

11. Find the length of the perpendicular

i) from the point (2, 5, 7) to the plane $6x + 6y + 3z = 11$

ii) from the origin on the plane $3x - 2y + 6z = 17$.

12. Find the distance between the following planes

i) $x - y + 2z - 4 = 0$ and $2x - 2y + 4z + 5 = 0$

ii) $3x + 2y - 6z + 1 = 0$ and $6x + 4y - 12z + 9 = 0$

13. A variable plane is at a constant distance $3p$ from the origin and meets the axes in the points A, B, C. Prove that the locus of the centroid of the triangle ABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

A. Prepare a model showing the coordinate planes and coordinate axes. Present it in the classroom.

a) What are the equations of the coordinate planes?

b) What are the direction cosines of the lines normal to coordinate planes.

Answers

1. a) 6, 4, 3 b) $\frac{x}{3}+\frac{2y}{3}+\frac{2z}{3}=\frac{3}{4};\frac{1}{3},\frac{2}{3},\frac{2}{3};\frac{3}{4}$ 2. a) $6x+4y+3z=12$ b) $x+y+z=9$

3. a) $2x+3y-3z=5$ 4. a) $3x-4y+5z=50$ 5. i) $\frac{\pi}{3}$ ii) $\frac{\pi}{2}$

7. a) $2x-y+3z=9$ b) $13x+3y-16z+35=0$

8. a) $ax+by+cz=a^2+b^2+c^2$ b) $x+5y-6z+19=0$ c) $3x-4y+2z=29$

9. a) $3x+4y-5z=9$ b) $x+3y-5z=7$

10. a) i) $3x+4y-z=10$ ii) $y-z=4$b) $x+7y+13z+96=0$ 11. i) $5\frac{7}{9}$ units ii) $2\frac{3}{7}$ units

12. $\frac{13}{2\sqrt{6}}$ units ii) $\frac{1}{2}$ unit

Additional Questions

1. A plane meets the coordinate axes in A, B, C. If (α, β, γ) be the coordinates of the centroid of the triangle ABC, find the equation of the plane.
2. A plane passes through the middle point of the join of A(–2, 5, 1) and B(6, 1, 5) and perpendicular to it. Find the equation of the plane.
3. PA and PB are drawn perpendicular from $P(\alpha, \beta, \gamma)$ to the coordinate planes $x = 0$ and $y = 0$. Show that the equation of the plane OAB is $\frac{x}{\alpha}+\frac{y}{\beta}-\frac{z}{\gamma}=0$
4. Show that $ax + by + d = 0$ represents a plane parallel to z-axis. Find the equation of the plane through the points (1, 2, 3) and (7, 5, 6) parallel to x-axis.

Answers

1. $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$ 2. $2x-y+z=4$ 4. $y-z+1=0$

Chapter 11
Product of Vectors

Products of Vectors

The multiplication operation between two vectors can be performed by the following two ways:

(a) Scalar product or Dot product

(b) Vector product or Cross product

Scalar Product of Two Vectors

Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be any two plane vectors, then the scalar product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \cdot \vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$$

Again, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ are two space vectors then the scalar product of two vectors is defined by

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Alternative Definition

The scalar product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \cdot \vec{b}$ is defined by

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = ab\cos\theta$$

where a and b are the magnitudes of the vectors $\vec{a}$ and $\vec{b}$ and, θ, the angle between the two vectors.

Vector Product of Two Vectors

Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be two vectors in the Cartesian plane (*i.e.*, *xy* plane) then the vector product of the two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is defined by

$$\vec{a} \times \vec{b} = (a_1 b_2 - a_2 b_1)\ \vec{k}$$

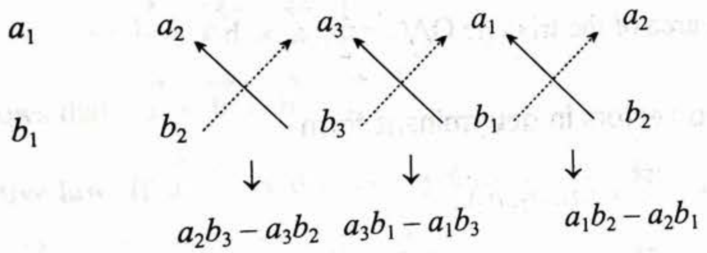
where $\vec{k}$ is the standard unit vector along the positive z-axis.

In other words, $\vec{a} \times \vec{b} = (a_1 b_2 - a_2 b_1)\ \vec{k}$
$= (a_1 b_2 - a_2 b_1)\ (0, 0, 1)$
$= (0, 0, a_1 b_2 - a_2 b_1)$

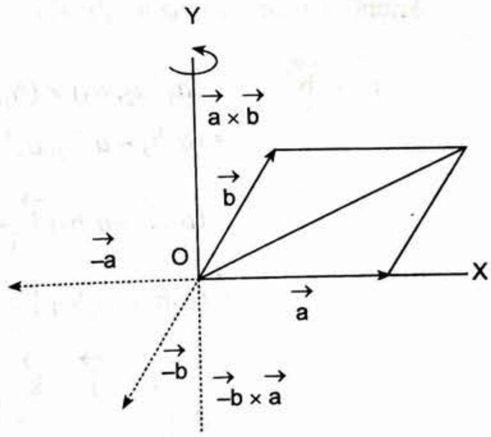
The vector product of two space vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ denoted by $\vec{a} \times \vec{b}$ is a vector defined by

$$\vec{a} \times \vec{b} = (a_1, a_2, a_3) \times (b_1, b_2, b_3)$$
$$= (a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1)$$

Later, we see that this vector is perpendicular to the plane of the vectors $\vec{a}$ and $\vec{b}$. The components of the vector $\vec{a} \times \vec{b}$ can be obtained by the following rule.



Alternative Definition



The vector product of two vectors $\vec{a}$ and $\vec{b}$ is a vector which is perpendicular to the plane of the vectors $\vec{a}$ and $\vec{b}$ and whose magnitude is $ab \sin\theta$ where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$ and θ, the angle between $\vec{a}$ and $\vec{b}$, the sense of rotation being from $\vec{a}$ to $\vec{b}$ as in a right handed screw.

Symbolically,

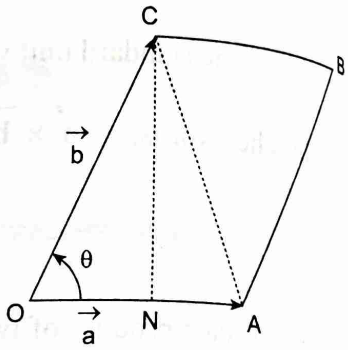
$$\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \vec{n}$$

where $\vec{n}$ is the unit vector normal to $\vec{a}$ and $\vec{b}$.

i) Geometrical Interpretation

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OC} = \vec{b}$ and $\angle AOC = \theta$.

Draw a parallelogram OABC with OA = a and OC = b as its adjacent sides. From C, draw CN perpendicular to OA.



Now, area of the parallelogram OABC

$= OA \times CN$

$= OA \times OC \sin\theta$

$= ab \sin\theta$

Also $|\vec{a} \times \vec{b}| = ab \sin\theta$

Hence the vector product of two vectors $\vec{a}$ and $\vec{b}$ is a vector normal to the plane of the vectors $\vec{a}$ and $\vec{b}$ and whose magnitude is equal to the area of the parallelogram with magnitudes of $\vec{a}$ and $\vec{b}$ as its adjacent sides.

Cor. 1. Since the area of the triangle is half the area of the parallelogram, on the same base having the same altitude, so area of the triangle OAC $= \frac{1}{2}|\vec{a} \times \vec{b}|$

Vector product of two vectors in determinant form

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = (a_1, a_2, a_3)$

and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k} = (b_1, b_2, b_3)$ be two space vectors.

Then,

$$\begin{aligned}\vec{a} \times \vec{b} &= (a_1, a_2, a_3) \times (b_1, b_2, b_3)\\ &= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)\\ &= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}\\ &= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}\\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\end{aligned}$$

ii) Properties of Vector Product

Vector products involving two or more vectors enjoy the following properties:

1. The vector product of two vectors is not **commutative**.

i.e. If $\vec{a}$ and $\vec{b}$ be two vectors then $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad (R_1 \leftrightarrow R_2)$$

$$= -\vec{b} \times \vec{a}$$

which shows that $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

2. **Distributive law:** If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors, then

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$ be three space vectors. Then,

$$\vec{b} + \vec{c} = (b_1, b_2, b_3) + (c_1, c_2, c_3)$$
$$= (b_1 + c_1, b_2 + c_2, b_3 + c_3)$$

$$\text{Now, } \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(Each element of R_3 being the sum of two terms)

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$\therefore \quad \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

3. For any scalar n,

$$(n\vec{a}) \times \vec{b} = n(\vec{a} \times \vec{b}) = \vec{a} \times n\vec{b}$$

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$.

Then, $n\vec{a} = (na_1, na_2, na_3)$

$$n\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ na_1 & na_2 & na_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = n \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= n(\vec{a} \times \vec{b})$$

Similarly, $\vec{a} \times n\vec{b} = n(\vec{a} \times \vec{b})$

$$\therefore n\vec{a} \times \vec{b} = n(\vec{a} \times \vec{b}) = \vec{a} \times n\vec{b}$$

4. The vector product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

Then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

$$(\vec{a} \times \vec{b}).\vec{a} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).(a_1, a_2, a_3)$$

$$= (a_2b_3 - a_3b_2)a_1 + (a_3b_1 - a_1b_3).a_2 + (a_1b_2 - a_2b_1)a_3$$

$$= 0$$

$\therefore$ $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Similarly, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{b}$.

iii) Expression for sin θ

Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$.

then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

and $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = ab \cos\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$.

$$\text{Now, } (\vec{a} \times \vec{b})^2 = (\vec{a} \times \vec{b}).(\vec{a} \times \vec{b})$$
$$= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= a^2b^2 - (\vec{a} \cdot \vec{b})^2$$

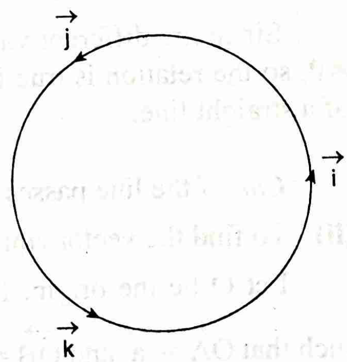
$$\text{or} \quad |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$
$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$$
$$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta$$

$$\text{or,} \quad |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$$

$$\therefore \quad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

iv) Unit Vectors

Let $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$ be three mutually orthogonal unit vectors forming a right handed screw system.



$$\text{Then,} \quad \vec{i} \times \vec{i} = (1, 0, 0) \times (1, 0, 0)$$
$$= (0, 0, 0) = 0$$

$$\text{Similarly,} \quad \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

$$\text{Again} \quad \vec{i} \times \vec{j} = (1, 0, 0) \times (0, 1, 0)$$
$$= (0 - 0, 0 - 0, 1 - 0)$$
$$= (0, 0, 1) = \vec{k}$$

$$\text{Similarly,} \quad \vec{j} \times \vec{k} = \vec{i} \text{ and } \vec{k} \times \vec{i} = \vec{j}$$

$$\text{Further,} \quad \vec{j} \times \vec{i} = (0, 1, 0) \times (1, 0, 0)$$
$$= (0, 0, -1) = -(0, 0, 1) = -\vec{k}$$

Similarly, $\vec{k} \times \vec{j} = -\vec{i}$ and $\vec{i} \times \vec{k} = -\vec{j}$

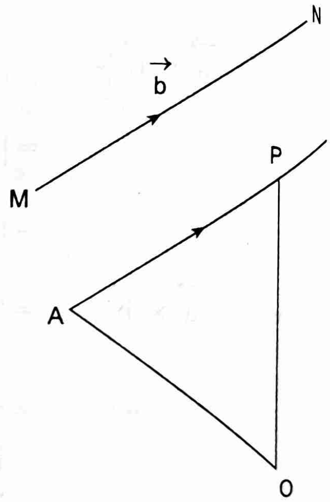
The value of a cross product of two mutually perpendicular unit vectors can be remembered by using the diagram circle.

Vector Equation of a Straight Line

(i) To find the vector equation of a straight line through a given point and parallel to a given vector.

Let O be the origin, A be the given point and $\vec{MN} = \vec{b}$ be the given vector.

Let P be any point on the line through A parallel to the given vector $\vec{b}$.



Let $\vec{OP} = \vec{r}$

Since $\vec{AP}$ and $\vec{MN}$ are parallel vectors, so $\vec{AP} = t\,\vec{MN}$ where t is a scalar.

Using vector additions,

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$\vec{r} = \vec{a} + t\vec{b}$$

Since the different values of t will give the position vectors of different points on the line AP, so the relation is true for any point on the line AP. Hence, it represents the vector equation of a straight line.

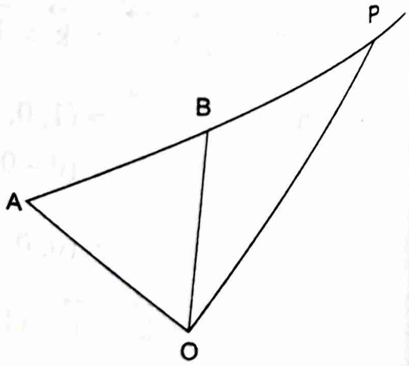
Cor. If the line passes through the origin, then $\vec{a} = (0, 0)$, so $\vec{r} = t\,\vec{b}$.

(ii) To find the vector equation of a straight line passing through two given points :

Let O be the origin. Let A and B be two given points such that $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Then, $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

Let P be any point on the line through A and B such that $\vec{OP} = \vec{r}$.



Since $\vec{AP}$ and $\vec{AB}$ are parallel vectors so,

$$\vec{AP} = t\,\vec{AB} = t(\vec{b} - \vec{a}) \quad \text{where } t \text{ is a scalar.}$$

Using vector addition

$$\vec{OP} = \vec{OA} + \vec{AP}$$
$$= \vec{a} + t(\vec{b} - \vec{a})$$
$$= (1 - t)\vec{a} + t\vec{b}$$

which represents the vector equation of a straight line through A and B.

Worked Out Examples

Example 1.

If $\vec{a} = \vec{i} + \vec{j} - 3\vec{k}$ and $\vec{b} = -\vec{i} - 2\vec{j} - 3\vec{k}$ are any two vectors :

(i) find $\vec{a} \times \vec{b}$

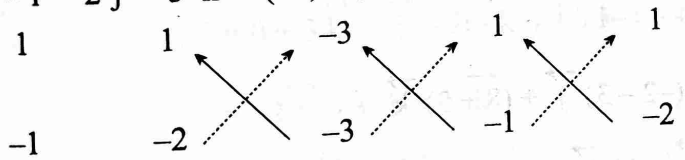
(ii) find a unit vector perpendicular to $\vec{a}$ and $\vec{b}$

(iii) find the sine of an angle between them.

Solution:

(i) $\vec{a} = \vec{i} + \vec{j} - 3\vec{k} = (1, 1, -3)$

$\vec{b} = -\vec{i} - 2\vec{j} - 3\vec{k} = (-1, -2, -3)$



$$\vec{a} \times \vec{b} = (1, 1, -3) \times (-1, -2, -3)$$
$$= (-3 - 6, 3 + 3, -2 + 1) = (-9, 6, -1)$$
$$= -9\vec{i} + 6\vec{j} - \vec{k}$$

$$|\vec{a} \times \vec{b}| = |-9\vec{i} + 6\vec{j} - \vec{k}|$$
$$= \sqrt{(-9)^2 + (6)^2 + (-1)^2}$$
$$= \sqrt{118}$$

(ii) Unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-9\vec{i} + 6\vec{j} - \vec{k}}{\sqrt{118}}$$

(iii) $|\vec{a}| = |\vec{i} + \vec{j} - 3\vec{k}| = \sqrt{1+1+9} = \sqrt{11}$

$|\vec{b}| = |-\vec{i} - 2\vec{j} - 3\vec{k}| = \sqrt{1+4+9} = \sqrt{14}$

Now, $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|a|\,|\vec{b}|} = \dfrac{\sqrt{118}}{\sqrt{11}\sqrt{14}} = \sqrt{\dfrac{118}{154}}$

$\therefore \sin\theta = \sqrt{\dfrac{59}{77}}$.

Example 2.

Prove that the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$ is $\dfrac{1}{\sqrt{155}}(-3\vec{i} + 5\vec{j} + 11\vec{k})$ and the sine of an angle between them is $\sqrt{\dfrac{155}{156}}$.

Solution :

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k} = (2, -1, 1)$ and, $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k} = (3, 4, -1)$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$

$= (1-4)\vec{i} - (-2-3)\vec{j} + (8+3)\vec{k}$

$= -3\vec{i} + 5\vec{j} + 11\vec{k}$

(ii) $|\vec{a} \times \vec{b}| = |-3\vec{i} + 5\vec{j} + 11\vec{k}|$

$= \sqrt{9+25+121}$

$= \sqrt{155}$

Unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$= \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

$= \dfrac{-3\vec{i} + 5\vec{j} + 11\vec{k}}{\sqrt{155}}$

Again, $|\vec{a}| = |2\vec{i} - \vec{j} + \vec{k}| = \sqrt{4+1+1} = \sqrt{6}$

and, $|\vec{b}| = |3\vec{i} + 4\vec{j} - \vec{k}| = \sqrt{9+16+1} = \sqrt{26}$

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}}$$

$$\therefore \quad \sin\theta = \sqrt{\frac{155}{156}}.$$

Example 3

Find the area of the parallelogram determined by the vectors $\vec{i} + \vec{j} + \vec{k}$ and $-2\vec{i} + 3\vec{j} + \vec{k}$.

Solution:

Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + 3\vec{j} + \vec{k}$

Now $\vec{a} \times \vec{b} = (\vec{i} + \vec{j} + \vec{k}) \times (-2\vec{i} + 3\vec{j} + \vec{k})$

$$= -2\vec{i} \times \vec{i} + 3\vec{i} \times \vec{j} + \vec{i} \times \vec{k} - 2\vec{j} \times \vec{i} + 3\vec{j} \times \vec{j} + \vec{j} \times \vec{k} - 2\vec{k} \times \vec{i} + 3\vec{k} \times \vec{j} + \vec{k} \times \vec{k}$$

$$= 0 + 3\vec{k} - \vec{j} - 2(-\vec{k}) + 0 + \vec{i} - 2\vec{j} - 3\vec{i} + 0$$

$$= -2\vec{i} - 3\vec{j} + 5\vec{k}$$

$$|\vec{a} \times \vec{b}| = |-2\vec{i} - 3\vec{j} + 5\vec{k}| = \sqrt{4+9+25} = \sqrt{38}$$

Area of the parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{38}$ sq. units.

Example 4

Find the area of a parallelogram whose diagonals are represented by the vectors $-3\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} + 3\vec{k}$.

Solution:

Let $\vec{d_1} = -3\vec{i} - 2\vec{j} + \vec{k} = (-3, -2, 1)$

and $\vec{d_2} = \vec{i} + 2\vec{j} + 3\vec{k} = (1, 2, 3)$

$$\vec{d_1} \times \vec{d_2} = (-3, -2, 1) \times (1, 2, 3)$$
$$= (-6-2, 1+9, -6+2)$$
$$= (-8, 10, -4)$$

$$|\vec{d_1} \times \vec{d_2}| = \sqrt{(-8)^2 + (10)^2 + (-4)^2}$$
$$= \sqrt{64 + 100 + 16}$$
$$= \sqrt{180} = 6\sqrt{5}$$

Area of the parallelogram

$$= \frac{1}{2}|\vec{d_1} \times \vec{d_2}| = \frac{1}{2}(6\sqrt{5})$$
$$= 3\sqrt{5} \text{ sq. units.}$$

Example 5

Find the area of the triangle formed by the points A (1, 1, 1), B (1, 2, 3) and C (2, 3, 1).

Solution:

Let O be the origin. Then

$$\vec{OA} = (1, 1, 1)$$
$$\vec{OB} = (1, 2, 3) \text{ and } \vec{OC} = (2, 3, 1)$$

Now, $\vec{AB} = \vec{OB} - \vec{OA}$

$$= (1, 2, 3) - (1, 1, 1) = (0, 1, 2)$$

$$\vec{AC} = \vec{OC} - \vec{OA}$$
$$= (2, 3, 1) - (1, 1, 1) = (1, 2, 0)$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= (0-4)\vec{i} - (0-2)\vec{j} + (0-1)\vec{k}$$
$$= -4\vec{i} + 2\vec{j} - \vec{k}$$

$$|\vec{AB} \times \vec{AC}| = |-4\vec{i} + 2\vec{j} - \vec{k}|$$
$$= \sqrt{16+4+1}$$
$$= \sqrt{21}$$

Area of the triangle $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

$$= \frac{1}{2}\sqrt{21} = \frac{\sqrt{21}}{2} \text{ sq. units.}$$

Example 6

Prove that $(\vec{a} . \vec{b})^2 + (\vec{a} \times \vec{b})^2 = a^2b^2$

Solution:

$$(\vec{a} . \vec{b})^2 = (ab \cos\theta)^2 = a^2b^2 \cos^2\theta$$

and $$(\vec{a} \times \vec{b})^2 = (\vec{a} \times \vec{b}) . (\vec{a} \times \vec{b})$$

$$= (ab \sin\theta\, \vec{n}). (ab \sin\theta\, \vec{n})$$

where $\vec{n}$ is the unit vector perpendicular to $\vec{a}$ and $\vec{b}$

$\therefore \quad (\vec{a} \times \vec{b})^2 = a^2b^2 \sin^2\theta$

Now, $(\vec{a} . \vec{b})^2 + (\vec{a} \times \vec{b})^2 = a^2b^2\cos^2\theta + a^2b^2 \sin^2\theta$

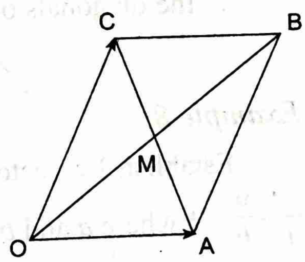
$$= a^2b^2 (\cos^2\theta + \sin^2\theta) = a^2b^2$$

$\therefore \quad (\vec{a} . \vec{b})^2 + (\vec{a} \times \vec{b})^2 = a^2b^2.$

Example 7

Show that the diagonals of a rhombus bisect each other at right angles (use vector method).

Solution :



Let $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OC} = \vec{b}$

Then, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OA} + \overrightarrow{OC} = \vec{a} + \vec{b}$

and $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}$

Now, the vector equation of the line OB is

$$\vec{r} = t(\vec{a} + \vec{b}) \ldots\ldots \text{(i)}$$

where t is a scalar.

Again the vector equation of the straight line AC is

$$\vec{r} = (1-s)\,\vec{a} + s\,\vec{b} \ldots\ldots \text{(ii)}$$

where s is a scalar.

If two diagonals OB and AC meet at M, then for M, we have

$$t(\vec{a}+\vec{b}) = (1-s)\vec{a} + s\vec{b}$$

Equating the coefficients of like vectors, $t = 1 - s$ and $t = s$.

Solving $t = s = \frac{1}{2}$

∴ the position vector of M

i.e. $\vec{OM} = \frac{1}{2}(\vec{a}+\vec{b}) = \frac{1}{2}\vec{OB}$

Also, $\vec{AM} = \vec{AO} + \vec{OM}$

$= -\vec{a} + \frac{1}{2}(\vec{a}+\vec{b})$

$= \frac{1}{2}(\vec{b}-\vec{a}) = \frac{1}{2}\vec{AC}$

∴ $\vec{AM} = \frac{1}{2}\vec{AC}$

Hence the diagonals bisect each other.

Again, $\vec{OB} \cdot \vec{AC} = (\vec{a}+\vec{b}) \cdot (\vec{b}-\vec{a})$

$= \vec{b}^2 - \vec{a}^2$

$= \vec{OC}^2 - \vec{OA}^2$

$= OC^2 - OA^2$

$= 0 \qquad (\because OC = OA)$

So, the diagonals of a rhombus are at right angles.

∴ the diagonals of a rhombus bisect each other at right angles.

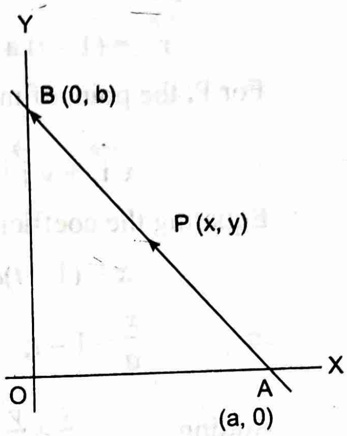
Example 8

Establish by vector method the equation of the straight line in the double intercept form $\frac{x}{a} + \frac{y}{b} = 1$ where a and b have their usual meanings.

Solution :

Let a straight line AB meet two mutually perpendicular straight lines OX and OY at A and B such that OA = a and OB = b.

Take O as the origin. Then the co-ordinates of A and B are $(a, 0)$ and $(0, b)$ respectively. Let P(x, y) be a point on the line AB.



Then, $\overrightarrow{AP} = (x - a, y)$

and, $\overrightarrow{AB} = (0 - a, b - 0) = (-a, b)$

Since $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are parallel, so

$\overrightarrow{AP} = k\ \overrightarrow{AB}$ where k is a scalar.

or, $(x - a, y) = (-ka, kb)$

By the definition of equal vectors,

$x - a = -ka$

or, $x = -ka + a$

$\therefore \dfrac{x}{a} = -k + 1$ ....(i)

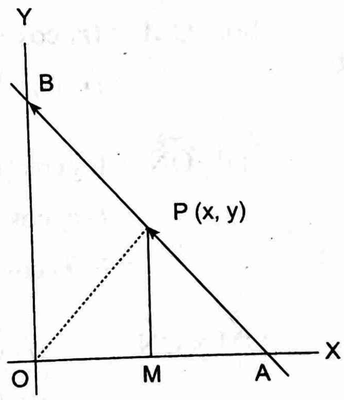
Also, $y = kb$

$\therefore \dfrac{y}{b} = k$ ....(ii)

Adding (i) and (ii), $\dfrac{x}{a} + \dfrac{y}{b} = 1.$

Alternative method

Let $\vec{i}$ and $\vec{j}$ be the unit vectors along two mutually perpendicular straight lines OX and OY respectively. Let OA = a and OB = b.



Then $\overrightarrow{OA} = a\ \vec{i}$ and $\overrightarrow{OB} = b\ \vec{j}$.

Let P(x, y) be a point on the line AB. From P, draw PM perpendicular to OA.

Then, $\overrightarrow{OM} = x\ \vec{i}$ and $\overrightarrow{MP} = y\ \vec{j}$

Now, join OP.

By vector addition,

$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = x\ \vec{i} + y\ \vec{j}$ ....(i)

Again, the vector equation of the straight line AB is

$$\vec{r} = (1-t)\,a\,\vec{i} + t\,b\,\vec{j} \quad \ldots\ldots\ldots\text{(ii)}$$

For P, the point of intersection of OP and AB, we have

$$x\vec{i} + y\vec{j} = (1-t)\,a\,\vec{i} + t\,b\,\vec{j}$$

Equating the coefficients of like vectors,

$$x = (1-t)a, \quad y = tb$$

$$\Rightarrow \quad \frac{x}{a} = 1 - t, \quad \frac{y}{b} = t$$

Adding, $\quad \dfrac{x}{a} + \dfrac{y}{b} = 1.$

Application of Vector Product to Plane Trigonometry

As an application of a vector product, we use it to establish the following formula

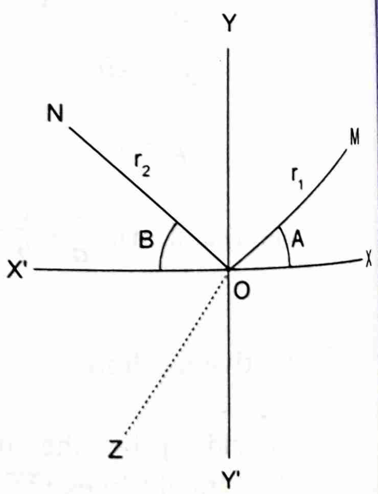
1. sin (A + B) = sin A cos B + cos A sin B

Let XOX' and YOY', the two mutually perpendicular straight lines represent x-axis and y-axis respectively.

Let $\angle XOM = A$ and $\angle NOX' = B$

so that $\angle MON = \pi - (A + B)$

Let OM = r_1 and ON = r_2. Then the co-ordinates of M and N are



$(r_1 \cos A, r_1 \sin A)$

and $(r_2 \cos(\pi - B), r_2 \sin(\pi - B))$

So, $\vec{OM} = (r_1 \cos A, r_1 \sin A)$

$= (r_1 \cos A, r_1 \sin A, 0)$

and $\vec{ON} = (r_2 \cos(\pi - B), r_2 \sin(\pi - B))$

$= (-r_2 \cos B, r_2 \sin B)$

$= (-r_2 \cos B, r_2 \sin B, 0)$

$\vec{OM} \times \vec{ON} = (r_1 \cos A, r_1 \sin A, 0) \times (-r_2 \cos B, r_2 \sin B, 0)$

$= (0, 0, r_1 \cos A\, r_2 \sin B + r_1 \sin A\, r_2 \cos B)$

$= (0, 0, r_1 r_2 \cos A \sin B + r_1 r_2 \sin A \cos B)$

$\therefore |\vec{OM} \times \vec{ON}| = r_1 r_2 (\cos A \sin B + \sin A \cos B)$

Since $\pi - (A + B)$ is the angle between OM and ON, so

$$\sin[\pi - (A + B)] = \frac{|\vec{OM} \times \vec{ON}|}{|\vec{OM}|\,|\vec{ON}|}$$

or $$\sin(A + B) = \frac{r_1 r_2 (\sin B \cos A + \sin A \cos B)}{r_1 r_2}$$

$\therefore$ $$\sin(A + B) = \sin A \cos B + \cos A \sin B.$$

EXERCISE

1. Find the vector and the unit vector perpendicular to each of the following pair of vectors.
   (i) (4, –2, 3) and (5, 1, –4)
   (ii) $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$.

2. (i) If $\vec{a} = 6\vec{i} + \vec{j} - 5\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$ and show by calculation that $\vec{a} \times \vec{b}$ is perpendicular to each of the constituent vectors $\vec{a}$ and $\vec{b}$ of the product.
   (ii) If $\vec{a} = (2, 3, 6)$, $\vec{b} = (3, -6, 2)$ and $\vec{c} = (6, 2, -3)$ show that $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular and that $\vec{a} \times \vec{b} = 7\vec{c}$

3. Find the sine of an angle between the following pair of vectors:
   (i) (2, 1, –4) and (3, –1, –1)
   (ii) $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} - 2\vec{j} + 4\vec{k}$.

4. Show that the unit vector perpendicular to each of the vectors $\vec{i} + 3\vec{j} + 2\vec{k}$ and $2\vec{i} - 4\vec{j} + \vec{k}$ is $\dfrac{11\vec{i} + 3\vec{j} - 10\vec{k}}{\sqrt{230}}$ and the sine of an angle between them is $\sqrt{\dfrac{115}{147}}$.

5. (i) Find the area of the parallelogram determined by the vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$.
   (ii) Prove that the area of the parallelogram whose three of four vertices are (1, 1, 2), (2, –1, 1) and (3, 2, –1) is $5\sqrt{3}$ sq units.
   (iii) Find the area of the parallelogram whose diagonals are represented by the vectors $3\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{i} + \vec{j} - 3\vec{k}$.

6. (i) Find the area of the triangle determined by the vectors $3\vec{i} + 4\vec{j}$ and $-5\vec{i} + 7\vec{j}$.
   (ii) Show that the area of the triangle PQR whose vertices are P (1, 2, 3), Q (3, 4, 5) and R (1, 4, 7) is $2\sqrt{6}$ sq. units.

7. Obtain the sides, angles and the area of the triangle formed by the points whose position vectors are $2\vec{i} - \vec{j} + 3\vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $\vec{i} - \vec{j} - 2\vec{k}$.

8. (a) If $\vec{a} \cdot \vec{b} = 48$, $|\vec{a}| = 15$ and $|\vec{b}| = 4$, find $|\vec{a} \times \vec{b}|$.
   (b) If $|\vec{a} \times \vec{b}| = 27$, $|\vec{a}| = 9$ and $|\vec{b}| = 5$, find $\vec{a} \cdot \vec{b}$.

9. (a) Prove that :
   (i) $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = 0$
   (ii) $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$ and interpret it.
   (b) If $\vec{a} + \vec{b} + \vec{c} = 0$, prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$ and interpret it geometrically.

10. (a) Using vector method, prove that $\sin(A - B) = \sin A \cos B - \cos A \sin B$
    (b) Prove, in any triangle, by vector method that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

11. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of the points A, B and C respectively, show that the vector area of the triangle ABC is

$$\frac{1}{2}(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$

A. Find the area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$. Verify this result by measurement.

B. Using cross-product of two vectors, prove that
   a) $\frac{x}{a} + \frac{y}{b} = 1$
   b) $x \cos \alpha + y \sin \alpha = p$

Answer

1. (i) (5, 31, 14), $\frac{5\vec{i} + 31\vec{j} + 14\vec{k}}{\sqrt{1182}}$ (ii) (8, –8, –8), $\frac{\vec{i} - \vec{j} - \vec{k}}{\sqrt{3}}$

3. (i) $\frac{5\sqrt{2}}{\sqrt{77}}$ (ii) $\frac{2}{\sqrt{7}}$

5. (i) $6\sqrt{5}$ sq. units (iii) $5\sqrt{2}$ sq. units

6. (i) 20.5 sq. units

7. $\sqrt{46}, \sqrt{10}, \sqrt{26}, \sin^{-1}\left(\sqrt{\frac{235}{1196}}\right), \sin^{-1}\left(\sqrt{\frac{47}{92}}\right), \sin^{-1}\left(\sqrt{\frac{47}{52}}\right), \frac{\sqrt{235}}{2}$ sq. units

8. a) 36 b) 36

Chapter 12
Correlation and Regression Analysis

Correlation

The various statistical methods discussed up to now consist of only one variable. But in practice, we may come across a number of problems consisting two or more variables. Distribution consisting of two variables are said to be bivariate distribution. In this chapter, we discuss various methods to determine if there exist any relationship between two variables. As for example; the amount of rainfall and the volume of production of certain commodity, age and the blood pressure, etc.

Two variables are said to have "correlation," when they are so related that the change in the value of one variable is accompanied by the change in the value of the other. For example i) the amount of rainfall to some extent is accompanied by an increase in the volume of production ii) the decrease in the price of a commodity is accompanied by the increase in the quantity demanded iii) increase in advertisement expenditure is accompanied by increase in sales. The measure of correlation called the 'correlation coefficient' summarizes in one figure, the degree and direction of movement. But the important thing that is to be noted here is that, correlation analysis only helps in determining the extent to which the two variables are correlated but it does not tell us about cause and effect relationship. Though, there is a high degree of correlation between two variables one cannot say which one is the cause and which one the effect.

Types of Correlation

Correlation may be of the following three types:

(i) Positive and negative

(ii) Linear and non-linear

(iii) Simple, multiple and partial

(i) Positive and negative correlation

If two variables vary in the same direction i.e. increase (or decrease) in the value of one variable results increase (or decrease) in the value of other variable, then the two variables are said to have positive correlation: For examples:

(a)

x:	10	20	25	50
y:	5	8	10	20

(b)

x:	100	50	30	10
y:	8	5	3	2

On the other hand, two variables are said to have negative correlation if two variables move in the opposite direction i.e. if one variable increases (or decreases) the second decreases (or increases). As for example:

(a)

x:	10	20	25	50
y:	50	20	10	8

(b)

x:	100	50	30	10
y:	1	2	3	7

(ii) Linear and non-liner correlation

The correlation between two variables is said to be linear when a unit change in one variable results a constant change in the other variable over the entire range of the values. As for example :

x:	1	2	3	4
y:	7	9	11	13

If corresponding to a unit change in one variable, there is no constant change in other variable, then the correlation is said to be non linear. As for example:

x:	1	2	3	4
y:	7	10	11	20

Methods of studying correlation

The following methods can be used to study the correlation between two variables.

(a) Scatter diagram

(b) Karl Pearson's correlation coefficient

(c) Spearman's rank correlation

a) Scatter diagram

It is a graphical method of studying correlation. The simplest method of ascertaining the correlation between two variables is the scatter diagram. For this let X and Y be two variables, each consisting the same number of values. Points are plotted with the values of X as the x-coordinates and the corresponding values of Y as y-coordinates. The points are represented by dots. The diagram consisting of set of dots thus formed is said to be the scatter diagram. On seeing the scatterness of the dots, an idea about the degree and the direction of correlation between two variables can be made. More the closeness of the dots to a straight line, higher will be the correlation between two variables. Greater the scatterness, less will be the correlation.

b) Karl Pearson's correlation coefficient

One of the widely used mathematical methods of calculating the correlation coefficient between two variables is Karl Pearson's correlation coefficient. It is also known as Pearsonian correlation coefficient. It is denoted by r_{xy} or simply r and is defined by

$$r = \frac{\text{Cov}(X,Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} \quad \text{.........(i)}$$

where $\quad Cov(X,Y) = \frac{1}{n}\sum(X-\overline{X})(Y-\overline{Y})$

$\overline{X}$, $\overline{Y}$ being the arithmetic averages of X-series and Y-series respectively. The formula (i) can be put in the following forms.

(i) can also be expressed as follows

$$r = \frac{\Sigma(X-\overline{X})(Y-\overline{Y})}{\sqrt{\Sigma(X-\overline{X})^2}\sqrt{\Sigma(Y-\overline{Y})^2}} \quad \ldots\ldots(ii)$$

If $x = X - \overline{X}$ and $y = Y - \overline{Y}$, then

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}} \quad \ldots\ldots(iii)$$

Also, $r = \frac{\Sigma xy}{n\sigma_x\sigma_y}$ $\quad \ldots\ldots(iv)$

On simplification, (ii) becomes

$$r = \frac{n\Sigma XY - \Sigma X.\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} \quad \ldots\ldots(v)$$

(v) can also be changed into the following form

$$r = \frac{\Sigma XY - n\overline{X}\,\overline{Y}}{\sqrt{\Sigma X^2 - n\overline{X}^2}\sqrt{\Sigma Y^2 - n\overline{Y}^2}} \quad \ldots\ldots(vi)$$

The formula (ii) or (iii) is also known as the product moment formula.

Computation of correlation coefficient using product moment formula will be tedious if the arithmetic mean be not a whole number. So, to avoid such a problem, we put $u = \frac{x-a}{h}$ and $v = \frac{y-b}{k}$ where a, b, h and k are constants. The correlation coefficient of two variables x and y will be same as the correlation coefficient between the two new variables u and v. We quote this property of the correlation below.

thus, $$r = \frac{n\Sigma uv - \Sigma u.\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}} \quad \ldots\ldots(vii)$$

This method is known as the deviation method of calculating the correlation coefficient between two variables.

Example 1

Calculate Karl Pearson's coefficient of correlation from the following data using product moment formula.

X :	12	9	8	10	11	13	7
Y :	14	8	6	9	11	12	3

Solution :

COMPUTATION OF CORRELATION COEFFICIENT

X	Y	$x = X - \bar{X}$	$y = Y - \bar{Y}$	x^2	y^2	xy
12	14	2	5	4	25	10
9	8	–1	1	1	1	1
8	6	–2	–3	4	9	6
10	9	0	0	0	0	0
11	11	1	2	1	4	1
13	12	3	3	9	9	9
7	3	–3	–6	9	36	18
$\sum X = 70$	$\sum Y = 63$			$\sum x^2 = 28$	$\sum y^2 = 84$	$\sum xy = 46$

$$\bar{X} = \frac{\sum X}{n} = \frac{70}{7} = 10; \qquad \bar{Y} = \frac{\sum Y}{n} = \frac{63}{7} = 9$$

Now,

$$r = \frac{\sum xy}{\sqrt{\sum x^2}\,\sqrt{\sum y^2}} = \frac{46}{\sqrt{28}\,\sqrt{84}}$$

$$= \frac{46}{\sqrt{2352}} = \frac{46}{48.497}$$

$$= 0.95$$

Properties of correlation coefficient

Important properties of the correlation coefficient are given below :

1. Correlation coefficient between two variables is independent of change of origin and scale.
2. The correlation coefficient between two variables lies in between –1 and +1. Symbolically $-1 \le r \le 1$.
3. The formula for the correlation coefficient between the two variables x and y is symmetrical i.e. $r_{xy} = r_{yx}$
4. Correlation coefficient is the geometrical mean between two regression coefficients.

Example 2

Calculate the coefficient of correlation from the following data of price and demand:

Price(Rs):	14	16	19	22	24	30
Demand (kg):	24	22	20	24	23	26

Solution :

COMPUTATION OF CORRELATION COEFFICIENT

Price (x)	u = x – 19	u^2	Demand (y)	v = y – 23	v^2	uv
14	–5	25	24	1	1	–5
16	–3	9	22	–1	1	3
19	0	0	20	–3	9	0
22	3	9	24	1	1	3
24	5	25	23	0	0	0
30	11	121	26	3	9	33
	$\sum u = 11$	$\sum u^2 = 189$		$\sum v = 1$	$\sum v^2 = 21$	$\sum uv = 34$

$$r = \frac{n\sum uv - \sum u \sum v}{\sqrt{\{n\sum u^2 - (\sum u)^2\}}\ \sqrt{\{n\sum v^2 - (\sum v)^2\}}}$$

$$= \frac{6 \times 34 - 11 \times 1}{\sqrt{\{6 \times 189 - (11)^2\}}\ \sqrt{\{6 \times 21 - (1)^2\}}}$$

$$= \frac{204 - 11}{\sqrt{\{1134 - 221\}}\ \sqrt{(126 - 1)}}$$

$$= \frac{193}{(1013 \times 125)^{1/2}}$$

$$= \frac{193}{(126625)^{1/2}}$$

$$= \frac{13}{355.84}$$

$$= 0.542$$

Interpretation of correlation coefficient (r)

From property no.1 of correlation coefficient, its value lies between –1 and +1. After getting the value of r, care should be taken to interpret, otherwise wrong conclusion may be obtained. However the following general rules are mentioned for interpreting the value of r.

(i) When r = 1, there is a positively perfect correlation between the two variables.

(ii) When r = –1, there is negatively perfect correlation between the two variables.

(iii) When r = 0, the variables are uncorrelated.

(iv) Nearer the value of r to +1, closer will be the relationship between two variables and nearer the value of r to 0, lesser will be the relationship.

Merits and limitations of Karl Pearson's Correlation coefficient

Merits :

(i) Karl Pearson's method of finding correlation coefficient is based on all the observations.

(ii) This method summaries in one figure the degree of relationship as well as direction.

Limitations:

(i) This method always assumes linear relationship between two variables whether this assumption is true or not.

(ii) It is affected by extreme values.

(iii) In comparison to other methods, it is most time consuming.

(iv) Interpretation of the value of r is not an easy attempt.

Worked Out Examples

Example 1

Find the correlation coefficient between the two variables from the following data

$n = 10, \quad \Sigma X = 18, \ \Sigma Y = 25, \ \Sigma X^2 = 90$

$\Sigma Y^2 = 120$ and $\Sigma XY = 65$

Solution:

Here, $$r = \frac{n\,\Sigma XY - \Sigma X.\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$$

$$= \frac{10 \times 65 - 18 \times 25}{\sqrt{10 \times 90 - (18)^2}\sqrt{10 \times 120 - (25)^2}}$$

$$= \frac{650 - 450}{\sqrt{576}\sqrt{575}} = \frac{200}{575.5}$$

$$= 0.35$$

Example 2

From the following data, compute the correlation coefficients between the two variables X and Y.

	Series X	Series Y
	13	13
No. if items:	25	18
Arithmetic mean:	3.2	3.3
s.d.	136	138
Sum of squares of deviation from mean:		

Sum of the product of the deviations of X and Y from their respective arithmetic mean = 122

Solution:

Here, n = 15, $\overline{X} = 25$, $\overline{Y} = 18$, $\sigma_x = 3.01$, $\sigma_y = 3.03$,

$\Sigma(X - \overline{X})^2 = 135$, $\Sigma(Y - \overline{Y})^2 = 138$ and $\Sigma(X - \overline{X})(Y - \overline{Y}) = 122$

$$\text{Now, } r = \frac{\Sigma(X - \overline{X}).(Y - \overline{Y})}{\sqrt{\Sigma(X - \overline{X})^2}\sqrt{\Sigma(Y - \overline{Y})^2}} = \frac{122}{\sqrt{136}\sqrt{138}}$$

$$= \frac{122}{\sqrt{136 \times 138}} = \frac{122}{136.996} = 0.89$$

Note: r can also be obtained using σ_x and σ_y.

Example 3

Calculate using product moment formula, the coefficient of correlation between the price and sales

Price:	25	21	28	26	20	18
Sales:	60	54	66	68	53	57

Solution:

Let X and Y represent the price and sales respectively.

X	Y	$x = X - \overline{X}$ $(\overline{X} = 23)$	$y = Y - \overline{Y}$ $(\overline{Y} = 60)$	x^2	y^2	xy
25	60	2	0	4	0	0
21	54	–2	–6	4	36	12
28	66	5	6	25	36	30
26	68	3	8	9	64	24
20	53	–3	–7	9	49	21
18	59	–5	–1	25	1	5
$\Sigma X = 138$	$\Sigma Y = 360$			$\Sigma x^2 = 76$	$\Sigma y^2 = 186$	$\Sigma xy = 92$

$\overline{X} = \frac{\Sigma X}{n} = \frac{138}{6} = 23$ and $\overline{Y} = \frac{\Sigma Y}{n} = \frac{360}{6} = 60$

$$\text{Now, } r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}}$$

$$= \frac{92}{\sqrt{76}\sqrt{186}} = \frac{92}{118.89}$$

$$= 0.77$$

Price and sales are positively correlated. That is, the degree of relationship between price and sales is 0.77 and the direction of their movement is same.

Example 4

Calculate the correlation coefficient between the two variables

X:	25	19	28	30	18	24
Y:	58	52	65	70	51	62

Solution:

Computation of Correlation Coefficient

X	Y	u = X − 23	v = Y − 62	u^2	v^2	uv
25	58	2	−4	4	16	−8
19	52	−4	−10	16	100	40
28	65	5	3	25	9	15
30	70	7	8	49	64	56
18	51	−5	−11	25	121	55
24	62	1	0	1	0	0
		$\Sigma u = 6$	$\Sigma v = -14$	$\Sigma u^2 = 120$	$\Sigma v^2 = 310$	$\Sigma uv = 158$

Now, $r = \dfrac{n\Sigma uv - \Sigma u.\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$

$= \dfrac{6 \times 158 - 6 \times (-14)}{\sqrt{6 \times 120 - (6)^2}\sqrt{6 \times 310 - (-14)^2}}$

$= \dfrac{1032}{\sqrt{684 \times 1664}} = \dfrac{1032}{1068.85} = 0.96$

Example 5

A computer while calculating correlation coefficient between two variables x and y from 25 pair of observations obtained the following results.

$n = 25$, $\Sigma x = 125$, $\Sigma x^2 = 650$, $\Sigma y = 100$, $\Sigma y^2 = 460$, $\Sigma xy = 508$.

It was however, later discovered at the time of checking that he had copied down pairs as

x	y
16	14
8	6

while the correct values were

x	y
8	12
6	8

. Obtain the correct value of the correlation coefficient.

Solution:

n = 25

Corrected $\Sigma x = 125 - 6 - 8 + 8 + 6 = 125$

" $\Sigma y = 100 - 14 - 6 + 12 + 8 = 100$

" $\Sigma x^2 = 650 - (6)^2 - (8)^2 + (8)^2 + (6)^2 = 650$

$$\text{,, } \sum y^2 = 460 - (14)^2 - (6)^2 + (12)^2 + (8)^2 = 436$$

$$\text{,, } \sum xy = 508 - 6 \times 14 - 8 \times 6 + 8 \times 12 + 6 \times 8 = 520$$

r_c = Corrected value or f = ?

Now, $r_c = \dfrac{n\sum xy - \sum x.\sum y}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$

$$= \frac{25 \times 520 - 125 \times 100}{\sqrt{25 \times 650 - (125)^2}\sqrt{25 \times 436 - (100)^2}}$$

$$= \frac{500}{(625 \times 900)^{1/2}} = \frac{500}{25 \times 30}$$

$$= \frac{2}{3}$$

EXERCISE

1. Find the correlation coefficient between the two variables under the following conditions
   a) Cov (X, Y) = covariance of the variables X and Y = 18
      Variance of X = 16 and the variance of Y = 81
   b) Cov (X, Y) = –16.5, Var. (X) = 2.89 and Var. (Y) = 100
   c) $\Sigma(X - \overline{X})^2 = 40$, $\Sigma(Y - \overline{Y})^2 = 63$ and $\Sigma(X - \overline{X})(Y - \overline{Y}) = 35$
   d) $n = 15$, $\sigma_X = 3.2$, $\sigma_Y = 3.4$ and $\Sigma(X - \overline{X})(Y - \overline{Y}) = 122$
   e) $n = 10$, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$
   f) $n = 10$, $\overline{X} = 5$, $\overline{Y} = 3$, $\Sigma X^2 = 290$, $\Sigma Y^2 = 300$, $\Sigma XY = 115$
   g)

	Series X	Series Y
No. of pair of observations	10	10
Standard deviation:	2.05	2.41
Sum of the squares of deviations from their respective means:	42	58

   Sum of the products of deviations of X and Y from their respective means = 36

2. a) Calculate Karl Pearson's correlation coefficient between the two variables height (in cm) and weight (in kg) from the data given below

Height :	160	162	165	161	162
Weight:	63	62	64	60	61

b) Find the correlation coefficient between the two variables X and Y from the following data

X :	5	7	1	3	4
Y :	2	3	4	5	6

c) Calculate Karl Pearson's correlation coefficient between the sales and expenses in thousand rupees of 5 firms.

Sales:	43	41	36	34	50
Expenses:	12	24	15	21	19

d) Determine the degree of relationship between the ages of the husbands and their wives from the following data

Age of the husband:	23	22	24	23	26	27
Age of the wives:	20	18	20	21	21	22

3. From the following table calculate Karl Pearson's correlation coefficient between the two variables:

X :	6	2	10	4	8
Y :	9	11	?	8	7

Arithmetic means of X and Y series are 6 and 8 respectively.

4. In order to find the correlation coefficient between the two variables X and Y from 12 pair of observations, the following calculations were made:
$\Sigma X = 30$, $\Sigma Y = 5$, $\Sigma X^2 = 670$, $\Sigma Y^2 = 285$, $\Sigma XY = 334$
On subsequent verification, it was found that the pair (X = 11, Y = 4) was copied wrong, the correct value being (X = 10, Y = 14). Find the correct value of correlation coefficient.

A. Student are asked to visit in the different departmental store to collect the information about the demand of the electric items and their respective prices. Prepare a note on the correlation between the demand of the electric items and the prices. Present the report with conclusion in the class.

Answers

1. a) 0.5 b) –0.97 c) 0.697 d) 0.75 e) 0.59 f) –0.38 g) 0.73
2. a) 0.51 b) –0.42 c) –0.10 d) 0.80 3. –0.92 4. 0.775

Rank Correlation

The degree of relationship between two variables with respect to their respective ranks is known as "Rank correlation coefficient". It is also known as Speraman's rank correlation coefficient.

Let R_1 and R_2 be the ranks of n individuals according to the characteristics say A and B respectively. If we assume that no two individuals have the same rank in a character then R_1 and

R_2 take numerical values from 1 to n. Now, Spearman's rank correlation coefficient denoted by R is given by the formula

$$R = 1 - \frac{6\,\Sigma d^2}{n(n^2 - 1)}$$

where $d = R_1 - R_2$, n = no. of pair of observations.

The value of R also lies between −1 and +1.

Methods of calculating rank correlation coefficient

We obtain rank correlation coefficient under the following conditions:

(a) When actual ranks are given

When ranks are given the following steps are to be used for calculating the rank correlation coefficient.

i) Find the difference between the ranks and denote it by d.

ii) Find d^2 and obtain Σd^2

iii) Use the formula, $R = 1 - \dfrac{6\,\Sigma d^2}{n(n^2 - 1)}$

Note: For correct ranking, $\Sigma d = 0$.

(b) When ranks are not given

If ranks be not given, we rank the given data by giving the highest value as 1 and next highest value 2 and so on. The same procedure is used for both variables under consideration. Rest procedure is same as in case (a).

Note: We can assign 1 to the lowest value and 2 to next lowest value and so on.

Repeated ranks

If two or more variate values are equal then each equal value is given the common rank. In such cases, each individual is given an average rank. For example, if two individuals are ranked equal at 4th place, they are each given the rank $\dfrac{4+5}{2} = 4.5$ and then the succeeding value the 6th rank and so on.

Some adjustments are to be made in the above Spearman's correlation formula for repeated ranks. In the above formula an extra factor $\dfrac{m(m^2-1)}{12}$ is to be added to Σd^2 where m is the no. of times a variate value repeats. This correlation factor is to be added for each repeated value in both series. After adjustment, Spearman's rank correlation coefficient R is

$$R = 1 - \frac{6\left\{\Sigma d^2 + \dfrac{m_1(m_1^2 - 1)}{12} + \dfrac{m_2(m_2^2 - 1)}{12} + \dots\right\}}{n(n^2 - 1)}$$

Interpretation of Rank Correlation Coefficient

The interpretation of rank correlation coefficient is same as that of Karl Pearson's correlation coefficient. Its value R lies between –1 and 1 i.e. $-1 \leq R \leq 1$. If R = 1, there is a perfect positive correlation between the two ranks of two judges. If R = 0, there is no relationship between the ranks of two judges.

Merits and limitations of rank correlation

Merits

i) This method is easier compared to Karl Pearson's method.

ii) Thi is the only method of finding the relationship when ranks are given.

iii) This method is appropriate for qualitative characteristics such as intelligence, beauty etc.

Limitations

i) When the number of observations is greater than 30, the calculation will be tedious and consume a lot of time.

ii) It can not be used for bivariate frequency distribution.

Worked Out Examples

Example 1

A firm wanted to employ some accountants. Six candidates appeared for an aptitude test and were assigned the following ranks by 2 examiners:

Candidates:	A	B	C	D	E	F
Rank by X:	1	3	2	5	4	6
Rank by Y:	2	1	3	6	4	5

Calculate the rank correlation coefficient.

Solution

Computation of Rank Correlation Coefficient

Candidate	Rank by X (R_1)	Rank by Y (R_2)	d $= R_1 - R_2$	d^2
A	1	2	–1	1
B	3	1	2	4
C	2	3	–1	1
D	5	6	–1	1
E	4	4	0	0
F	6	5	1	1
			$\Sigma d = 0$	$\Sigma d^2 = 8$
n = 7				

Here n = 6, $\Sigma d^2 = 8$, R = ?

Now R $= 1 - \dfrac{6\Sigma d^2}{n(n^2-1)}$

$= 1 - \dfrac{6 \times 8}{6 \times (36-1)} = 1 - \dfrac{48}{6 \times 35}$

$= 0.77$

Example 2

Calculate the coefficient of rank correlation between price and supply for the following data:

Price:	8	10	12	6	9	14	18	16
Supply:	15	25	18	20	16	21	10	12

Solution

Calculation of Rank Correlation Coefficient

Price	Supply	Rank of Price (R_1)	Rank of Supply (R_2)	d $= R_1 - R_2$	d^2
8	15	7	6	1	1
10	25	5	1	4	16
12	18	4	4	0	0
6	20	8	3	5	25
9	16	6	5	1	1
14	21	3	2	1	1
18	10	1	8	–7	49
16	12	2	7	–5	25
				$\Sigma d = 0$	$\Sigma d^2 = 118$

Here, n = 8, $\Sigma d^2 = 118$, R = ?

Now R $= 1 - \dfrac{6\Sigma d^2}{n(n^2-1)}$

$= 1 - \dfrac{6 \times 118}{8\,(64-1)} = 1 - \dfrac{6 \times 118}{6 \times 63}$

$= -0.405.$

Example 3

From the following data of the marks obtained by 8 students in computer and Mathematics paper, compute Spearman rank correlation.

Marks in Computer (X)	25	68	45	50	80	74	50	68
Marks in Mathematics (Y)	36	40	57	40	72	75	60	40

Solution:

X	Y	Rank of Comp. (R_1)	Rank of Math. (R_2)	d $= R_1 - R_2$	d^2
25	36	8	8	0	0
68	40	3.5	6	–2.5	6.25
45	57	7	4	3.0	9.0
50	40	5.5	6	–0.5	0.25
80	72	1	2	–1.0	1.0
74	75	2	1	1.0	1.0
50	60	5.5	3	2.5	6.25
68	40	3.5	6	–2.5	6.25
				$\Sigma d = 0$	$\Sigma d^2 = 30$

Here, $n = 8$, $\Sigma d^2 = 30$, $m_1 = 2$, $m_2 = 2$, $m_3 = 3$, $R = ?$

$$\text{Now, } R = 1 - \frac{6\left\{\Sigma d^2 + \frac{m_1(m_1^2 - 1)}{12} + \frac{m_2(m_2^2 - 1)}{12} + \frac{m_3(m_3^2 - 1)}{12}\right\}}{n(n^2 - 1)}$$

$$= 1 - \frac{6\left\{30 + \frac{2(4 - 1)}{12} + \frac{2(4 - 1)}{12} + \frac{3(9 - 1)}{12}\right\}}{8(64 - 1)}$$

$$= 1 - \frac{6 \times 33}{8 \times 63} = 0.61$$

Example 4

Ten competitors in a beauty contest are ranked by three judges in the following data:

1st Judge:	1	6	5	10	3	2	4	9	7	8
2nd Judge:	3	5	8	4	7	10	2	1	6	9
3rd Judge:	6	4	9	8	1	2	3	10	5	7

Use the rank correlation coefficient to determine which pair of judges has the nearest approach to common test in beauty.

Solution

Calculation of Rank Correlation Coefficient

R_1	R_2	R_3	$d_{12} = R_1 - R_2$	$d_{23} = R_2 - R_3$	$d_{13} = R_1 - R_3$	d_{12}^2	d_{23}^2	d_{13}^2
1	3	6	–2	–3	–5	4	9	25
6	5	4	1	1	2	1	1	4
5	8	9	–3	–1	–4	9	1	16

10	4	8	6	–4	2	36	16	4
3	7	1	–4	6	2	16	36	4
2	10	2	–8	8	0	64	64	0
4	2	3	2	–1	1	4	1	1
9	1	10	8	–9	–1	64	81	1
7	6	5	1	1	2	1	1	4
8	9	7	–1	2	1	1	4	1
						$\Sigma d_{12}^2 = 200$	$\Sigma d_{23}^2 = 214$	$\Sigma d_{13}^2 = 60$

From R_1 and R_2:

Here, $n = 10$, $\Sigma d_{12}^2 = 200$, $R_{12} = ?$

Now, $R_{12} = 1 - \dfrac{6\Sigma d_{12}^2}{n(n^2-1)}$

$= 1 - \dfrac{6 \times 200}{10(100-1)} = 1 - \dfrac{6 \times 200}{10 \times 99}$

$= -0.212$

From R_2 and R_3:

Here $n = 10$, $\Sigma d_{23}^2 = 214$, $R_{23} = ?$

Now $R_{23} = 1 - \dfrac{6\Sigma d_{23}^2}{n(n^2-1)}$

$= 1 - \dfrac{6 \times 214}{10(100-1)} = 1 - \dfrac{6 \times 214}{10 \times 99}$

$= -0.297$

From R_1 and R_3:

Here, $n = 10$, $\Sigma d_{13}^2 = 60$, $R_{13} = ?$

Now, $R_{13} = 1 - \dfrac{6\,\Sigma d_{13}^2}{n(n^2-1)}$

$= 1 - \dfrac{6 \times 60}{10(100-1)} = 1 - \dfrac{6 \times 60}{10 \times 99}$

$= 0.636$

From the values of R_{12}, R_{23}, R_{13} we can say that the ranks given by 1st and 3rd judges have the nearest approach to common test in beauty.

Example 5

The rank correlation coefficient of the marks obtained by 10 students in English and Marketing was found to be 0.4. It was later discovered that the difference in ranks of two subjects obtained by one student was wrongly copied as 9 instead of 6. Find the correct coefficient of rank correlation.

Solution:

Here, $n = 10$, $R = 0.4$, $\Sigma d^2 = ?$

Now, $R = 1 - \dfrac{6\,\Sigma d^2}{n(n^2-1)}$

$\Rightarrow\ 0.4 = 1 - \dfrac{6\,\Sigma d^2}{10(100-1)}$

$\Rightarrow\ \dfrac{6\,\Sigma d^2}{10 \times 99} = 0.6$

$\Rightarrow\ \Sigma d^2 = \dfrac{0.6 \times 10 \times 99}{6} = 99$

Correct value of $\Sigma d^2 = 99 - (9)^2 + (6)^2 = 54$

Now, correct value of $R = 1 - \dfrac{6 \times \text{correct value of } \Sigma d^2}{n(n^2-1)}$

$= 1 - \dfrac{6 \times 54}{10(100-1)} = 1 - \dfrac{6 \times 54}{10 \times 99}$

$= 0.67$

EXERCISE

1. a) In a drawing competition, two judges have given the following ranks for 8 competitors

S.No.	1	2	3	4	5	6	7	8
Judge A:	3	5	4	7	8	6	1	2
Judge B:	6	4	2	8	7	5	1	3

Find the rank correlation coefficient between the ranks of two judges.

b) Ten items ranked by two experts are presented below

Items	A	B	C	D	E	F	G	H	I	J
Expert X	7	5	4	8	6	3	1	2	9	10
Expert Y	6	1	4	3	5	9	8	7	10	2

Compute Spearman's rank coefficient.

2. a) The IQ of 6 students and their respective marks in a certain examination are presented below

Student	A	B	C	D	E	F
IQ	125	110	140	130	90	100
Exam. Marks	82	86	90	75	70	87

b) The marks obtained by 8 students in Statistics and Accountancy are given below

S.No.	1	2	3	4	5	6	7	8
Marks in Stat:	30	50	25	60	70	80	65	75
Marks in Acc:	50	60	30	40	82	90	70	73

Compute Spearman's correlation coefficient between the marks in Statistics and Accountancy.

c) Calculate Spearman rank correlation between advertisement cost (in thousnd Rs.) and sales (in Lakhs Rs.) from the following data

Advt. cost	57	45	72	78	53	63	86	98	59	71
Sales	78	37	41	84	56	85	77	87	70	59

3. a) Calculate Spearman rank correlation for the relationship between the demand and supply of 6 goods

Items	1	2	3	4	5	6
Demand	100	150	160	200	160	180
Supply	90	120	130	120	150	160

b) The marks obtained by 8 students in Mathematics and Physics are given below

S.No.	1	2	3	4	5	6	7	8
Marks in Math	40	60	35	68	70	96	70	84
Marks in Phy	48	62	28	52	85	90	52	73

Find the rank correlation coefficient between the marks in Mathematics and Phsyics.

c) The following data gives the scores in psychological test (X) and the arithmetic ability (Y) of 10 children

Child	A	B	C	D	E	F	G	H	I	J
X:	20	25	60	45	80	25	55	65	25	75
Y:	52	50	55	50	60	70	72	78	80	63

Calculate the correlation coefficient between the psychological test and the arithmetic ability in terms of their ranks.

4. The following table presents the ranks of 8 competitors given by three judges in a certain music contest.

Judge I	4	7	8	1	2	5	6	3
Judge II	5	8	4	7	3	2	1	6
Judge III	6	5	3	1	2	4	8	7

Use the method of rank correlation to examine which pair of judges have the nearest approach to music.

5. Spearman rank correlation of marks obtained by 10 students in Mathematics and Computer was found to be 0.6. It was later discovered that difference in rank in two subjects obtained

by one student was incorrectly taken as 9 instead of 7. Find the correct rank correlation coefficient.

Answers

1. a) 0.786 b) –0.32 2. a) 0.43 b) 0.905 c) 0.56
3. a) 0.51 b) 0.85 c) 0.19 4. I and III 5. 0.79

Meaning of regression

The simple meaning of regression is returning back to the original position. The theory of regression was first developed by Sir F. Galton. He studied the heights of a group of father and their sons. He found that the taller fathers have taller sons and dwarf fathers have dwarf sons. He also found that the average height of the sons of the taller father is less than the average height of the taller fathers and the average height of the dwarf sons is greater than the average height of their fathers. But now-a-days, regression analysis is used as a tool to present the strength of the relationship between the two variables. Thus it is considered as a device which is used to predict the value of one variable when the value of other variable is known. The variable whose value is known is the independent variable and the variable whose value is to be determined is known as the dependent variable. The analysis which is used to describe the average relationship between the two variables is known as a simple linear regresion analysis.

Difference between correlation and regression

The following are the differences between the correlation and regression.

i) Correlation means the relationship between the two variables so that the change in the value of one variable results change in the value of the other variable. But the regression means the returning back to the average value.

ii) There is no need of cause and effect relationship between the two variables in case of correlation but there must be cause and effect relationship between two variables in case of regression.

iii) Correlation analysis presents the extent to which the two variables are correlated and also the direction of their movements. But regression analysis aims to study the nature of the relationship between the two variables so that we may able to find the value of one variable when the value of other variable is known.

iv) Correlation coefficient is independent of change of origin and scale but regression coefficients are independent of change of origin but not of scale.

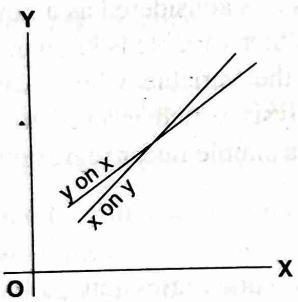
v) The correlation coefficient between the two variables can not exceed 1 but one regression coefficient can exceed 1 and other regression coefficient less than 1 making their product less than or equal to 1.

Lines of regression

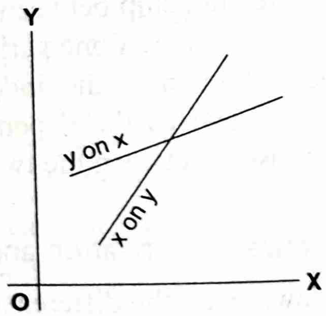
Whenever there shows a relationship between the two variables, the dots of the scatter diagram will concentrate around a certain curve. If the curve is a straight line, then it is known as the line of regression.

A line of regression gives the best estimate (in the sense of least square method) of one variable for a given value of the other variable. So, there are in general two lines of regression. One is the line of regression of y on x giving the best estimated value of y whenever the value of x is known. Other is the line of regression of x on y giving the best estimated value of x whenever the value of y is known. The two lines of regression intersect at the point $(\bar{x}, \bar{y})$, $\bar{x}$ and $\bar{y}$ being the averages of x and y. If there is a wide gap between the two lines of regression, the correlation between the two variables is less. If the two lines of regression are near enough, then the correlation between the two variables is high. Whenever the two lines of regression coincides there is a perfect correlation between the two variables. On the other hand, if the two lines of regression intersect at right angles, there is no correlation between the two variables.

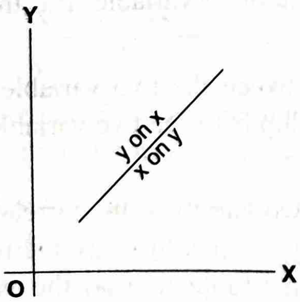
Let us see the following graphs of the lines of regression presenting the correlation between the two variables.



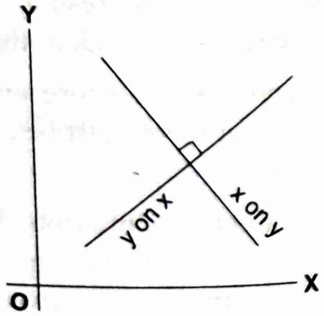
High degree correlation



Low degree correlation



Perfect correlation



No correlation

Regression Equations and regression coefficients

Regression lines expressed in terms of algebraic relations are known as the regression equations. Since there are two regression lines, so there are two regression equations.

i) The regression equation of y on x expresses the variation of y for a change in the value of x.

ii) The regression equation of x on y expresses the variabtion of x for a change in the value of y.

Regression equation of y on x and x on y

Let the regression equation of y on x be

$$y = a + bx \qquad \ldots\ldots \text{(i)}$$

where a and b are constants to be determined. b is the slope of the line (i)

The constants a and b can be obtained by solving the following two normal equations obtained by the method of least squares.

$$\Sigma y = na + b\Sigma x \qquad \ldots\ldots \text{(ii)}$$

and

$$\Sigma xy = a\,\Sigma x + b\,\Sigma x^2 \qquad \ldots\ldots\text{(iii)}$$

After getting the values of a and b, we substitute in equation (i) which will give the regression equation of y on x.

Instead of solving the above two equations, we can use the direct method as well. For this, multiplying equation (ii) by Σx and (iii) by n then subtracting, we get

$$b = \frac{n\,\Sigma xy - \Sigma x.\Sigma y}{(\Sigma x)^2 - n\Sigma x^2}$$

This is known as the regression coefficient of y on x; so it is also denoted by b_{yx}.

$$\therefore \quad b = b_{yx} = \frac{n\,\Sigma xy - \Sigma x.\Sigma y}{(\Sigma x)^2 - n\Sigma x^2} = r\frac{\sigma_y}{\sigma_x}$$

Again from (ii),

$$\frac{\Sigma y}{n} = a + b\frac{\Sigma x}{n}$$

$$\Rightarrow \quad \overline{y} = a + b\,\overline{x} \qquad \ldots\ldots\text{(iv)}$$

Subtracting (iv) from (i), we have

$$y - \overline{y} = b(x - \overline{x})$$

$$\Rightarrow \quad y - \overline{y} = b_{yx}(x - \overline{x}) \qquad \ldots\ldots\text{(v)}$$

which is the required regression equation of y on x.

Similarly the regression equation of x on y is

$$x - \overline{x} = b_{xy}(y - \overline{y})$$

where b_{xy} = regression coefficient of x on y

$$= \frac{n\,\Sigma xy - \Sigma x.\Sigma y}{(\Sigma y)^2 - x\,\Sigma y^2} = r\frac{\sigma_x}{\sigma_y}$$

The above regression equation (v) shows that it passes through the point $(\overline{x}, \overline{y})$, $\overline{x}$ and $\overline{y}$ being the arithmetic means of x and y series respectively.

Note: Equations (ii) and (iii) can easily be solved if we can make $\Sigma x = 0$ in which case $a = \frac{\Sigma y}{n}$ and $b = \frac{\Sigma xy}{\Sigma x^2}$.

The regression equations can easily be obtained when the deviations of the items are taken from the assumed mean.

If $u = x - a$ and $v = y - b$ i.e. if the deviations of the items of x-series and y-series be taken from the assumed means a and b respectively, then

$$\bar{x} = a + \frac{\Sigma u}{n}, \qquad \bar{y} = b + \frac{\Sigma v}{n}$$

and $$b_{yx} = \frac{n\,\Sigma uv - \Sigma u.\Sigma v}{n\,\Sigma u^2 - (\Sigma u)^2}, \qquad b_{xy} = \frac{n\,\Sigma uv - \Sigma u.\Sigma v}{n\,\Sigma v^2 - (\Sigma v)^2}$$

Now the equation can be obtained from using equation (iii) and (iv).

Properties of regression coefficients

Regression coefficients follow the following properties:

i) The regression coefficients are independent of change of origin but not of scale.

ii) The correlation coefficient between the two variables is the geometric mean between the two regression coefficients.

iii) The product of the two regression coefficient is less than or equal to 1.

iv) The two regression coefficients have the same sign. The sign of the correlation coefficients depends upon the sign of the regression coefficients. That is, if both regression coefficients are positive, the correlation coefficient is positive and if both regression coefficients are negative, then the correlation coefficient is negative.

v) The arithmetic average of the two regression coefficients is greater than or equal to the correlation coefficient.

Worked Out Examples

Example 1

Find the correlation coefficient between the two variables x and y when the regression coefficients of y on x and x on y are 0.84 and 1.15 respectively.

Solution:

By given, $b_{yx} = 0.84$ and $b_{xy} = 1.15$

$$r = \sqrt{b_{yx}.b_{xy}} = \sqrt{0.84 \times 1.15} = 0.98$$

Example 2

From the following results, find the regression coefficients:

$\sigma_x = 20$, $\sigma_y = 15$ and $r = 0.48$

Solution:

b_{yx} = regression coefficient of y on x

$$= r\frac{\sigma_y}{\sigma_x} = 0.48 \times \frac{15}{20} = 0.36$$

and b_{xy} = regression coefficient of x on y

$$= r\frac{\sigma_x}{\sigma_y} = 0.48 \times \frac{20}{15} = 0.64$$

Example 3

Two lines of regression are x + 2y = 5 and 2x + 3y = 8 and $\sigma_x^2 = 12$. Calculate $\bar{x}$, $\bar{y}$, b_{yx}, b_{xy}, r and σ_y^2.

Solution:

Since the two lines of regression pass through the point $(\bar{x}, \bar{y})$, $\bar{x}$, $\bar{y}$ being the average of x and y series, so

$$\bar{x} + 2\bar{y} = 5 \quad \text{......(i)}$$

and

$$2\bar{x} + 3\bar{y} = 8 \quad \text{......(ii)}$$

Solving (i) and (ii),

$$\bar{x} = 1, \quad \bar{y} = 2$$

From the regression equation x + 2y = 5, we have

$$y = -\frac{1}{2}x + \frac{5}{2} \qquad \therefore \quad b_{yx} = -\frac{1}{2}$$

Again from the regression equation 2x + 3y = 5, we have

$$x = -\frac{3}{2}y + \frac{5}{2} \qquad \therefore \quad b_{xy} = -\frac{3}{2}$$

Now, $r = \sqrt{b_{yx}.b_{xy}}$

$$= \sqrt{-\frac{1}{2} \times \left(-\frac{3}{2}\right)} = -\frac{\sqrt{3}}{2} = -0.87$$

Again, $b_{yx} = -\frac{1}{2}$

$$\Rightarrow \quad r\frac{\sigma_y}{\sigma_x} = -\frac{1}{2}$$

$$\Rightarrow \quad r^2\frac{\sigma_y^2}{\sigma_x^2} = \frac{1}{4}$$

$$\Rightarrow \quad \frac{3}{4} \times \frac{\sigma_y^2}{12} = \frac{1}{4}$$

$$\therefore \quad \sigma_y^2 = 4$$

Example 4

The regression coefficients of x on y and y on x are 1.5 and 0.65 respectively. If the arithmetic means $\bar{x}$ and $\bar{y}$ are 36 and 52 respectively, find the two regression equations. Also, find the value of y when x = 60.

Solution:

Here, $b_{xy} = 1.5$ and $b_{yx} = 0.65$

Also, $\bar{x} = 36$ and $\bar{y} = 52$

The regression equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$$\Rightarrow y - 52 = 0.65(x - 36)$$

$$\Rightarrow y - 52 = 0.65x - 23.4$$

$$\Rightarrow y = 0.65x + 28.6 \quad \text{......(i)}$$

Again the regression equation of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$$\Rightarrow x - 36 = 1.5(y - 52)$$

$$\Rightarrow x - 36 = 1.5y - 78$$

$$\Rightarrow x = 1.5y - 42 \quad \text{......(ii)}$$

(i) and (ii) are the equations of two regression lines.

When x = 60, $y = 0.65 \times 60 + 28.6 = 67.6$

So, when x = 60, the value of y = 67.6

Example 5

The following table gives the normal weight of a baby during the first six months of life:

Age in month:	0	2	3	5	6
Weight:	5	7	8	10	12

Estimate the weight of a baby at the age of 4 months.

Solution:

Let x and y represent the age and the weight respectively.

Computation of Regression Equation

x	y	u = x – 3	v = y – 8	u^2	uv
0	5	–3	–3	9	9
2	7	–1	–1	1	1
3	8	0	0	0	0
5	10	2	2	4	4

6	12	3	4	9	12
		$\Sigma u = 1$	$\Sigma v = 2$	$\Sigma u^2 = 23$	$\Sigma uv = 26$

Here, $n = 5, \Sigma u = 1, \Sigma v = 2, \Sigma u^2 = 23, \Sigma uv = 26$

$$\bar{x} = a + \frac{\Sigma u}{n} = 3 + \frac{1}{5} = 3.2$$

$$\bar{y} = b + \frac{\Sigma v}{n} = 8 + \frac{2}{5} = 8.4$$

$$b_{yx} = \frac{n\,\Sigma uv - \Sigma u.\Sigma v}{n\Sigma u^2 - (\Sigma u)^2}$$

$$= \frac{5 \times 26 - 1 \times 2}{5 \times 23 - (1)^2} = \frac{128}{114} = 1.12$$

The regression equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

$\Rightarrow$ $y - 8.4 = 1.12(x - 3.2)$

$\Rightarrow$ $y - 8.4 = 1.12x - 3.584$

$\Rightarrow$ $y = 1.12x + 4.82$

When $x = 4$, $y = 1.12 \times 4 + 4.82 = 9.3$

$\therefore$ at the age of 4 months, the weight of the baby is 9.3

Example 6

Given the following data, calculate the mark in mathematics obtained by a student who has secured 80 marks in English

	Math	English
Mean mark:	80	64
s.d. of mark:	3	4

Correlation coefficient between the marks of Mathematics and English = –0.40

Solution:

Let x and y represent the marks in Math and English respectively.

Then, $\bar{x} = 80$, $\bar{y} = 64$, $\sigma_x = 3$, $\sigma_y = 4$ and $r = -0.40$

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = -0.4 \times \frac{3}{4} = -0.3$$

The regression equation of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

$\Rightarrow$ $x - 80 = -0.3(y - 64)$

$\Rightarrow$ $x - 80 = -0.3y + 19.2$

$\Rightarrow \quad x = -0.3y + 99.2$

When $y = 80$, $x = -0.3 \times 80 + 99.2 = 75.2$

$\therefore$ the mark in mathematics is 75.2

EXERCISE

1. What is regression? Distinguish between correlation and regression.

2. What are the lines of regression? Why are there in general two lines of regression?

3. Find the correlation coefficients between the two variables under the following conditions (if possible)
   i) $b_{xy} = 1.8$ and $b_{yx} = 0.35$
   ii) $b_{xy} = -0.24$ and $b_{yx} = -3.25$
   iii) the two regression coefficients are 1.36 and 0.8
   iv) the regression coefficient of x on y is 0.56 and that of y on x is 1.24.

4. Find the two regression coefficients from the following results
   a) $\sigma_x = 6, \sigma_y = 12, r = 0.8$ b) $\sigma_x = 8, \sigma_y = 10, r = -0.6$

5. From the following pair of regression equations, find the regression coefficients, correlation coefficients and the means of x and y series
   a) $4x - 5y + 33 = 0$
   $20x - 9y - 107 = 0$
   b) $3x + 2y - 26 = 0$
   $6x + y - 31 = 0$

6. a) The regression coefficient of x on y is 0.32 and that of y on x is 0.73. If the arithmetic means $\overline{x}$ and $\overline{y}$ are 40 and 35 respectively, find out the
   i) correlation coefficient between the variable x and y.
   ii) regression equations of y on x and x on y.

   b) The regression coefficients of x on y and y on x are 0.84 and 0.32 respectively. If the arithmetic means of x and y series are 42 and 26 respectively, find two equations of lines of regression. Estimate the value of y when x = 20 and the value of x when y = 30.

7. a) Find the regression equation of y on x when
   i) $\overline{x} = 28$, $\overline{y} = 36$ and $b_{yx} = 0.5$
   ii) $\Sigma x = 15, \Sigma y = 25, \Sigma x^2 = 55, \Sigma y^2 = 140, \Sigma xy = 78, n = 5$

   b) Find the regression equation of x on y when
   i) $\overline{x} = 30$, $\overline{y} = 45$ and $b_{xy} = 0.32$
   ii) $\overline{x} = 6, \overline{y} = 11, \Sigma xy = 306, \Sigma x^2 = 164, \Sigma y^2 = 574, n = 4$

8. a) In a correlation study, the following values were obtained

	X	Y
Mean :	64	72
Standard Deviation:	2.4	3.2

Coefficient of correlation = 0.72

Find the two regression equations that are associated with the above values.

b) From the following data of rainfall and production of rice, find the most likely production corresponding to the rainfall of 40 mm

	Rainfall (mm)	Production (quintals)
Mean :	35	50
s.d. :	5	8

Coefficient of correlation = 0.8

c) Given the following data relating the price and supply, estimate the supply when the price is 50

Mean price = 100, Mean supply = 103

Variance of price = 64, Variance of supply = 16

Correlation coefficient between the price and the supply = –0.65.

9. Find the regression equation of x on y from the following data

x :	5	9	13	17	21
y :	3	8	13	18	23

Estimate the value of x when y = 18.

10. Find the regression equation of y on x from the following data:

x :	2	4	5	6	8	11
y :	18	12	10	8	7	5

Estimate the value of y when x = 12.

A. Make a group of 8 students in each group. Ask them to visit in different colleges, office to collect their ages and the respective blood pressure.

a) Find the correlation coefficient between the ages and blood pressures of the people.

b) Find the regression equation relating the ages and the blood pressures of the people.

c) Estimate the blood pressure when particular age of the person be given.

Answers

3. i) 0.79 ii) –0.88 iii) not possible iv) 0.83 4. a) 0.4, 1.6 b) –0.48, –0.75

5. a) 4/5, 9/20, 0.6, 13, 17 b) $-\frac{3}{2}, -\frac{1}{6}$, –0.5, 4, 7

6. a) i) 0.48 ii) y = 0.73x + 5.8, x = 0.32y + 28.8 b) i) x = 0.32y + 15.6 ii) x = 0.47y + 0.83
b) y = 0.32x + 12.56, x = 0.84y + 20.16, 18.96, 45.36
7. a) i) y = 0.5x + 22 ii) y = 0.3x + 4.1 b) 56.4 quintals c) 119.25
8. a) Y = 0.96X + 10.56, X = 0.54Y + 25.12
9. x = 0.8y + 2.6, 17 10. y = –1.34x + 18.04, 1.96

Chapter 13
Probability

Introduction

We have already discussed about the basic things of probability in grade XI. We hav also solved some of the problems of probability without using combination. Now in grade XII, we deal and sholve the problem of probability using combination and we also deal the dependent events and then the conditional probability. For easiness we review some of the related terms of the probability which are given below.

Mutually exclusive cases (events)

Two or more events are said to be mutually exclusive if their simultaneous occurrence is not possible. If a coin is tossed either head or tail will occur, so head and tail are two *mutually exclusive events*.

Favourable cases

The cases or the outcomes of a random experiment which entail the happening of an event are known as the cases *favourable* to that event.

In throwing a die, the cases favourable to "getting an odd numbers" are 3.

Independent cases (Events)

Two events are said to be *independent* if the occurrence of one event does not effect the occurrence of the other. If a coin and a die are thrown, the turning head up in a coin will not effect the getting 1 on the die.

Permutation and Combination

Permutation

Permutation of a set of objects means the arrangement of objects in some order. Below, we list some of the important formula without their proofs.

(a) The total number of permutation of a set of n different objects taken r at a time denoted by P (n, r) or ${}^nP_r$ is given by

$$^nP_r = P(n, r) = \frac{n!}{(n-r)!} \quad (r \le n)$$

where $n! =$ factorial $n = 1.2.3. \ldots\ldots\ldots n.$

In particular, $5! = $ the continued product from 1 to 5

$= 1.2.3.4.5$

$= 5 \times 4! = 5 \times 4 \times 3!$

Also $\quad {}^nP_n = P(n, n) = n!\ (0! = 1)$

(b) The number of permutations of a set of n objects taken all of them at a time where p of them are of one kind, q of them the second kind, r of them of the third kind is

$$\frac{n!}{p!\,q!\,r!}$$

Combination

Combination of a set of objects means the selection of objects without regard to any order of arrangement.

(a) The total number of selections of a set of n different objects taken r at a time denoted by $C(n, r)$ or ${}^nC_r$ is given by

i) ${}^nC_r = C(n, r) = \dfrac{n!}{(n-r)!\,r!} \quad (r \le n)$

ii) ${}^nC_r = {}^nC_{n-r}$

iii) ${}^nC_0 = C(n, 0) = 1$

iv) ${}^nC_n = C(n, n) = 1$

While solving the problems of probability, combination must be used of objects are taken more than 1 at a time. We can use combinaion in case of the problem in which the object is taken one at at time also.

Definition of Probability

If there are n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to an event E, then the probability of the happening of an event E denoted by P (E) is defined by

$$P(E) = \frac{m}{n}$$

The probability P (E) of happening of an event E satisfies the following property

$$0 \le P(E) \le 1.$$

This definition is also known as 'Mathematical definition of probability.

Cor. 1 (i) If E is an impossible event then $P(E) = 0$

(ii) If E is a sure event then $P(E) = 1$.

Cor. 2 The sum of the probabilities of the occurrence and non-occurrence of an event is unity.

If the probability of occurrence of an event is denoted by P (E), the probability of its non-occurrence is denoted by $P(\overline{E})$

$$\therefore P(E) + P(\overline{E}) = 1$$

Here E and $\overline{E}$, the two mutually exclusive events, the sum of whose probabilities is unity, are known as complementary events.

Dependent Events

Two events are said to be dependent if the probability of the occurrence of one event in a trial affects the probability of the occurrence of the event in the other trial.

Let us see the following example.

A bag contains 4 red balls and 6 white balls. A ball is drawn and found to be red. Now, we find its probability.

1st draw:

n = total no. of balls = 4 + 6 = 10

m = no. of red balls = 4

P (red ball) = ?

$$P\text{ (red ball)} = \frac{m}{n} = \frac{4}{10} = \frac{2}{5}$$

If the ball drawn is not replaced, then the balls left in the bag are 3 red balls and 6 white balls. Again we draw a ball, and found to be red.

2nd draw:

n = total no. of balls = 3 + 6 = 9

m = no. of red balls = 3

P (second red ball) = ?

$$P\text{ (second red ball)} = \frac{m}{n} = \frac{3}{9} = \frac{1}{3}$$

Here the probability of getting red ball in 2^{nd} **draw** i.e. P(2^{nd} red ball) is different from the probability of getting red ball in first draw i.e. P(1^{st} red ball).

This experiment shows that the probability of getting second red ball depends upon the occurrence of the first red ball. So, this is the case of dependent events.

Conditional Probability

Let A and B be two dependent events. Then the probability of occurrence of event A when it is given that the event B has already occurred is known as the conditional probability of the event A. It is denoted by P(A/B). Similarly, the probability of the event B given that the event A has already occurred, is denoted by P(B/A). P(A/B) and P(B/A) are calculated by the following formulae.

$$P(A/B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

and $$P(B/A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) \neq 0$$

when $P(A \cap B)$ is the probability of the simultaneous occurrence of the events A and B.

Multiplicative law of probability of dependent events

Let A and B be two dependent events with their respective probabilities P(A) and P(B). Then the probability of their simultaneous occurrence denoted by P(A and B) or $P(A \cap B)$ or P(AB) is given by

$$P(A \cap B) = P(A).P(B/A)$$

where P(B/A) is the conditional probability of B given that the event A has already occurred.

Also, $$P(A \cap B) = P(B).P(A/B)$$

when P(A/B) is the conditional probability of A given that the event B has already occurred.

Note: If A and B are independent events then P(B/A) = P(B) and P(A/B) = P(A) so that

$$P(A \cap B) = P(A).P(B)$$

Let us have the following example:

From a well shuffled pack of 52 cards, two cards are drawn one after another (in succession). Find the probability of getting first red card and second black card. It is given that the first card is not replaced before the second card is drawn.

Let us define the following terms.

R : Event of getting red card

B : Event of getting black card

There are 52 cards, 26 of them are red and 26 are black.

1st draw:

n = total no. of cards = 52

m = no. of red cards = 26

P(R) = P(red card) = ?

$$P(R) = \frac{m}{n} = \frac{26}{52} = \frac{1}{2}$$

Before drawing the second card, first card is not replaced. So, the number of cards left = 52 – 1 = 51 and the no. of black cards = 26

2nd draw:

n = total no. of cards = 51

m = no. of black cards = 26

P(black card) = P(B/R) = ?

$$P(B/R) = \frac{m}{n} = \frac{26}{51}$$

Now, $P(R \cap B)$ = Prob. of getting first red card and second black card

$$= P(R) \, . \, P(B/R) = \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{51}$$

Tree Diagram

Here, we deal with the simultaneous occurrence of two or more events. TWwe consider their different possible outcomes and their respective probabilities. These different possible outcomes and their respective probabilities can be presented in a diagram known as the probability tree diagram or simply tree diagram. This diagram consists of a set of line segments, each line segment in each branch shows one of the possible outcomes with its probability. Also with the help of the tree-diagram, the probability of the simultaneous occurrence of the events of the random experiment can be obtained as the product of the probability of the individual events using the path of the line segment representing the event.

Worked Out Examples

Example 1

A bag contains 4 white and 8 red marbles. If a marble is drawn, what is the probability that (i) it is a white marble (ii) it is a red marble.

Solution:

No. of white marbles = 4, No. of red marbles = 8

Total no. of marbles = 4 + 8 = 12

n = total no. of possible cases = $^{12}C_1 = C(12, 1) = 12$

i) m = no. of favourable cases = C(4, 1) = 4

$$P(\text{a white marble}) = \frac{m}{n} = \frac{4}{12} = \frac{1}{3}$$

ii) m = no. of favourable cases = C(8, 1) = 8

$$P(\text{a red marble}) = \frac{m}{n} = \frac{8}{12} = \frac{2}{3}$$

Example 2

From 24 tickets numbered from 1 to 24, one ticket is drawn at random. Find the probability that it is a) an odd number b) a multiple of 4 or 6.

Solution:

Total no. of tickets = 24

No. of ticket taken at a time = 1

n = total no. of possible cases = C(24, 1) = 24

a) **For odd number**

No. of odd numbers = 12

m = no. of favourable cases = C(24, 1) = 24

P(an odd number) = $\frac{m}{n} = \frac{12}{24} = \frac{1}{2}$

b) **Multiple of 4 or 6**

The multiple of 4 are 4, 8, 12, 16, 20, 24

The multiple of 6 are 6, 12, 18, 24

The multiples of 4 or 6 are 4, 6, 8, 12, 16, 18, 20, 24 i.e. 8 numbers

m = no. of favourable cases = C(8, 1) = 8

P(a multiple of 4 or 6) = $\frac{m}{n} = \frac{8}{24} = \frac{1}{3}$

Example 3

A bag contains 4 red and 6 white balls. Two balls are drawn at random. Find the probability that a) both are red b) both are white c) they are different colours.

Solution:

No. of red balls = 4, No. of white balls = 6

Total no. of balls = 4 + 6 = 10 No. of balls taken at a time = 2

n = total no. of possible cases = C(10, 2) = $\frac{10 \times 9}{2 \times 1} = 45$

a) **For both red balls**

m = no. of favourable cases = C(4, 2) = $\frac{4 \times 3}{2 \times 1} = 6$

P(both red balls) = $\frac{m}{n} = \frac{6}{45} = \frac{2}{15}$

b) **For both white balls**

m = no. of favourable cases = C(6, 2) = $\frac{6 \times 5}{2} = 15$

P(both white balls) = $\frac{m}{n} = \frac{15}{45} = \frac{1}{3}$

c) **For different colours i.e. one red and one white**

m = no. of favourable cases = C(4, 1) × C(6, 1) = 4 × 6

P(two different colour balls) = $\frac{m}{n} = \frac{4 \times 6}{45} = \frac{8}{15}$

Example 4

Suppose 3 cards are drawn from a well shuffled deck of 52 cards. What is the probability of getting a) all black cards b) all are spades c) two are aces.

Solution:

Total no. of cards = 52, No. of black cards = 26

No. of spades = 13, No. of aces = 4

No. of cards drawn at a time = 3

$$n = \text{total no. of cards} = C(52, 3) = \frac{52 \times 51 \times 50}{3 \times 2 \times 1} = 26 \times 17 \times 50$$

a) **For all black cards**

$$m = \text{no. of favourable cases} = C(26, 3) = \frac{26 \times 25 \times 24}{3 \times 2 \times 1} = 26 \times 25 \times 4$$

$$P(\text{all black cards}) = \frac{m}{n} = \frac{26 \times 25 \times 4}{26 \times 17 \times 50} = \frac{2}{17}$$

b) **For all spades**

$$m = \text{no. of favourable cases} = C(13, 3) = \frac{13 \times 12 \times 11}{3 \times 2 \times 1} = 13 \times 2 \times 11$$

$$P(\text{all three spades}) = \frac{m}{n} = \frac{13 \times 2 \times 1}{26 \times 17 \times 50} = \frac{11}{850}$$

c) **For two aces:** i.e. two aces and one other than ace

m = no. of favourable cases

= two aces from 4 aces and rest 1 from 48 other cards

$= C(4, 2) \times C(48, 1)$

$= \frac{4 \times 3}{2 \times 1} \times 48 = 6 \times 48$

$$P(\text{two aces}) = \frac{m}{n} = \frac{6 \times 48}{26 \times 17 \times 50} = \frac{72}{5525}$$

Example 5

A committee of 5 is to be formed out of a goup of 8 men and 6 women. Find the probability that in the committee there will be a) 3 men and 2 women b) 2 men and 3 women c) 3 men and 2 women or 2 men and 3 women

Solution:

No. of men = 8, No. of women = 6

Total no. of persons = 8 + 6 = 14

No. of persons taken at a time = 5

$$n = \text{total no. of possible cases} = C(14, 5) = \frac{14 \times 13 \times 12 \times 11 \times 10}{5 \times 4 \times 3 \times 2} = 14 \times 13 \times 11$$

a) **For 3 men and 2 women**

3 men from 8 and 2 women from 6 are to be selected.

m = no. of favourable cases

$= C(8, 3) \times C(6, 2) = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{6 \times 5}{2 \times 1} = 56 \times 15$

$P(\text{3 men and 2 women}) = \frac{m}{n} = \frac{56 \times 15}{14 \times 13 \times 11} = \frac{60}{143}$

b) **For 2 men and 3 women**

2 men from 8 and 3 women from 6 are to be selected.

m = no. of favourable cases

$= C(8, 2) \times C(6, 3) = \frac{8 \times 7}{2 \times 1} \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 28 \times 20$

$P(\text{2 men and 3 women}) = \frac{m}{n} =$

Example 6

A and B are two events such that P(A) = 0.45, P(B) = 0.60 and P(A ∪ B) = 0.90, find P(A/B) and P(B/A).

Solution:

Here, P(A) = 0.45, P(B) = 0.60 and P(A ∪ B) = 0.90

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.90 = 0.45 + 0.60 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 0.45 + 0.60 - 0.90 = 0.15$

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.60} = \frac{1}{4}$

$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.15}{0.45} = \frac{1}{3}$

Example 7

If P(A) = 0.40, P(B) = 0.32 and P(B/A) = 0.5; find P(A/B).

Solution:

Here, P(A) = 0.4, P(B) = 0.32 and P(B/A) = 0.5

$P(B/A) = \frac{P(A \cap B)}{P(A)}$

$\Rightarrow 0.5 = \frac{P(A \cap B)}{0.4}$

$\Rightarrow P(A \cap B) = 0.5 \times 0.4 = 0.2$

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.32} = \frac{5}{8}$

(A ∪ B) = 0.90

Example 8

The probability that a student passes in Economics is $\frac{3}{4}$ and the probability that he/she passes in Economics and Marketing is $\frac{7}{12}$. Find the probability that he/she will pass in Marketing given that he/she has passed in Economics.

Solution:

P(E) = Prob. of passing a student in Economics $= \frac{3}{4}$

$P(E \cap M)$ = Prob. of passing in Economics and Marketing $= \frac{7}{12}$

P(M/E) = Prob. of passing in Marketing given that he/she passed in Economics

Now, $P(M/E) = \frac{P(M \cap E)}{P(E)} = \frac{7/12}{3/4} = \frac{7}{12} \times \frac{4}{3} = \frac{7}{9}$

Example 9

In a certain college, 80% of the students passed in English, 75% passed in Mathematics and 60% passed in both English and Mathematics. A student is selected at random

a) What is the probability that he/she passed in Mathematics given that he/she passed in English.

b) Find the probability that he/she will pass in English if he/she pass in Mathematics

Solution:

E : Event that the student passed in English

M : Event that the student passed in Mathmatics

By given, P(E) = 80% = 0.80, P(M) = 75% = 0.75 and $P(E \cap M)$ = 60% = 0.60

a) $P(M/E) = \frac{P(M \cap E)}{P(E)} = \frac{0.60}{0.80} = \frac{3}{4}$

b) $P(E/M) = \frac{P(M \cap E)}{P(M)} = \frac{0.60}{0.75} = \frac{4}{5}$

Example 10

A coin has tossed twice and possible outcomes are assumed to be equally likely. If E represents the event that both coins show the same face and F, the event that at least one head is observed. Find P(E), P(F), P(E/F) and P(F/E).

Solution:

A coin has two faces head (H) and tail (T). So, when two coins are tossed, the possible outcomes are HH, TH, HT, TT

∴ sample space (S) = {HH, HT, TH, TT} ⇒ n(S) = 4

E = {HH, TT} ⇒ n(E) = 2

$F = \{HT, TH, HH\} \Rightarrow n(F) = 3$

$E \cap F = \{HH\} \Rightarrow n(E \cap F) = 1$

$P(E) = \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$, $P(F) = \frac{n(F)}{n(S)} = \frac{3}{4}$, $P(E \cap F) = \frac{n(E \cap F)}{n(S)} = \frac{1}{4}$

Now, $P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$

and $P(F/E) = \frac{P(E \cap F)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}$

Example 11

A die is rolled two times. Find the conditional probability that the number in each face shows odd numbers given that the sum of the numbers appeared in the faces of die is observed to be 8.

Solution:

The die has six faces and marked 1, 2, 3, 4, 5, 6 in the faces.

$E = \{1, 2, 3, 4, 5, 6\} \Rightarrow n(E) = 6$

$n(S)$ = total number of elements when die is rolled two times $= 6 \times 6 = 36$

A = event that the number in each face shows odd number

$= \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

$\Rightarrow n(A) = 9$

B = event that the sum of the number in the faces being 8

$= \{(2, 6), (6, 2), (3, 5), (5, 3) (4, 4)\}$

$\Rightarrow n(B) = 5$

$A \cap B = \{(3, 5), (5, 3)\} \Rightarrow n(A \cap B) = 2$

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/36}{5/36} = \frac{2}{5}$

Example 12

There are 6 red balls and 3 white balls in a box. Two balls are drawn randomly one after the other without replacing the first drawn ball. Find the probability that a) both balls are white b) first is white and second is red.

Solution:

a) W_1 : event of getting white ball in first draw

W_2 : event of getting white ball in second draw

1st draw:

n = total no. of balls = 6 + 3 = 9

m = no. of white balls = 3

$P(W_1) = P(\text{first white ball}) = ?$

$$P(W_1) = \frac{m}{n} = \frac{3}{9} = \frac{1}{3}$$

Since the ball drawn is not replaced, so no. of balls left = 9 – 1 = 8

and no. of white balls = 3 – 1 = 2

2nd draw:

n = total no. of balls = 8

m = no. of white balls = 2

$P(W_2/W_1) = ?$

$$P(W_2/W_1) = \frac{m}{n} = \frac{2}{8} = \frac{1}{4}$$

Now, $P(W_1 \cap W_2)$ = Prob. of getting both white balls

$$= P(W_1).P(W_2/W_1)$$

$$= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

b) W_1 : Event of getting white ball in first draw

R_2 : Event of getting red ball in second draw

1st draw:

$$P(W_1) = \frac{m}{n} = \frac{3}{9} = \frac{1}{3}$$

2nd draw:

n = total no. of balls = 9 – 1 = 8

m = no. of red balls = 6

$P(R_2/W_1) = ?$

$$P(R_2/W_1) = \frac{m}{n} = \frac{6}{8} = \frac{3}{4}$$

$P(W_1 \cap R_2)$ = Prob. of getting first white and second white balls

$$= P(W_1).P(R_2/W_1)$$

$$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

Alternatively: After understanding the question, we can solve the above problem in the following short way

a) $P(W_1)$ = Prob. of getting white ball in first draw

$$= \frac{3}{9} = \frac{1}{3}$$

Again $P(W_2/W_1) = ?$

$P(W_2/W_1) = \frac{2}{8} = \frac{1}{4}$

$P(W_1 \cap W_2) = P(W_1).P(W_2/W_1) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

b) $P(W_1) = \frac{1}{3}$

$P(R_2/W_1)$ = Prob. of getting red ball in second draw given that in first draw it is white

$= \frac{6}{8} = \frac{3}{4}$

$P(W_1 \cap R_2)$ = Prob. of getting first white and second red ball = $P(W_1).P(R_2/W_1)$

$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$

Example 13

Two cards are drawn from a well-shuffled deck of 52 cards. The cards drawn are one at a time and before the second draw, the first is not replaced. Find the probability that a) both are aces b) first is red card and second black card.

Solution:

Let A_1: Event of getting an ace in first draw A_2: Event of getting an ace in second draw

a) Total number of cards = 52

No. of aces = 4

1st draw:

n = total no. of cards = 52

m = no. of aces = 4

$P(A_1) = \frac{m}{n} = \frac{4}{52} = \frac{1}{13}$

$P(A_1)$ = Prob. of getting first ace = ?

The card drawn is not replaced. So, no. of cards left = 52 – 1 = 51

2nd draw:

n = total no. of cards = 51

m = no. of aces = 4 – 1 = 3

$P(A_2/A_1)$ = ?

$P(A_2/A_1) = \frac{m}{n} = \frac{3}{51} = \frac{1}{17}$

$P(A_1 \cap A_2)$ = Prob. of getting both aces

$= P(A_1)\, P(A_2/A_1) = \frac{1}{13} \cdot \frac{1}{17} = \frac{1}{221}$

b) Let R_1: Red card in first draw, R_2: Black card in second draw

1st draw:

n = total no. of cards = 52

m = no. of red cards = 26

$P(R_1)$ = Prob. of getting first red card = ?

$$P(R_1) = \frac{m}{n} = \frac{26}{52} = \frac{1}{2}$$

The first card drawn is not replaced.

2nd draw:

n = total no. of cards = 52 – 1 = 51

m = no. of black cards = 26

$P(B_2/R_1)$ = ?

$$P(B_2/R_1) = \frac{m}{n} = \frac{26}{51}$$

Now, $P(R_1 \cap B_2)$ = Prob. of getting first red and second black card

$$= P(R_1).P(B_2/R_1) = \frac{1}{2} \cdot \frac{26}{51} = \frac{13}{51}$$

Alternatively: The solution can be made short in the following way.

a) $P(A_1)$ = Prob. of drawing an ace in first draw = $\frac{4}{52}$

$P(A_2/A_1)$ = ?

$$P(A_2/A_1) = \frac{3}{51}$$

$$P(A_1 \cap A_2) = P(A_1).P(A_2/A_1) = \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$$

Similarly (b) can be solved.

Example 14

From the following table

	Male	Female	Total
Smokers	48	6	54
Non-smokers	112	34	146
	160	40	200

Compute the probabilty of selecting

a) male given that they are smokers

b) female given that they are non-smokers.

Solution:

M : Male, F : Female, S : Smoker; N : Non-smoker

Total no. of people = n(U) = 200

No. of male smokers = $n(M \cap S) = 48$

No. of female non-smokers = $n(F \cap N) = 34$

No. of smokers = n(S) = 54

No. of non-smokers = n(N) = 146

$$\because P(M \cap S) = \frac{n(M \cap S)}{n(U)} = \frac{48}{200} \text{ etc.}$$

Now,

a) $P(M/S) = \frac{P(M \cap S)}{P(S)} = \frac{48/200}{54/200} = \frac{8}{9}$

b) $P(F/N) = \frac{P(F \cap N)}{P(N)} = \frac{34/200}{146/200} = \frac{17}{73}$

Example 15

What is the probability that a couple's second child is a boy given that their first child is a girl?

Solution:

A couple has two children. The child may be a boy or a girl. Let B and G represent Boy and Girl respectively.

$\therefore$ sample space = S = {BB, BG, GB, GG} = n(S) = 4

E : Event of first child is a girl

E = {GB, GG} $\Rightarrow$ n(E) = 2

F : Event of second child is a boy

= {BB, GB} $\Rightarrow$ n(F) = 2

$E \cap F$ = {GB} $\Rightarrow$ $n(E \cap F) = 1$

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{4}, \quad P(F) = \frac{2}{4} \text{ and } P(E \cap F) = \frac{1}{4}$$

$$\text{Now, } P(F/E) = \frac{P(E \cap F)}{P(E)} = \frac{1/4}{2/4} = \frac{1}{2}$$

Example 16

A box contains 8 red and 6 white balls. Two successive drawing of 2 balls are made such that the balls are not replaced before the second trial. Find the probability that the first drawing will give 2 red balls and second 2 white balls.

Solution:

No. of red balls = 8, No. of white balls = 6

Total no. of balls = 8 + 6 = 14

No. of balls drawn at a time = 2

1st draw: For red balls

n = total no. of possible cases = $C(14, 2) = \frac{14 \times 13}{2 \times 1} = 7 \times 13$

m = no. of favourable cases = $C(8, 2) = \frac{8 \times 7}{2 \times 1} = 4 \times 7$

P(first 2 red balls) = $P(R_1) = \frac{m}{n} = \frac{4 \times 7}{7 \times 13} = \frac{4}{13}$

The two red bvalls drawn are not replaced.

2nd draw: For white balls

Total no. of balls = 14 – 2 = 12

n = total no. of possible cases = $C(12, 2) = \frac{12 \times 11}{2 \times 1} = 6 \times 11$

m = no. of favourable cases = $C(6, 2) = \frac{6 \times 5}{2 \times 1} = 3 \times 5$

$$P(W_2/R_1) = \frac{m}{n} = \frac{3 \times 5}{6 \times 11} = \frac{5}{22}$$

Now, $P(R_1 \cap W_2) = P(R_1) . P(W_2/R_1)$

$$= \frac{4}{13} \times \frac{5}{22} = \frac{10}{143}$$

EXERCISE

(Use combination for problem no. 1 also)

1. a) A card is drawn from a well shuffled deck of 52 cards. What is the probability that it is (i) a spade (ii) an ace (iii) a red card?
   b) A bag contains 9 red, 7 white and 4 black balls. A ball is drawn at random. What is the probability of drawing (i) a red ball (ii) a red or a white ball?
   c) From 20 tickets marked from 1 to 20, one ticket is drawn at random. Find the probabillity that it is (i) an even number (ii) a multiple of 4 or 5.
2. a) An urn contains 8 white and 4 red balls. If two balls are drawn at random, find the probability of getting (i) both white balls (ii) both red balls (iii) they are of different colours.
   b) A class consists of 60 boys and 40 girls. If two students are chosen at random, what is the probability that (i) both are boys (ii) both are girls (iii) one boy and one girl?
   c) Two cards are drawn from a well shuffled pack of 52 cards. What is the probability of getting (i) both red cards (ii) both kings (iii) one red and one black card?

3. a) A committee of 3 members are to be formed from 7 men and 6 women. What is the probability that there will be (i) 2 men and 1 woman (ii) 2 women and 1 man (iii) 2 men and 1 woman or 2 women and 1 man.

b) Suppose 4 cards are drawn at random from a well shuffled deck of 52 cards. What is the probability that (i) all 4 are spades (ii) all 4 are black (iii) three of the cards are red.

c) An urn contains 4 white, 6 red and 5 green marbles. If three marbles are dawn, find the probability of getting (i) 2 white and 1 red marble (ii) one of each colour (iii) no red marbles (iv) all are of the same colour.

4. a) If P(B) = 0.60, P(A ∩ B) = 0.48, find P(A/B).

b) If P(A) = 0.56, P(A ∩ B) = 0.21, find P(B/A).

c) If $P(A) = \frac{5}{12}$, $P(B) = \frac{1}{4}$ and $P(A \cap B) = \frac{1}{6}$, find P(A/B) and P(B/A).

d) If P(A) = 0.42, P(B) = 0.30, P(A ∪ B) = 0.48, find P(A/B), P(B/A).

5. a) If P(A) = 0.5, P(B) = 0.8 and P(B/A) = 0.4, find P(A/B).

b) If P(A) = 0.36, P(B) = 0.5 and P(A/B) = 0.60, find P(B/A).

6. a) The probability that a person uses NTC mobile is $\frac{5}{8}$ and the probability that a person uses both Ncell and NTC mobile is $\frac{1}{4}$. What is the probability that a person uses Ncell given that he/she has used NTC mobile?

b) In a certain college, 75% of the students passed in Marketing, 60% of the students passed in Account and 45% of the students passed in both Marketing and Account. A student is selected at random. Find the probability that

i) he/she passes in Marketing given that he/she passes in Account.

ii) he/she passes in Account if he/she passes in Marketing.

c) The probability that a man likes badminton is 0.60, the probability that a man likes table tennis is 0.5 and he likes both badminton and table tennis is 0.25. Find the probability that

i) he likes badminton given that he liked table tennis

ii) he likes table tennis if he liked badminton.

7. a) A coin is tossed two times and the possible outcomes are assumed to be equally likely. If E represents the events that both head and tail have occurred and F represents the events that at least one tail is observed, find P(E), P(F), P(E/F) and P(F/E).

b) i) A die is rolled once. What is the conditional probability that face turned up is a prime number given that the outcome is an even number.

ii) Two dice are thrown once. Find the conditional probability that the numbers in each face show the same number if the sum of the numbers appeared in the faces of the dice is observed to be greater than 9.

8. a) A lot containing 9 items of which 3 are defective. Two items are chosen from the lot at random one after another without replacement. Find the probability that

i) both of them are defective

ii) both of them are non-defective

iii) first is defective and second not.

b) A box contain 4 red and 6 white balls. Two balls are drawn one after another. If first ball is not replaced before the second ball is drawn, find the probability that i) both are white ii) both are red iii) first red and second white iv) one red and one white

c) A box contains 3 red, 4 white and 5 blue balls. From this box 3 balls are drawn in succession (i.e. one by one). Find the probability that they are drawn in order red, white and blue if each ball is not replaced before other ball is drawn.

d) Two cards are drawn successively one after the other from a well-shuffled pack of 52 cards. If the first card drawn is not replaced, find the probability that i) both cards are black ii) first is black and second red iii) both cards are kings.

9. a) The following table gives the information about the skilled and unskilled workers of a factory

Workers	Male	Female	Total
Skilled	75	50	125
Unskilled	50	25	75
Total	125	75	200

A worker is selected at random. Find the probability that the selected worker is a i) male given that he is skilled ii) female given that she is unskilled.

b) A factory has some automatic and some mechanical machines. Some of these machines are new and some old.

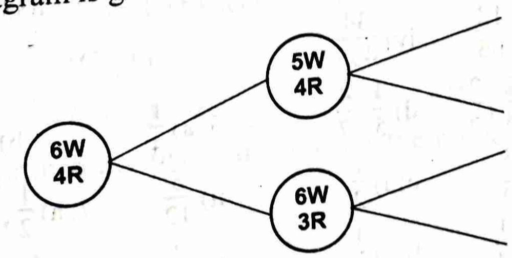
Machine	Automatic	Mechanical	Total
New	40	30	70
Old	20	10	30
Total	60	40	100

A machine is selected. Find the probability that

i) the machine is new given that it is automatic.

ii) the machine is old given that it is mechanical.

A. An incomplete tree-diagram is given below (two balls are drawn in succession)



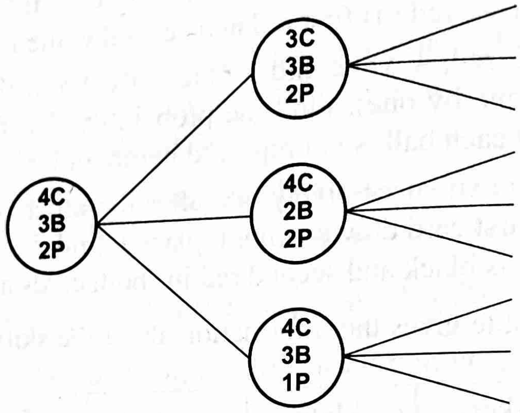
a) Complete the above tree diagram.

b) Show that the total probability is one.

c) Find the probability $P(W_1 \cap W_2)$ ($W_1 \rightarrow$ white ball in first draw)

d) Find the probability $P(R_1 \cap W_2)$.

B. A bag contains 4 copies, 3 books and 2 pencils. An incomplete tree-diagram showing the drawing of each of the three items in succession is presented below



a) Complete the tree-diagram.

b) Find the sample space.

c) Show that the total probability is one.

d) Find the probability $P(C_1 \cap B_2)$ ($C_1 \rightarrow$ copy drawn in 1st draw)

e) Find the probability $P(B_1 \cap B_2)$

f) Find the probability $P(P_1 \cap C_2)$

Answers

1. a) i) $\frac{1}{4}$ ii) $\frac{1}{13}$ iii) $\frac{1}{2}$ b) i) $\frac{9}{20}$ ii) $\frac{4}{5}$ c) i) $\frac{1}{2}$ ii) $\frac{2}{5}$

2. a) i) $\frac{14}{33}$ ii) $\frac{1}{11}$ iii) $\frac{16}{33}$ b) i) $\frac{59}{165}$ ii) $\frac{26}{165}$ iii) $\frac{16}{33}$

c) i) $\frac{25}{102}$ ii) $\frac{1}{221}$ iii) $\frac{26}{51}$

3. a) i) $\frac{63}{143}$ ii) $\frac{105}{286}$ iii) $\frac{231}{286}$ b) i) $\frac{11}{4165}$ ii) $\frac{46}{833}$ iii) $\frac{208}{833}$

c) i) $\frac{36}{455}$ ii) $\frac{24}{91}$ iii) $\frac{12}{65}$ iv) $\frac{34}{455}$

4. a) $\frac{4}{5}$ b) $\frac{3}{8}$ c) $\frac{2}{3}, \frac{2}{5}$ d) $\frac{4}{5}, \frac{4}{7}$

5. a) $\frac{1}{4}$ b) $\frac{5}{6}$

6. a) $\frac{2}{5}$ b) i) $\frac{3}{4}$ ii) $\frac{3}{5}$ c) i) $\frac{1}{2}$ ii) $\frac{5}{12}$

7. a) $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}, 1$ b) i) $\frac{1}{3}$ ii) $\frac{1}{3}$

8. a) i) $\frac{1}{12}$ ii) $\frac{5}{12}$ iii) $\frac{1}{4}$ b) i) $\frac{1}{3}$ ii) $\frac{2}{15}$ iii) $\frac{4}{15}$ iv) $\frac{8}{15}$

c) $\frac{1}{22}$ d) i) $\frac{25}{102}$ ii) $\frac{13}{51}$ iii) $\frac{1}{221}$

9. a) i) $\frac{3}{5}$ ii) $\frac{1}{3}$ b) i) $\frac{2}{3}$ ii) $\frac{1}{4}$

Binomial Distribution

When a coin is tossed either head or tail will turn up. The probability of getting each of head or tail when it is tossed is $\frac{1}{2}$. So, if a coin is tossed 100 times, the expected no. of heads turn up will be $\frac{1}{2} \times 100 = 50$ times. This is said to be the theoretical frequency of head. The distribution formed by such an assumption is known as the theoretical distribution. But in practical when 100 times a coin is tossed, only 10 times head or 80 times head may occur.

An experiment with only two types of outcomes is known as Bernnoully process. The two outcomes are known as success and failure. For example: tossing a coin, result of an examination, production of bulls etc.

Binomial Distribution

The discrete distribution derived from Bernoully process is known as Binomial distribution. The distribution is also known as Bernoully distribution. The basic assumption under which Binomial distribution is based is given below:

a) Random experiment should be performed for the fixed number of times.

b) The experiment should have only two mutually exclusive outcomeds 'success' and 'failure'.

c) All the experiment performed should be independent of one another.

d) The probability of success should be constant for every experiment.

Probability function of the Binomial distribution

If the probability of a success in one trial is known, the probabilities of success of one, two, three, ... n trials can be known.

Let n = no. of trails performed

p = probability of a success in a trial

q = probability of a failure in a trial

such that $p + q = 1$

r = no. of successes in n trials

then $P(r) = P(x = r)$ = Prob. of r successes in n trials

$$= {}^nC_r\, p^r q^{n-r}$$

$$= C(n, r)\, p^r q^{n-r} \quad \text{......(i)}$$

This is known as the probability mass function for the random variable X; n and p (or q) being the two parameters.

The probabilities of 0, 1, 2, 3, ..., n successes obtained by putting r = 0, 1, 2, 3, ..., n in (i) are listed below.

r (No. of successes)	Prob. of r successes i.e. P(r)
0	$P(0) = q^n$
1	$P(1) = {}^nC_1\, p^1 q^{n-1}$
2	$P(2) = {}^nC_2\, p^2 q^{n-2}$
:	: :
n	$P(n) = {}^nC_n\, p^n q^0 = p^n$

The probabilities of 0, 1, 2, 3, ..., n successes in n trials listed above are the successive terms of the binomial expansion of $(q + p)^n$. Hence the distribution is known as the **Binomial distribution** or **Binomial probability distribution**.

Mean and Standard deviation of Binomial distribution

If n = no. of trials,

p = probability of a success in one trial

q = 1 – p = probability of a failure, then

mean of the binomial distribution = np

and the variance of the binomial distribution = $\sigma^2 = npq$

so that the standard deviation $(\sigma) = \sqrt{npq}$

where the two constants n and p (or q) are known as the parameters.

Characteristics of Binomial distribution

Binomial distribution has the following characteristics

a) Since in a binomial distribution, the random variable X assumes only the integral values 0, 1, 2, 3, ... so the distribution is the discrete probability distribution.

b) Knowing the two parameters n and p, binomial distribution can completely be determined.

c) The variance of the binomial distribution is less than the mean i.e. npq < np as p < 1 and q < 1.

d) If $p = q = \frac{1}{2}$, the binomial distribution is symmetrical.

e) If $p < \frac{1}{2}$, the binomial distribution is positively skewed and if $p > \frac{1}{2}$, it is negatively skewed.

Worked Out Examples

Example 1

A symmetrical die is rolled 144 times. Getting an even number is assumed to be a success.

a) Find the mean and standard deviation for the binomial distribution.

b) Also verify that variance is less than the mean.

Solution:

a) There are six faces marked with 1, 2, 3, 4, 5, 6 in a die. 3 faces of them have even numbers.

$p = \text{prob. of getting an even number} = \frac{3}{6} = \frac{1}{2}$

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$n = \text{no. of experiments} = 144$

Mean of the binomial distribution $= np = \frac{1}{2} \times 144 = 72$

$\text{s.d.} = \sigma = \sqrt{npq} = \sqrt{144 \times \frac{1}{2} \times \frac{1}{2}} = 6$

b) Variance $= \sigma^2 = (t)^2 = 36$

and mean $= \bar{X} = 72$

Thus, variance < mean.

Example 2

The mean of the binomial distribution is 42 and the variance is 28.

a) Is the above statement consistent?

b) if consistent, find p, q and r.

Solution:

a) Here, mean $= \bar{X} = 42$, variance $= \sigma^2 = 28$

Also, mean = np $\Rightarrow$ $42 = np$ ......(i)

Again, variance = npq $\Rightarrow$ $28 = npq$ ......(ii)

From (i) and (ii)

$\frac{42}{28} = \frac{np}{npq}$

$\Rightarrow \frac{3}{2} = \frac{1}{q}$ $\quad \therefore q = \frac{2}{3} < 1$, so the given statement is consistent.

b) $q = \frac{2}{3}$

$p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

From (i), $42 = n \times \frac{1}{3}$

$\therefore \quad n = 126$

Example 3

For a binomial distribution, n = 3, p = 0.4, find P(X = 0), P(X = 2) and P(X ≥ 1).

Solution:

Here $n = 3,\ p = 0.4,\ q = 1 - p = 1 - 0.4 = 0.6$

$P(X = r) = P(r) \quad = C(n, r)p^r q^{n-r}$

$P(X = 0) = P(0) \quad = C(3, 0)\,(0.4)^0\,(0.6)^{3-0}$

$= 1.1.(0.6)^3 = 0.216$

$P(X = 2) = P(2) \quad = C(3, 2)\,(0.4)^2\,(0.6)^{3-2}$

$= \frac{3 \times 2}{2 \times 1} = (0.16) \times (0.6) = 0.288$

$P(X = r \geq 1) = P(r \geq 1) = 1 - P(0)$

$= 1 - 0.216 = 0.784$

Example 4

If 20% of the bulbs produced by a machine are defective, determine the probability that out of 4 bulbs chosen at random i) no bulb is defective ii) one is defective.

Solution:

n = no. of bulbs = 4

p = prob. of a defective bulb = 20% = 0.2

q = prob. of a non-defective bulb = 1 – 0.2 = 0.8

$P(r) = C(n, r)\, p^r q^{n-r}$

i) r = no. of defective bulb = 0

$P(r = 0) = C(4, 0)\,(0.2)^0\,(0.8)^{4-0}$

$= 1.1.(0.8)^4 = 0.4096$

ii) r = no. of defective bulb = 1

$P(r = 1) = C(4, 1)\,(0.2)^2\,(0.8)^{4-1}$

$= 4.\,(0.2)\,(0.8)^3 = 0.4096$

Example 5

A die is rolled 4 times. If getting 2 or 4 is a success, find the probability of getting (i) only one success ii) at least one success iii) not more than two successes.

Solution:

n = no. of trials = 4

$p = \text{prob. of success} = \frac{2}{6} = \frac{1}{3}$

$q = \text{prob. of failure} = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

$P(X = r) = P(r) = \text{Prob. of r successes in n trials}$
$= C(n, r)\, p^r q^{n-r}$

i) $P(r = 1) = C(4, 1) \cdot \left(\frac{1}{3}\right)^1 \cdot \left(\frac{2}{3}\right)^{4-1}$

$= 4 \times \frac{1}{3} \times \frac{8}{27} = \frac{32}{81}$

ii) $P(r = 0) = P(0) = \text{Prob. of no success}$

$= C(4, 0) \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{4-0}$

$= 1.1.\frac{16}{81} = \frac{16}{81}$

$P(\text{at least one success}) = P(r \geq 1)$

$= 1 - P(0) = 1 - \frac{16}{81} = \frac{65}{81}$

iii) $P(r = 2) = P(2) = C(4, 2) \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{4-2}$

$= \frac{4 \times 3}{2 \times 1} \times \frac{1}{9} \times \frac{4}{9} = \frac{8}{27}$

Now, P(not more than two successes)

$= P(r \leq 2) = P(0) + P(1) + P(2)$

$= \frac{16}{81} + \frac{32}{81} + \frac{8}{27} = \frac{8}{9}$

Example 6

Four coins are tossed simultaneously. What is the probability of getting a) two heads b) all 4 heads c) no head d) at least two heads.

Solution:

n = no. of coins = 4

$p = \text{prob. of getting a head} = \frac{1}{2}$

$q = \text{prob. of getting a tail} = \frac{1}{2}$

$P(r) = \text{Prob. of r successes} = C(n, r)\, p^r q^{n-r}$

a) No. of heads = r = 2

$$P(r = 2) = P(2) = C(4, 2)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{4-2}$$

$$= \frac{4 \times 3}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{8}$$

b) No. of heads = r = 4

$$P(r = 4) = P(4) = C(4, 4)\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{4-4}$$

$$= 1 \cdot \frac{1}{16} \cdot 1 = \frac{1}{16}$$

c) No. of head = r = 0

$$P(r = 0) = P(0) = C(4, 0)\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{4-0}$$

$$= 1 \cdot 1 \cdot \frac{1}{16} = \frac{1}{16}$$

d) No. of head = r = 1

$$P(r = 1) = P(1) = C(4, 1)\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{4-1}$$

$$= 4 \cdot \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{4}$$

$$P(r \geq 2) = 1 - P(0) - P(1)$$

$$= 1 - \frac{1}{16} - \frac{1}{4} = \frac{11}{16}$$

Example 7.

The average percentage of a failure in a certain examination is 25%. What is the probability that out of 5 students a) 1 will pass the examination ii) two or more students will pass the examination.

Solution:

n = no. of students = 5

p = prob. of a success = 75% = $\frac{3}{4}$

q = prob. of a failure = 25% = $\frac{1}{4}$

$P(r) = C(n, r)\, p^r q^{n-r}$

a) No. of students passed = 1

$$P(r = 1) = P(1) = C(5, 1)\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{5-1}$$

$= 5 \times \frac{3}{4} \times \frac{1}{256} = \frac{15}{1024}$

b) $P(r = 0) = P(0) = C(5, 0)\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^{5-0}$

$= 1 \cdot 1 \cdot \frac{1}{1024} = \frac{1}{1024}$

Now, $P(r \geq 2) = 1 - P(0) - P(1)$

$= 1 - \frac{1}{1024} - \frac{15}{1024} = \frac{63}{64}$

Example 8

A bionomial distribution has 5 independent trials. If the probabilities of 1 and 2 successes are $\frac{1}{4}$ and $\frac{1}{3}$ respectively, find p and q. Also find $P(r = 3)$.

Solution:

Here, $P(r = 1) = P(1) = \frac{1}{4}$, $P(r = 2) = P(2) = \frac{1}{3}$, $n = 5$

p = prob. of a success , q = prob. of failure

$P(r) = C(n, r)\, p^r q^{n-r}$

$\Rightarrow P(1) = C(5, 1)\, p^1 q^{5-1}$

$\Rightarrow \frac{1}{4} = 5pq^4$ ......(i)

Again $P(2) = C(5, 2)p^2q^{5-2}$

$\Rightarrow \frac{1}{3} = \frac{5 \times 4}{2} \times p^2q^3$

$\Rightarrow \frac{1}{3} = 10p^2q^3$ ......(ii)

From (i) and (ii)

$\frac{1/4}{1/3} = \frac{5pq^4}{10p^2q^3}$

$\Rightarrow \frac{3}{4} = \frac{q}{2p}$

$\Rightarrow \frac{3}{4} = \frac{1-p}{2p}$ $(\because q = 1 - p)$

$\Rightarrow 3p = 2 - 2p$

$\Rightarrow 5p = 2$

$\therefore p = \frac{2}{5}$

and $q = 1 - p = 1 - \frac{2}{5} = \frac{3}{5}$

Now, $P(3) = C(5, 3)\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^{5-3}$

$= \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times \frac{8}{125} \times \frac{9}{25} = \frac{144}{625}$

Example 9

Find the probability that a family with 3 children has a) 2 boys and 1 girl b) 2 girls and 1 boy c) no girls. Assume that boy and girl are equally likely.

Solution:

n = no. of children = 3

p = prob. of a boy = $\frac{1}{2}$

q = prob. of a girl = $\frac{1}{2}$

P(r) = Prob. of r successes = $C(n, r)\, p^r q^{n-r}$

a) r = no. of boys = 2

$P(r = 2) = P(2) = C(3, 2)\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{3-2}$

$= 3 \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}$

b) r = no. of boys = 1

$P(r = 1) = P(1) = C(3, 1)\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{3-1}$

$= 3 \times \frac{1}{2} \times \frac{1}{4} = \frac{3}{8}$

c) No girl means all 3 boys, so

r = no. of boys = 3

$P(r = 3) = P(3) = C(3, 3)\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{3-3}$

$= 1 . \frac{1}{8} . 1 = \frac{1}{8}$

Example 10

In a vacancy of 20 seats of accountants, 16 candidates with BBA and rest BBS have applied. If 5 candidates are selected, find the probability that a) 3 of them are BBA holders b) 3 of them are BBS holders.

Solution:

n = no. of candidates = 20

m = no. of BBA holders = 16

p = prob. of a success (i.e. BBA holders) = $\frac{16}{20} = \frac{4}{5}$

$q = 1 - p = 1 - \frac{4}{5} = \frac{1}{5}$

a) n = no. of selected candidates = 5

$p = \frac{4}{5}, q = \frac{1}{5}$

r = no. of BBA holder candidates = 3

$P(r) = C(n, r)\, p^r q^{n-r}$

$P(3) = C(5, 3) \left(\frac{4}{5}\right)^3 \left(\frac{1}{5}\right)^{5-3}$

$= \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times \frac{64}{125} \times \frac{1}{25} = \frac{128}{625}$

b) n = no. of selected candidates = 5

p = prob. of a success (i.e. BBS holder) = $\frac{4}{20} = \frac{1}{5}$

$q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

r = no. of BBS holder candidate = 3

$P(3) = C(5, 3) \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^{5-3}$

$= \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \times \frac{1}{125} \times \frac{16}{25} = \frac{32}{625}$

EXERCISE

1. a) The mean and the variance of the binomial distribution are 6 and 4 respectively. Find p, q and n.

   b) In a binomial distribution, a student obtained the results that mean = 3.2 and s.d. (σ) = $\sqrt{4.8}$. Comment on the result obtained.

   c) The mean of the binomial distribution is 20 and standard deviation is 4. Is there any inconsistency in the statement?

      i) If no inconsistency, find p, q and n.

      ii) Also find the binomial distribution.

2. a) Find the mean and standard deviation for binomial distribution if
   i) $n = 40$ and $q = 0.50$ ii) $n = 600, p = 0.4$
   b) A coin is tossed 16 times. Getting a head is a success. Find the mean and the standard deviation for binomial distribution.
   c) A die is thrown 36 times. Getting an odd number is a success. Find mean and the standard deviation of binomial distribution.
3. a) A coin is tossed 6 times. Find the probability of getting i) one head ii) 3 heads iii) heads, an odd number of times.
   b) Five coins are tossed simultaneously. Find the probability of getting i) no head ii) only one head iii) exactly two heads iv) at least one head.
   c) A die is thrown 4 times. Getting a '5' or '6' is considered to be a success. Find the probability of getting i) no success ii) one success iii) exactly 3 success.
   d) A die is rolled 4 times. Getting an even number is a success. Find the probability of getting i) two successes ii) four successes.
   e) If 3 dice are thrown simultaneously, what is the probability of getting i) no six ii) two sixes iii) 3 sixes.
4. a) A certain manufacturing process produces electric fuses of which 10% are defective. Find the probability that in a sample of 4 fuses selected at random, there will be i) no defective fuse ii) at least one defective fuse iii) at most one defective fuse.
   b) Suppose that in a certain city 60% of all recorded births are male. Suppose we select 5 births record from population, what is the probability that i) three of them are males ii) less or equal to 2 are males.
   c) The incident of occupation disease in an industry is such that the workmen have a 20% chance of suffering from it. What is the probability that out of six workmen four or more will contract the disease?
5. a) Out of 32 students in a class 8 are girls. If 3 students are selected, find the probability that i) one student is a boy ii) 2 are boys and 1 girl iii) at least one student is a boy.
   b) Six men in a group of 8 are skilled. If 3 men are selected find the probability that i) 2 men are skilled ii) 1 skilled and 2 are unskilled.
6. In a binomial distribution with $n = 4$, if $P(r = 2) = P(r = 3)$, find p and q, the probabilities of a success and a failure in a trial. Also, find $P(r = 1)$.
7. The probability of hitting a target is $\frac{1}{4}$. If 5 hittings are made, find the probability that i) none will strike the target ii) exactly one will strike the target iii) the target is destroyed if two hittings are enough to destroy the target.

A. In an objective true-false test, the following six questions are given. Write T for true and F for false.

a) Saturday comes after Friday. (……)

b) Zero is a real number. (……)

c) Increase in demand ⇒ Increase in price. (……)

d) C.V. is the absolute measure of dispersion. (……)

e) Correlation coefficient gives the degree of relationship between two variables. (……)

f) When demand and supply balance, the equilibriuim condition will be satisfied. (……)

Three questions are to be answered.
From the answers given by the student himself, find the probability

i) no answer is correct
ii) one answer is correct
iii) two answers are correct
iv) at least one answer is correct
v) at most two answers are correct.

Answers

1. a) , $\frac{2}{3}$, 18   b) Inconsistent   c) i) $\frac{4}{5}, \frac{1}{5}$, 100   ii) $\left(\frac{4}{5}+\frac{1}{5}\right)^{100}$

2. a) i) 20, 3.16   ii) 240, 12   b) 8, 2   c) 18, 3

3. a) i) $\frac{3}{32}$ ii) $\frac{5}{16}$ iii) $\frac{1}{2}$ b) i) $\frac{1}{32}$ ii) $\frac{5}{32}$ iii) $\frac{5}{16}$ iv) $\frac{31}{32}$
c) i) $\frac{16}{81}$ ii) $\frac{32}{81}$ iii) $\frac{8}{81}$ d) i) $\frac{3}{8}$ ii) $\frac{1}{16}$ e) i) $\frac{125}{216}$ ii) $\frac{5}{72}$ iii) $\frac{1}{216}$

4. a) i) 0.6561   ii) 0.3439   iii) 0.9477   b) i) $\frac{216}{625}$ ii) $\frac{992}{3125}$ c) $\frac{53}{3125}$

5. a) i) $\frac{9}{64}$ ii) $\frac{27}{64}$ iii) $\frac{63}{64}$ b) i) $\frac{27}{64}$ ii) $\frac{9}{64}$ 6. $\frac{3}{5}, \frac{2}{5}, \frac{96}{625}$

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Chapter 14
Derivatives

Limit and Continuity

The concepts of limit and continuity are inherent in the definition of the derivative of a function. So it will be proper to introduce briefly the ideas of limit and continuity of function, before we actually deal with the subject matter of this chapter.

a) Limit

A function $f(x)$ is said to have the *limit* A at a point $x = a$, if given a small positive number ε, there exists another positive number δ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - A| < \varepsilon$. This means to say that when x approaches a within a distance of δ, $f(x)$ approaches A within a distance of ε. Symbolically, we express this idea by writing

$$\lim_{x \to a} f(x) = A$$

We know that $|x - a| = x - a$, if $x - a > 0$ and $|x - a| = -(x - a) = a - x$, if $x - a < 0$. So, in the definition of the limit of a function $f(x)$, if we replace $0 < |x - a| < \delta$ by $0 < x - a < \delta$, we say the $\lim f(x) = A$ as x approaches a from above (or the right side of a) and write

$$\lim_{x \to a+} f(x) = A \qquad \text{or} \qquad \lim_{h \to 0} f(a + h) = A$$

Similarly, if we replace $0 < |x - a| < \delta$ by $0 < a - x < \delta$, we say that $\lim f(x) = A$ as x approaches a from below (or the left side of a) and write

$$\lim_{x \to a-} f(x) = A \qquad \text{or} \qquad \lim_{h \to 0} f(a - h) = A$$

The limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to a-} f(x)$ are respectively called *right hand limit* and *left hand limit* of $f(x)$ at $x = a$.

Thus we can also say that a function $f(x)$ has a *limit* at $x = a$, if and only if the right hand limit $\lim\limits_{x \to a+} f(x)$ is equal to the left hand limit $\lim\limits_{x \to a-} f(x)$.

Worked Out Examples

Example 1

Find the limit of $f(x) = \sin\frac{1}{x}$ at the point $x = 0$, if it exists.

Solution:

As $x \to 0-$, $\frac{1}{x} \to -\infty$ and as $x \to 0+$, $\frac{1}{x} \to \infty$. In such a case $\sin\frac{1}{x}$ can take any value lying between -1 and 1. As $\sin\frac{1}{x}$ cannot have any definite value, $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist.

Example 2

Find the limit of $f(x) = x \sin\frac{1}{x}$ at the point $x = 0$.

Solution :

We have seen in the Example 3 that for a sufficiently small value of x or a sufficiently large value of $\frac{1}{x}$, $\sin\frac{1}{x}$ can take any value lying between -1 and 1. So, $x.\sin\frac{1}{x}$, being a product of a sufficiently small x and a finite $\sin\frac{1}{x}$, tends to zero, as x tends to zero.

$\therefore \lim\limits_{x\to 0-} f(x) = 0$ and $\lim\limits_{x\to 0+} f(x) = 0$

Now as $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0+} f(x)$,

$\lim\limits_{x\to 0} f(x)$ exists and is equal to zero.

Example 3

Show that $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$.

Solution:

We have,

$$\frac{e^x - 1}{x} = \frac{1}{x}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots - 1\right)$$

$$= 1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots$$

$$\therefore \lim_{x\to 0-} \frac{e^x - 1}{x} = \lim_{x\to 0+} \frac{e^x - 1}{x} = 1$$

Hence $\lim\limits_{x\to 0} \frac{e^x - 1}{x} = 1$

Example 4

Show that $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$

Solution:

We have

$$\frac{\log(1+x)}{x} = \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)$$

$$= 1 - \frac{x}{2} + \frac{x^2}{3} - \dots\dots$$

$$\therefore \quad \lim_{x\to 0-} \frac{\log(1+x)}{x} = 1$$

and $\lim_{x\to 0+} \frac{\log(1+x)}{x} = 1$

Hence, as $\lim_{x\to 0-} \frac{\log(1+x)}{x} = \lim_{x\to 0+} \frac{\log(1+x)}{x}$,

we get $\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$

b) Continuity

A function $f(x)$ is said to be *continuous* at $x = a$, if $\lim_{x\to a} f(x) = f(a)$

This definition shows that the limit $\lim_{x\to a} f(x)$ must exist for $f(x)$ to be continuous. But it may happen that though $\lim_{x\to a} f(x)$ exists, the value of $\lim_{x\to a} f(x)$ may not be equal to $f(a)$. In that case, $f(x)$ is not continuous at $x = a$. So we can say that the existence of the limit of $f(x)$ at $x = a$ is the necessary but not the sufficient condition for the continuity of the function $f(x)$ at $x = a$.

The function of Example 1 is continuous at $x = 0$, because $f(0) = 1$ and so

$$\lim_{x\to 0} f(x) = f(0).$$

The functions of Examples 2 and 3 are not continuous at $x = 0$, because $\lim_{x\to 0} f(x)$ does not exist in both cases.

The limits of the functions of Examples 4 and 5 exist. But both of the functions are not defined at $x = 0$. So the functions are not continuous. By defining the functions suitably at $x = 0$, we can make the functions continuous at $x = 0$. So the continuity of such functions depend on how we define the functions at $x = 0$. Let us consider the following examples.

Derivative

A function $y = f(x)$ is said to be *differentiable* with respect to x, if the limit

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists. This limit, if it exists, is called the *derivative* or the *differential coefficient* of $f(x)$ with respect to x. The derivative or the differential coefficient of $f(x)$ is denoted by $f'(x)$, $\frac{dy}{dx}$ or $\frac{df}{dx}$.

The differential coefficient or the derivative of $f(x)$ for the value $x = a$ is denoted by $f'(a)$.

So

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

Relation between continuity and differentiability

Now let us see what we can expect of a function $f(x)$, if $f'(x)$ for $x = a$ exists, i.e. if $f'(a)$ is finite.

We can write

$$\lim_{h\to 0} f(a+h) - f(a) = \lim_{h\to 0} \{f(a+h) - f(a)\}$$

$$= \lim_{h\to 0} \left\{\frac{f(a+h)-f(a)}{h} . h\right\}$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} . \lim_{h\to 0} h$$

$$= f'(a) . 0 = 0$$

or, $\lim_{h\to 0} f(a+h) = f(a)$

$\therefore$ $f(x)$ is continuous at $x = a$.

Thus we have seen that the *differentiability of a function at a point implies the continuity of the function at that point.*

But the converse is *not necessarily* true, i.e. the continuity of a function at a point does not generally imply the differentiability of the function at that point. This we can see with the help of an example.

Consider the function $f(x)$ defined as $f(x) = \begin{cases} x \sin\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Since for all $x \neq 0$, $\sin\frac{1}{x}$ lies between -1 and $+1$, so

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x \sin\frac{1}{x} = 0$$

and $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x \sin\frac{1}{x} = 0$

$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin\frac{1}{x} = 0$

Also, by given $f(0) = 0$

$\therefore \lim_{x\to 0} f(x) = f(0)$

which shows that $f(x) = x \sin\frac{1}{x}$ is continuous at $x = 0$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} \qquad (\because f(0) = 0)$$

$$= \lim_{h\to 0} \frac{h \sin\frac{1}{h}}{h} = \lim_{h\to 0} \sin\frac{1}{h}$$

which is not defined. So $f'(0)$ does not exist.

$\therefore f(x)$ is continuous but not differentiable at $x = 0$.

Now what we have seen can be stated as —"*The continuity of a function at a point is the necessary but not the sufficient condition for the existence of the derivative of the function at that point.*"

Now let us find the derivatives of some functions.

Use of logarithm in differentiating a function

Sometimes, in the process of differentiation, we come across a function of the types $a^{f(x)}$ or $\{f(x)\}^{g(x)}$ etc. where 'a' is a constant. In differentating such a function we first take the logarithm and then only differentiate.

Worked Out Examples

Example 1

Find, from definition, the derivative of $\sin^{-1} x$.

Solution:

Let $f(x) = \sin^{-1}x$.

Then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

or $\frac{d(\sin^{-1}x)}{dx} = \lim_{h\to 0} \frac{\sin^{-1}(x+h) - \sin^{-1}x}{h}$ ...... (i)

Let $\sin^{-1} x = y$ and $\sin^{-1}(x+h) = y+k$ such that as $h \to 0, k \to 0$.

Then $\sin y = x$ and $\sin(y+k) = x+h$

$\therefore$ $h = (x+h) - x = \sin(y+k) - \sin y$

and $\sin^{-1}(x+h) - \sin^{-1} x = y + k - y = k$

Then (i) becomes

$$\frac{d(\sin^{-1} x)}{dx} = \lim_{k\to 0} \frac{k}{\sin(y+k) - \sin y}$$

$$= \lim_{k\to 0} \frac{k}{2\cos\frac{2y+k}{2}\sin\frac{k}{2}}$$

$$= \lim_{k\to 0} \frac{1}{\cos\frac{2y+k}{2}}\left(\frac{k/2}{\sin k/2}\right)$$

$$= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Example 2

Find, from first principles, the differential coefficient of a^x.

Solution:

Let $f(x) = a^x = e^{\log a^x} = e^{x \log a}$

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{e^{(x+h)\log a} - e^{x \log a}}{h}$$

$$= \lim_{h\to 0} e^{x\log a} \frac{(e^{h\log a} - 1)}{h}$$

$$= \lim_{h\to 0} e^{x\log a} \left\{\frac{e^{h\log a} - 1}{h\log a}\right\} \log a$$

$$= e^{x\log a}.\ \log a = a^x \log a.$$

Example 3

Find, from first principles, the differential coefficient of $e^{\cos x}$.

Solution:

Let $f(x) = e^{\cos x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

or, $$\frac{d(e^{\cos x})}{dx} = \lim_{h\to 0} \frac{e^{\cos(x+h)} - e^{\cos x}}{h} \quad \ldots\ldots \text{(i)}$$

Let $\cos x = y$ and $\cos(x+h) = y + k$ such that as $h\to 0$, $k\to 0$. Then (i) becomes

$$\frac{d(e^{\cos x})}{dx} = \lim_{h\to 0}\left(\frac{e^{y+k} - e^{y}}{h}\right)$$

$$= \lim_{h\to 0}\left(e^{y} \cdot \frac{e^{k}-1}{k} \cdot \frac{k}{h}\right)$$

$$= e^{y} \lim_{k\to 0}\frac{e^{k}-1}{k} \cdot \lim_{h\to 0}\frac{k}{h}$$

$$= e^{y} \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= e^{y} \lim_{h\to 0}\frac{2\sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(-\frac{h}{2}\right)}{h}$$

$$= e^{y} \lim_{h\to 0}\left(-\sin\frac{2x+h}{2} \cdot \frac{\sin h/2}{h/2}\right)$$

$$= -e^{y}\sin x$$

$$= -e^{\cos x}\sin x$$

Example 4

Find, from definition, the derivative of $x \log x$.

Solution:

Let $f(x) = x\log x$, then

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

or, $$\frac{d(x\log x)}{dx} = \lim_{h\to 0}\frac{(x+h)\log(x+h) - x\log x}{h}$$

$$= \lim_{h\to 0}\frac{x\{\log(x+h) - \log x\} + h\log x}{h}$$

$$= \lim_{h\to 0}\left(\frac{\log(1+\frac{h}{x})}{h/x} + \log x\right)$$

$$= 1 + \log x. \qquad \left(\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$$

Example 5

Find, from definition, the differential coefficient of $\log(\sec x)$.

Solution:

Let $f(x) = \log(\sec x)$,

then, $f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$

or, $\frac{d(\log \sec x)}{dx} = \lim_{h\to0} \frac{\log \sec(x+h) - \log \sec x}{h}$ ...... (i)

Let $y = \sec x$ and $y + k = \sec(x+h)$

such that as $h\to0$, $k\to0$. Then (i) becomes

$$\frac{d}{dx}(\log \sec x) = \lim_{h\to0}\left(\frac{\log(y+k)-\log y}{k}\cdot\frac{k}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{\frac{1}{y}\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}}\,\frac{\sec(x+h)-\sec x}{h}\right)$$

$$= \frac{1}{y}\lim_{h\to0}\left(\frac{\cos x-\cos(x+h)}{h\cos(x+h)\cos x}\right)$$

$$= \frac{1}{y}\lim_{h\to0}\left(\frac{\sin\frac{2x+h}{2}}{\cos(x+h)\cos x}\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)$$

$$= \frac{1}{\sec x}\cdot\frac{\sin x}{\cos x\cos x}$$

$$= \tan x.$$

Example 6

Find from defintion, the derivative of $\tan\sqrt{x}$.

Solution:

Let $f(x) = \tan\sqrt{x}$ then $f(x+h) = \tan\sqrt{x+h}$

Now, $f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow \frac{d}{dx}(\tan\sqrt{x}) = \lim_{h\to0}\frac{\tan\sqrt{x+h}-\tan\sqrt{x}}{h}$ ...... (i)

Put $y = \sqrt{x}$ and $y+k = \sqrt{x+h}$ so that $k = \sqrt{x+h}-\sqrt{x}$

When $h\to0$, $k\to0$.

Now (i) becomes

$$\frac{d}{dx}(\tan\sqrt{x}) = \lim_{h\to0}\frac{\tan(y+k)-\tan y}{k}\cdot\frac{k}{h}$$

$$= \lim_{k\to0}\frac{\tan(y+k)-\tan y}{k}\cdot\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{k\to 0} \frac{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}{k} \cdot \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

(Rationalizing numerator and denominator by $\sqrt{x+h}+\sqrt{x}$)

$$= \lim_{k\to 0} \frac{\sin(y+k)\cos y - \cos(y+k)\sin y}{k\cos y\cos(y+k)} \cdot \lim_{h\to 0} \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

$$= \lim_{k\to 0} \frac{\sin(y+k-y)}{k\cos y\cos(y+k)} \cdot \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{k\to 0} \left(\frac{\sin k}{k} \cdot \frac{1}{\cos y\, y\cos(y+k)}\right) \cdot \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= 1 \cdot \frac{1}{\cos^2 y} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\sec^2 y}{2\sqrt{x}} = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$$

Example 7

Show that the function $f(x) = |x-1|$ is continuous but not differentiable at $x = 1$.

Solution:

Here $f(x) = |x-1|$

$$|x-1| = \begin{cases} (x-1) \text{ if } x \geq 1 \\ -(x-1) \text{ if } x < 1 \end{cases}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} |x-1|$$

$$= \lim_{x\to 1^+} (x-1) = 0$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} |x-1|$$

$$= \lim_{x\to 1^-} (1-x) = 0$$

Also $f(1) = 0$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

Hence $f(x)$ is continuous at $x = 1$.

Again $Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$$= \lim_{h\to 0} \frac{(1+h-1)-0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

and $\quad Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h \to 0} \frac{-(1-h-1) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

$$Lf'(1) \neq Rf'(1)$$

$\therefore f(x)$ is not differentiable at $x = 1$.

Example 8

Find $\frac{dy}{dx}$ if $y = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2+a^2}\right)$

Solution:

$$y = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2+a^2}\right)$$

Differentiating both sides w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2}\log\left(x + \sqrt{x^2+a^2}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{x\, d\sqrt{x^2+a^2}}{d(x^2+a^2)} \cdot \frac{d(x^2+a^2)}{dx} + \sqrt{x^2+a^2}\right\} + \frac{a^2}{2}\frac{d\log\left(x+\sqrt{x^2+a^2}\right)}{d\left(x+\sqrt{x^2+a^2}\right)} \cdot \frac{d\left(x+\sqrt{x^2+a^2}\right)}{dx}$$

$$= \frac{1}{2}\left\{x \cdot \frac{1}{2\sqrt{x^2+a^2}} \cdot 2x + \sqrt{x^2+a^2}\right\} + \frac{a^2}{2} \cdot \frac{1}{x+\sqrt{x^2+a^2}}\left(1 + \frac{1}{2\sqrt{x^2+a^2}} \cdot 2x\right)$$

$$= \frac{x^2+x^2+a^2}{2\sqrt{x^2+a^2}} + \frac{a^2}{2}\frac{x+\sqrt{x^2+a^2}}{\left(x+\sqrt{x^2+a^2}\right)\sqrt{x^2+a^2}}$$

$$= \frac{2(x^2+a^2)}{2\sqrt{x^2+a^2}}$$

$$= \sqrt{x^2+a^2}$$

Example 9

Find $\frac{dy}{dx}$ if $\quad y = \tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$

Solution:

Put $x^2 = \cos 2\theta$

Then, $\quad y = \tan^{-1}\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}$

$$= \tan^{-1}\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$y = \frac{\pi}{4} + \theta = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$$

Differentiation both sides w.r.t. 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2\right)$$

$$= \frac{1}{2}\frac{d(\cos^{-1} x^2)}{d(x^2)}\frac{d(x^2)}{dx}$$

$$= -\frac{1}{2}\frac{1}{\sqrt{1-x^4}} \cdot 2x$$

$$= -\frac{x}{\sqrt{1-x^4}}$$

Example 10

Find the derivatives of a) $x^{\cos x}$ b) $(\tan x)^{\log x}$

Solution:

a) Let $y = x^{\cos x}$

Taking log on both sides,

$$\log y = \cos x \log x$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\cos x \log x)$$

$$\frac{d(\log y)}{dy}\frac{dy}{dx} = \cos x\frac{d(\log x)}{dx} + \log x\frac{d}{dx}(\cos x)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{1}{x} + \log x(-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{x} - \sin x \log x\right]$$

$$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right]$$

b) Let $y = (\tan x)^{\log x}$

Taking log on both sides,

$$\log y = \log(\tan x)^{\log x}$$

$$\Rightarrow \log y = \log x \log(\tan x)$$

$$\frac{d(\log y)}{dy}\frac{dy}{dx} = \log x \frac{d}{dx}(\log \tan x) + \log(\tan x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \log x \frac{d(\log \tan x)}{d(\tan x)}\cdot\frac{d(\tan x)}{dx} + \log(\tan x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x \frac{1}{\tan x}\sec^2 x + \log(\tan x)\frac{1}{x}$$

$$\frac{dy}{dx} = y\left[\frac{\sec^2 x}{\tan x}\log x + \frac{\log(\tan x)}{x}\right]$$

$$= (\tan x)^{\log x}\left[\frac{\sec^2 x}{\tan x}\log x + \frac{\log(\tan x)}{x}\right]$$

Example 11

Find the derivative of $y = x^y$

Solution:

$$y = x^y$$

Taking logarithm both sides,

$$\log y = y \log x$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{d(\log y)}{dy}\cdot\frac{dy}{dx} = y\frac{d}{dx}(\log x) + \frac{dy}{dx}\log x$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = y\cdot\frac{1}{x} + \log x\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y} - \log x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

Example 12

Find the derivative of $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

Solution:

Let $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$ ....(i)

$= u + v$

where $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$

$\log u = \cos x \log \sin x$

$$\frac{d}{dx}(\log u) = \frac{d}{dx}(\cos x \log \sin x)$$

$$\frac{1}{u}\cdot\frac{du}{dx} = \cos x \cdot \frac{1}{\sin x}\cos x + \log \sin x\,(-\sin x)$$

$$\frac{du}{dx} = u\,(\cos x \cot x - \sin x \log \sin x)$$

$$= (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x)$$

Again, $\log v = \sin x \log \cos x$

$$\frac{d}{dx}(\log v) = \frac{d}{dx}(\sin x \log \cos x)$$

$$\frac{1}{v}\cdot\frac{dv}{dx} = -\sin x \cdot \frac{1}{\cos x}\sin x + \log \cos x \,.\, \cos x$$

$$\frac{dv}{dx} = v\,(\cos x \log \cos x - \sin x \,.\, \tan x)$$

$$= (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x)$$

Now from (i),

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\sin x)^{\cos x}(\cos x \cot x - \sin x \log \sin x) + (\cos x)^{\sin x}(\cos x \log \cos x - \sin x \tan x)$$

EXERCISE

Find, from first principles, the derivatives of: (Ex 1 — 5)

1. (i) $e^{\sqrt{x}}$ (ii) $e^{\sin x}$ (iii) $e^{\tan x}$ (iv) e^{x^2}
2. (i) $\log\left(\sin \frac{x}{a}\right)$ (ii) $\log(\tan x)$ (iii) $\log(\sec x^2)$
3. (i) $\cos^{-1}x$ (ii) $\tan^{-1}x$ (iii) $\log \cos^{-1}x$ (iv) $e^{\sin^{-1}x}$
4. (i) x^x (ii) 2^{x^2} (iii) $\log x^x$
5. (i) $\sin x^2$ (ii) $\sin(\log x)$ (iii) $\sqrt{\tan x}$ (iv) $\sin e^x$
6. Find, from first principles, the derivatives of
   (i) $f(x) = e^{\cos x}$ at $x = 0$
   (ii) $f(x) = \log \cos x$ at $x = 0$
7. (i) If $f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$. Find $f'(0)$

(ii) If $f(x) = \begin{cases} 3 + 2x & \text{for } -\dfrac{3}{2} \le x \le 0 \\ 3 - 2x & \text{for } 0 < x < \dfrac{3}{2} \end{cases}$

Show that $f(x)$ is continuous at $x = 0$, but $f'(0)$ does not exist.

(iii) If $f(x) = |x|$, show that $f'(0)$ does not exist.

8. Find $\dfrac{dy}{dx}$ if

(i) $y = \dfrac{x\sqrt{x^2 - a^2}}{2} - \dfrac{a^2}{2} \log (x + \sqrt{x^2 - a^2})$

(ii) $y = \dfrac{x\sqrt{a^2 - x^2}}{2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a}$

(iii) $y = 2 \tan^{-1} \dfrac{\sqrt{1 + x^2} - 1}{x}$

9. (i) If $x^y = e^{x-y}$, prove that $\dfrac{dy}{dx} = (\log x)(\log ex)^{-2}$

(ii) If $\sin y = x \sin (a + y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2 (a + y)}{\sin a}$

10. Find the derivatives of

a) $x^{\sin x}$ b) $(\sin x)^x$ c) $(\sin x)^{\log x}$

d) e^{x^x} e) x^{e^x} f) $(\log x)^{\tan x}$

g) $x^{\sec x}$ h) $(\sin x)^{\cos x}$

11. Find $\dfrac{dy}{dx}$ when

a) $x^y.y^x = 1$ b) $x^m y^n = (x + y)^{m+n}$ c) $e^{\sin x} + e^{\sin y} = 1$

d) $x^y = y^x$ e) $x^{\sin x} = y^{\sin y}$

12. Find the derivatives of

(i) $x^{\tan x} + (\tan x)^x$ (ii) $(\tan x)^{\cot x} + (\cot x)^{\tan x}$

Answer

1. (i) $\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ (ii) $e^{\sin x} \cos x$ (iii) $e^{\tan x} \sec^2 x$ (iv) $2xe^{x^2}$

2. (i) $\dfrac{1}{a} \cot \dfrac{x}{a}$ (ii) $2 \operatorname{cosec} 2x$ (iii) $2x \tan x^2$

3. (i) $-\dfrac{1}{\sqrt{1 - x^2}}$ (ii) $\dfrac{1}{1 + x^2}$ (iii) $-\dfrac{1}{\cos^{-1} x \sqrt{1 - x^2}}$ (iv) $\dfrac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}}$

4. (i) $x^x (1 + \log x)$ (ii) $2x \log 2 . 2^{x^2}$ (iii) $1 + \log x$

5. (i) $2x \cos x^2$ (ii) $\dfrac{1}{x} \cos (\log x)$ (iii) $\dfrac{\sec^2 x}{2\sqrt{\tan x}}$ (iv) $e^x \cos e^{e^x}$

6. (i) 0 (ii) 0

7. (i) 0. (iii) $\frac{1}{1+x^2}$

8. (i) $\sqrt{x^2 - a^2}$ (ii) $\sqrt{a^2 - x^2}$

10. a) $x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right)$ b) $(\sin x)^x (x \cot x + \log \sin x)$

c) $(\sin x)^{\log x}\left(\frac{1}{x}\log \sin x + \cot x \log x\right)$ d) $e^{x^x}.x^x (1 + \log x)$

e) $x^{e^x}.e^x\left(\log x + \frac{1}{x}\right)$ f) $(\log x)^{\tan x}\left(\sec^2 x \log (\log x) + \frac{\tan x}{x \log x}\right)$

g) $x^{\sec x} \sec x\left(\tan x \log x + \frac{1}{x}\right)$ h) $(\sin x)^{\cos x} (\cos x \cot x - \sin x \log (\sin x))$

11. a) $-\frac{y\,(y + x \log y)}{x\,(y \log x + x)}$ b) $\frac{y}{x}$ c) $-\frac{e^{\sin x} \cos x}{e^{\sin y} \cos y}$

d) $\frac{y\,(x \log y - y)}{x\,(y \log x - x)}$ e) $\frac{y\,(x \cos x \log x + \sin x)}{x\,(y \cos y \log y + \sin y)}$.

12. (i) $x^{\tan x}\left(\frac{\tan x}{x} + \sec^2 x \log x\right) + (\tan x)^x (2x \operatorname{cosec} 2x + \log \tan x)$

(ii) $(\tan x)^{\cot x} \{\operatorname{cosec}^2 x\,(1 - \log \tan x)\} + (\cot x)^{\tan x} \{\sec^2 x\,(\log \cot x - 1)\}$

Derivatives of Hyperbolic Functions

a) Hyperbolic functions

The hyperbolic functions are defined for a real number x as follows:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}) \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

cosech x, sech x and coth x are respectively the reciprocals of sinh x, cosh x, and tanh x and are defined accordingly. In the cases of cosech x and coth x, x cannot be zero.

From these relations, we can easily derive the following relations

$\sinh(-x) = -\sinh x$

$\cosh(-x) = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x + \sinh^2 x = \cosh 2x$

$\sinh 2x = 2 \sinh x \cosh x.$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

b) Derivatives of hyperbolic functions

(a) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d(e^x - e^{-x})}{dx} = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

i.e. $\frac{d(\sinh x)}{dx} = \cosh x$

Similarly, we can derive

(b) $\frac{d(\cosh x)}{dx} = \sinh x$

(c) Let $y = \tanh x = \frac{\sinh x}{\cosh x}$

$$\therefore \frac{dy}{dx} = \frac{\cosh x \cdot \frac{d(\sinh x)}{dx} - \sinh x \cdot \frac{d(\cosh x)}{dx}}{\cosh^2 x}$$

$$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x} = \text{sech}^2 x$$

i.e. $\frac{d(\tanh x)}{dx} = \text{sech}^2 x$

Similarly, we can derive

(d) $\frac{d(\coth x)}{dx} = -\text{cosech}^2 x$

(e) $\frac{d(\text{sech } x)}{dx} = -\text{sech } x \tanh x$

(f) $\frac{d(\text{cosech } x)}{dx} = -\text{cosech } x \coth x$

c) Derivatives of inverse hyperbolic functions

(a) Let $y = \sinh^{-1} x$

or $x = \sinh y$

$$\therefore \frac{dx}{dy} = \cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$$

or $\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$

or $\frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{1 + x^2}}$

Similarly, we can derive

(b) $\frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 - 1}}$ $(x > 1)$

(c) $\frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1 - x^2}$ $(x < 1)$

(d) $\frac{d(\text{cosech}^{-1} x)}{dx} = -\frac{1}{x\sqrt{x^2 + 1}}$

(e) $\frac{d(\text{sech}^{-1} x)}{dx} = -\frac{1}{x\sqrt{1 - x^2}}$ $(x < 1)$

(f) $\frac{d(\coth^{-1} x)}{dx} = -\frac{1}{x^2 - 1}$ $(x > 1)$

Worked Out Examples

Example 1

Differentiate : $2\tan^{-1}\left(\tanh\frac{x}{2}\right)$ w.r.t. 'x'

Solution:

Let $y = 2\tan^{-1}\left(\tanh\frac{x}{2}\right)$

Differentiating both sides w.r.t. 'x', we have

$$\frac{dy}{dx} = \frac{d}{dx}\left[2\tan^{-1}\left(\tanh\frac{x}{2}\right)\right]$$

$$= \frac{2d\left[\tan^{-1}\tanh\left(\frac{x}{2}\right)\right]}{d\left(\tanh\frac{x}{2}\right)} \cdot \frac{d\left(\tanh\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)} \cdot \frac{d\left(\frac{x}{2}\right)}{dx}$$

$$= 2.\frac{1}{1 + \tanh^2\frac{x}{2}} \cdot \text{sech}^2\frac{x}{2} \cdot \frac{1}{2}$$

$$= \frac{1 - \tanh^2\frac{x}{2}}{1 + \tanh^2\frac{x}{2}} = \text{sech}\, x$$

Example 2

Find the derivative of Arc sinh $(\cosh x)$

Solution:

Let $y = \text{Arc sinh}(\cosh x) = \sinh^{-1}\cosh x$

Differentiating both sides w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}(\sinh^{-1}(\cosh x))$$

$$= \frac{d(\sinh^{-1}(\cosh x))}{d(\cosh x)} \cdot \frac{d(\cosh x)}{dx}$$

$$= \frac{1}{\sqrt{1+\cosh^2 x}} \cdot \sinh x$$

$$= \frac{\sinh x}{\sqrt{1+\cosh^2 x}}$$

Example 3

Find the derivative of $x^{\sinh^2 x/a}$

Solution:

Let $y = x^{\sin^2 x/a}$

Taking logarithm on both sides

$$\log y = \sinh^2\frac{x}{a} \cdot \log x$$

Differentiating both sides w.r.t. x

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left(\sinh^2\frac{x}{a}\log x\right)$$

$$\Rightarrow \frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \sinh^2\frac{x}{a} \cdot \frac{d}{dx}(\log x) + \log x \frac{d(\sinh^2\frac{x}{a})}{d(\sinh\frac{x}{a})} \cdot \frac{d\sinh\frac{x}{a}}{d(\frac{x}{a})} \cdot \frac{d(\frac{x}{a})}{dx}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x} \cdot \sinh^2\frac{x}{a} + \log x\, 2\sinh\frac{x}{a}\cosh\frac{x}{a} \cdot \frac{1}{a}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x}\sinh^2\frac{x}{a} + \frac{1}{a}\sinh 2x \log x\right)$$

$$= x^{\sinh^2 x/a}\left(\frac{1}{x}\sinh^2\frac{x}{a} + \frac{1}{a}\sinh 2x \log x\right)$$

Example 4

Differentiate $(\sinh x)^{\cosh^{-1} x}$ w.r.t. 'x'

Solution:

Let $y = (\sinh x)^{\cosh^{-1} x}$

or, $\log y = \cosh^{-1} x \log \sinh x$

Differentiating both sides w.r.t. 'x', we have

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\cosh^{-1} x \log \sinh x)$$

$$\frac{d(\log y)}{dy} \cdot \frac{dy}{dx} = \cosh^{-1} x \frac{d(\log \sinh x)}{d(\sinh x)} \cdot \frac{d(\sinh x)}{dx} + \log \sinh x \frac{d}{dx}(\cosh^{-1} x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cosh^{-1} x \frac{1}{\sinh x} \cosh x + \log \sinh x \frac{1}{\sqrt{x^2-1}}$$

$$\therefore \frac{dy}{dx} = y\left[\cosh^{-1} x \coth x + \log \sinh x \cdot \frac{1}{\sqrt{x^2-1}}\right]$$

$$= (\sinh x)^{\cosh^{-1} x}\left[\cosh^{-1} x \coth x + \frac{1}{\sqrt{x^2-1}} \log \sinh x\right]$$

EXERCISE

Find the derivatives of

1. $\log(\tanh x)$
2. $\log \sinh \frac{x}{a}$
3. $e^{\sinh x}$
4. $e^{\cosh^{-1} x/a}$
5. $\text{sech}(\tan^{-1} x)$
6. $\text{sech}^{-1} x - \cosh^{-1} x$
7. Arc tan sinh x
8. $2 \tanh^{-1}\left(\tan \frac{1}{2} x\right)$
9. $x^{\cosh x/a}$
10. $x^{\sinh x^2/a}$
11. $x^{\cosh^2 \frac{x}{a}}$
12. $\left(\sinh \frac{x}{a}\right)^{x^2}$
13. $\left(\cosh \frac{x}{a}\right)^{\log x}$
14. $(\cosh x)^{\sinh^{-1} x}$
15. $\left(\sinh \frac{x}{a} + \cosh \frac{x}{a}\right)^{nx}$

Answer

1. $2 \text{ cosech } 2x$
2. $\frac{1}{a} \coth \frac{x}{a}$
3. $e^{\sinh x} \cosh x$
4. $\frac{e^{\cosh^{-1} x/a}}{\sqrt{x^2 - a^2}}$
5. $-\frac{\text{sech}(\tan^{-1} x)\tanh(\tan^{-1} x)}{1+x^2}$
6. $-\frac{1}{x\sqrt{1-x^2}} - \frac{1}{\sqrt{x^2-1}}$
7. $\text{sech } x$
8. $\sec x$
9. $x^{\cosh x/a}\left(\frac{1}{x}\cosh \frac{x}{a} + \frac{1}{a}\log x \sinh \frac{x}{a}\right)$
10. $x^{\sinh x^2/a}\left(\frac{\sinh \frac{x^2}{a}}{x} + \frac{2x \log x}{a}\cosh \frac{x^2}{a}\right)$

11. $x^{\cosh^2 x/a}\left[\frac{1}{x}\cosh^2 x/a + \frac{1}{a}\log x \sinh\frac{2x}{a}\right]$

12. $\left(\sinh\frac{x}{a}\right)^{x^2}\left[\frac{x^2}{a}\coth\frac{x}{a} + 2x\log\sinh\frac{x}{a}\right]$

13. $\left(\cosh\frac{x}{a}\right)^{\log x}\left[\frac{1}{a}\log x\tanh\frac{x}{a} + \frac{1}{x}\log\cosh\frac{x}{a}\right]$

14. $(\cosh x)^{\sinh^{-1} x}\,(\sinh^{-1} x\tanh x + \frac{1}{\sqrt{1+x^2}}\log\cosh x)$

15. $n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left[\frac{x}{a} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$

Chapter 15
Applications of Derivatives

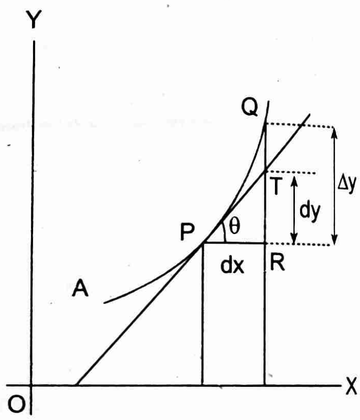
Differentials

Let $y = f(x)$ be a function of x. Then

i) the differential, dx, of the independent variable x, is an arbitrary increment of x; that is, $dx = \Delta x$;

ii) the differential, dy of the dependent variable y is $dy = f'(x)\, dx$, where $f'(x)$ is the derivative of $f(x)$. While the differential dx of the independent variable is an increment Δx, the differential dy of the dependent variable is not, in general, equal to the corresponding increment Δy. Because

$$\Delta y = f(x + \Delta x) - f(x) \qquad \text{and} \qquad dy = f'(x)\, dx$$



Let P(x, y) and Q$(x + \Delta x, y + \Delta y)$ be two neighbouring points on the graph of $y = f(x)$,

we have

$$\tan\theta = f'(x)$$

$$\frac{TR}{PR} = f'(x)$$

$$TR = f'(x)\, dx$$

$$dy = f'(x)\, dx$$

$$TR = dy$$

$$QR = \Delta y$$

Difference between Δy and dy is TQ. TQ can be made as small as we please by taking Δx sufficiently small, so that dy will approximate Δy.

This is also known as the *tangent line approximation* as dy is the tangent line increment and Δy is the curve increment for an increment Δx or dx in x.

As an application consider the area, y of a square of side x so that $y = x^2$

Here $\Delta y = (x + \Delta x)^2 - x^2$

$= 2x.\, \Delta x + (\Delta x)^2 = \Delta x\,(2x + \Delta x)$

and $dy = f'(x)\, dx = 2x dx$

and $dx = \Delta x$

Let the side x change from 2 to 2.01,

$\Delta x = dx = .01$

so $\Delta y = 0.01\ (2 \times 2 + 0.01) = 0.0401$, the actual change in area

then $dy = 2 \times 2 \times 0.01 = 0.04$, the approximate change in area

and

and so the error $= 0.0401 - 0.04 = 0.0001$

Worked Out Examples

Example 1

The edge of a cube increases from 10 cm to 10.025 cm. Find the approximate increments in the volume and the surface area of the cube. Also find the actual increments and the percentage error in the approximation.

Solution :

$$V = x^3$$

Approximate increase in volume

$$dV = 3x^2dx = 3 \times 10^2 \times 0.025 = 7.5$$

Actual increase in volume

$$\Delta V = (x + \Delta x)^3 - x^3$$

$$= (10.025)^3 - 10^3$$

$$= 1007.5187 - 1000 = 7.5187$$

$$\text{error} = .0187$$

$$\%\ \text{error} = \frac{0.0187}{10^3} \times 100 = 0.00187\%$$

$$S = 6x^2$$

Approximate increase in surface area

$$dS = 12xdx = 12 \times 10 \times 0.025 = 3$$

Actual increase in surface area

$$\Delta S = 6\ [(x + \Delta x)^2 - x^2]$$

$$= 6\ [(10.025)^2 - 10^2]$$

$$= 6\ [0.50062]$$

$$= 3.00372$$

$$\text{error} = 0.00372$$

$$\%\ \text{error} = \frac{0.0037}{600} \times 100 = 0.00062\%$$

EXERCISE

1. Compute Δy, dy and $\Delta y - dy$ when $y = \frac{x^2}{2} + 3x$, $x = 2$ and $dx = 0.5$.
2. Find an approximate change in the volume of a cube of side x m, caused by increasing the sides by 1%. What is the percentage increment in the volume?
3. Use differentials to approximate the change in x^3 as x changes from 5 to 5.01.
4. Find an approximate change in $1/x$ as x changes from 1 to 0.98.
5. A circular copper plate is heated so that its radius increases from 5 cm to 5.06 cm. Find the approximate increase in area and also the actual increase in area.
6. Find the approximate increase in the surface area of a cube if the edge increases from 10 to 10.01 cm. Also calculate the percentage error in the use of differential approximation.
7. Find the approximate increase in the volume of a sphere when its radius increases from 2 to 2.1. Find also the actual increase and compare the two values.

Answers

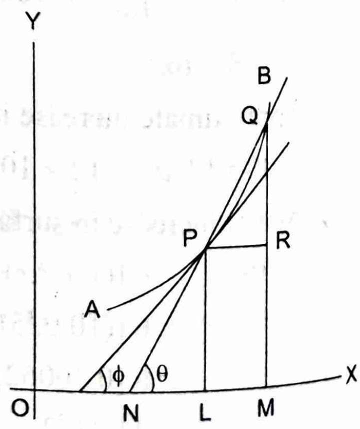
1. 2.625, 2.5, 0.125 2. $0.03x^3$ m^3, 3% 3. 0.75 4. 0.02 5. 0.6π cm^2, 0.6036π cm^2

6. 1.2 cm^2, 0.0001% 7. 1.6π, $\frac{5.044}{3}\pi$, 0.9516

Tangents and Normals

Geometrical Interpretation

Let AB be a continuous curve given by $y = f(x)$ and P, Q be any two points in it. Let the coordinates of P and Q be (x, y) and (x', y'). When a point moves along the curve from the point P to the point Q, it moves horizontally through the distance PR and vertically through the distance RQ.



$$PR = LM = OM - OL = x' - x$$

$$RQ = QM - RM = y' - PL = y' - y$$

These quantities $x' - x$ and $y' - y$ are the increments in x and y respectively and are denoted by Δx and Δy, i.e.

$$\Delta x = x' - x \quad \text{and} \quad \Delta y = y' - y$$

Also $\Delta y = f(x') - f(x) = f(x + \Delta x) - f(x)$

If we join the points P and Q, we get secant PQ which makes an angle θ with the x-axis, i.e. $\angle QNM = \theta$. So $\angle QPR = \angle QNM = \theta$ and $\tan\theta = \frac{QR}{PR} = \frac{\Delta y}{\Delta x}$, which is the slope of the secant PQ. As Q moves along the curve and approaches P, the secant rotates about P. The limiting position of the secant , when Q ultimately coincides with P, is the tangent at P, making the angle ø with the x-axis. In that situation, Δx, Δy tend to zero. So,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \tan\theta = \tan ø$$

or, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \tan ø$$

Thus $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

or $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

gives the slope of a tangent to the curve represented by the function f.

a) Equations of Tangent and Normal

Let a function $y = f(x)$ have a finite derivative $f'(x_1)$ at $x = x_1$.

[$f'(x_1)$ is the value of dy/dx when $x = x_1$], the curve $y = f(x)$ has a tangent at P (x_1, y_1) whose slope or gradient is $m = \tan\theta = f'(x_1) = \left(\frac{dy}{dx}\right)_{x = x_1}$

The equation of the tangent is $y - y_1 = m(x - x_1)$

If $m = 0$, the curve has a tangent parallel to the x-axis (horizontal tangent) at that point; its equation is $y = y_1$. If $f(x)$ is continuous at $x = x_1$ but $\lim_{x \to x_1} f'(x) = \infty$, the curve has a tangent parallel to y-axis (vertical tangent) at that point and its equation is $x = x_1$.

The normal to a curve at any point P (x_1, y_1) on it is the line through the point and perpendicular to the tangent at that point. So the equation to the normal at P (x_1, y_1) is

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

or $x = x_1$, if the tangent is horizontal,

and $y = y_1$, if the tangent is vertical.

b) The angle of intersection of two curves

If two curves intersect, the angle between the tangents to the curve at their point of intersection is the angle between the curves at the point.

Hence to determine the angle of intersection of two curves

a) solve the equations simultaneously to get the points of intersection.

b) find the slopes m_1 and m_2 of the tangents to the two curves at each point of intersection (these are the values of the derivatives at the points).

c) if $m_1 = m_2$, the angle of intersection ø = 0°; if $m_1 = -1/m_2$, the angle of intersection is ø = 90°; otherwise, $\tan ø = \frac{m_1 - m_2}{1 + m_1 m_2}$

Worked Out Examples

Example 1

For the curve $y = 6x - x^2$ find the slopes at the points (x_1, y_1), $(0, 0)$ and $(1, 5)$. At what point on the curve is the tangent parallel to the x-axis ?

Solution :

Here $y = 6x - x^2$

$\therefore \quad \dfrac{dy}{dx} = 6 - 2x$

$\therefore$ slope at $(x_1, y_1) = 6 - 2x_1$

slope at $(0, 0) = 6$

slope at $(1, 5) = 6 - 2 = 4$

For the tangent to be parallel to the x – axis, slope is zero.

i.e., $6 - 2x = 0$ or $x = 3$

and when $x = 3, y = 6 \times 3 - 3^2 = 18 - 9 = 9$

Hence the point is (3, 9).

Example 2

Find the equations of the tangent and normal to $y = x^3 - 2x^2 + 4$ at (2, 4).

Solution :

Here $\dfrac{dy}{dx}$ or $f'(x) = 3x^2 - 4x$

The slope of the tangent at $(2, 4) = 3 \times 2^2 - 4 \times 2 = 4$

The slope of the normal at $(2, 4) = -1/4$

The equation of the tangent is

$y - 4 = 4(x - 2)$

or $y = 4x - 4$

The equation of the normal is

$y - 4 = -\dfrac{1}{4}(x - 2)$

or $x + 4y = 18$

Example 3

Find the equations of the tangents to the curve $y = (x - 1)(x - 2)$ at the points where the curve meets the x-axis.

Solution:

$y = (x - 1)(x - 2).$

The curve meets the x-axis at the points where $y = 0$

$0 = (x-1)(x-2)$

$\therefore \quad x = 1, 2$

$\therefore$ the curve meets the x-axis at the points (1, 0) and (2, 0)

Again, $\quad y = (x-1)(x-2)$

$\Rightarrow \quad y = x^2 - 3x + 2$

$\Rightarrow \quad \frac{dy}{dx} = 2x - 3$

$\frac{dy}{dx}$ at $(1, 0) = 2 \times 1 - 3 = -1$

The equation of the tangent at (1, 0) is

$y - 0 = -1(x-1) \quad \Rightarrow \quad x + y = 1$

Again, $\frac{dy}{dx}$ at $(2, 0) = 2 \times 2 - 3 = 1$

The equation of the tangent at (2, 0) is

$y - 0 = 1.(x-2) \quad \Rightarrow \quad x - y = 2$

Example 4

Show that the tangents to the curve $x^2 = 3y + 1$ at the points (1, 0) and (–9/4, 65/48) are perpendicular to each other.

Solution:

$x^2 = 3y + 1$

$\Rightarrow \quad 2x = 3\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2x}{3}$

$\frac{dy}{dx}$ at $(1, 0) = m_1 = \frac{2}{3} \times 1 = \frac{2}{3}$

Again, $\quad \frac{dy}{dx}$ at $(-9/4, 65/48) = m_2 = \frac{2}{3} \times \frac{-9}{4} = -\frac{3}{2}$

Since $m_1 \times m_2 = \frac{2}{3} \times \left(-\frac{3}{2}\right) = -1$

so the two tangents are perpendicular to each other.

Example 5

Find the point on the curve $y = 3x^2 + 4x - 5$ where the tangent is parallel to the line $16x + 2y = 3$.

Solution:

Let (α, β) be a point on the given curve

$$y = 3x^2 + 4x - 5 \qquad \ldots\ldots(i)$$

$$\therefore \quad \beta = 3\alpha^2 + 4\alpha - 5 \qquad \ldots\ldots(ii)$$

From (i)

$$\frac{dy}{dx} = 6x + 4$$

$$\frac{dy}{dx} \text{ at } (\alpha, \beta) = 6\alpha + 4$$

Again, $16x + 2y = 3$

$$\Rightarrow \quad 16 \times 1 + 2\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -8$$

If the tangent is parallel to the line, then

$$6\alpha + 4 = -8$$

$$\Rightarrow \quad 6\alpha = -12$$

$$\therefore \quad \alpha = -2$$

Substituting the value of α in (ii),

$$\beta = 12 + 4 \times (-2) - 5 = -1$$

$\therefore$ the required point is (α, β) i.e. $(-2, -1)$

Example 6

Find the angle of intersection of the curves $y = x^2$ and $x = y^2$.

Solution :

Solving simultaneously the two equations, we see that the curves intersect at the points O(0, 0) and P(1, 1).

Also differentiating the equations we have

$$\frac{dy}{dx} = 2x \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{2y} \quad \text{respectively}$$

At O (0, 0), slope of the first curve = 0

slope of the second curve = ∞

Hence, the angle of intersection is $90°$

At P (1, 1), slope of the first curve = 2

slope of the second curve = 1/2.

Hence, the angle between them is

$$\tan^{-1}\left(\frac{2-\frac{1}{2}}{1+2.\frac{1}{2}}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

EXERCISE

1. Find the slope and the inclination with the x-axis of the tangent of
   a) $2y = 2 - x^2$ at $x = 1$
   b) $y = -3x - x^4$ at $x = -1$
   c) $x^2 + y^2 = 25$ at $(-3, 4)$
   d) $2x^2 + 3y^2 + 8x + 3 = 0$ at $(-1, 1)$
2. At what angle does the curve $y(1 + x) = x$ cut the x-axis ?
3. Find the equations of the tangents and normals to the curve
   a) $y = 2x^3 - 5x^2 + 8$ at $(2, 4)$
   b) $x^2 - y^2 = 7$ at $(4, -3)$
   c) $y^2 = 2x$ at $(8, 4)$
   d) $x^2 + y^2 = 20$ at $(4, -2)$
4. Find the points on the curve where the tangents are parallel to the x-axis
   a) $y = 2x - x^2$
   b) $y = x^3 - 3x^2 + 1$
   c) $4y = x^4 - 8x^2$
   d) $x^2 + y^2 - 2x - 8 = 0$
5. Find the points on the circle $x^2 + y^2 = 16$ at which the tangents are parallel to the
   a) x – axis
   b) y – axis
6. a) Find the point on the curve $4y = x^2$ where the tangent drawn makes angle 45° with the x-axis.
   b) Find the point on the curve $x^2 = 3y + 1$ at which the tangent is parallel to the line $4x + 3y + 5 = 0$.
   c) Find the point on the curve $y^2 = 4x + 1$ at which the tangent is perpendicular to the line $7x + 2y = 10$.
7. Show that the equation of the tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (a, b) is

$$\frac{x}{a} + \frac{y}{b} = 2.$$

8. Find the angle of intersection of the curves
   a) $y = 6 - x^2$ and $x^2 = 2y$
   b) $4y = x^2 + 12$ and $y^2 = 8x$ at $(2, 4)$
   c) $2y = x^2$ and $2x^2y = 1$
9. a) Prove that the tangents to the curve $y = x^2 - 3x + 4$ at $(1, 2)$ and $(2, 1)$ are perpendicular to each other.
   b) Prove the tangents to the curve $y = x^3 - 5$ at $(1, 5)$ and $(-1, 5)$ are parallel.

at the points

Answers

1. a) $-1, \frac{3\pi}{4}$ b) $1, \frac{\pi}{4}$ c) $\frac{3}{4}, \tan^{-1}\left(\frac{3}{4}\right)$ d) $-\frac{2}{3}, \tan^{-1}\left(-\frac{2}{3}\right)$ 2. $\frac{\pi}{4}$

3. a) $4x - y = 4, x + 4y = 18$ b) $4x + 3y = 7, 3x - 4y = 24$ c) $x - 4y + 8 = 0, 4x + y = 36$
   d) $2x - y = 10, x + 2y = 0$

4. a) (1, 1) b) (0, 1), (2, –3) c) (0, 0), (2, –4), (–2, –4) d) (1, 3) and (1, –3)

5. a) (0, 4), (0, –4) b) (4, 0), (–4, 0)

6. a) (2, 1) b) (–2, 1) c) (12, 7)

8. a) $\tan^{-1}\left(\frac{6}{7}\right)$ at (2, 2), $\tan^{-1}\left(-\frac{6}{7}\right)$ at (–2, 2)
   b) They have a common tangent. c) They are orthogonal.

L Hospital's Rule

If $f(x) = x^2 - 4$ and $g(x) = x - 2$ so that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$ giving $\frac{f(x)}{g(x)} = \frac{0}{0}$ when $x\to 2$, then the form is known as indeterminante form. Besides this, we have other indeterminate forms $\frac{\infty}{\infty}$, $\infty - \infty$ etc. as well. But here we consider only the indeterminate form $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Now, in this section, we find the limiting values of these functions which take the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ when $x\to a$ (say).

The form $\frac{0}{0}$ (L Hospital's rule)

If $f(x)$ and $g(x)$ and also their derivatives $f'(x)$ and $g'(x)$ are continuous at $x = a$, and if $f(a) = g(a) = 0$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)} = \frac{f'(a)}{g'(a)}$$

provided that $g'(a) \neq 0$.

If $f'(a)$ and $g'(a)$ are both zero then the above theorem can further be used. Thus,

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)} = \frac{f''(a)}{g''(a)}$$

provided that $f''(x)$ and $g''(x)$ both are continuous at $x = a$ and $g''(a) \neq 0$.

The above theorem can further be used if $f''(a) = 0$ and $g''(a) = 0$.

Example

Find $\lim_{x\to a} \frac{x^n - a^n}{x - a}$

Solution:

$\lim_{x\to a} \frac{x^n - a^n}{x - a}$ (form $\frac{0}{0}$)

$= \lim_{x\to a} \frac{n\,x^{n-1}}{1} = n \times a^{n-1} = na^{n-1}$

$\therefore \lim_{x\to a} \frac{x^n - a^n}{x - a} = n.a^{n-1}$

The form $\frac{\infty}{\infty}$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ so that $\lim_{x\to a} \frac{f(x)}{g(x)}$ will be in the form of $\frac{\infty}{\infty}$ then in such a form, we express $\frac{f(x)}{g(x)}$ into $\frac{1/g(x)}{1/f(x)}$ so that each of the numerator and denominator will tend to zero as $x\to a$. Thus we have changed the form $\frac{\infty}{\infty}$ into the form $\frac{0}{0}$ after which L Hospital's rule can be applied to evaluate the limit.

Example

Evaluate $\lim_{x\to\infty} \frac{x^3}{e^x}$

Solution:

$\lim_{x\to\infty} \frac{x^3}{e^x}$ (form $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{3x^2}{e^x}$ (form $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{6x}{e^x}$ (form $\frac{\infty}{\infty}$)

$= \lim_{x\to\infty} \frac{6}{e^x} = 0$

Worked Out Examples

Example 1

Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$

Solution:

$$\lim_{x\to 0}\frac{\sin x}{x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$

Example 2

Evaluate $\lim\limits_{x\to a}\dfrac{x^2 - a^2}{x^{4/3} - a^{4/3}}$

Solution:

$$\lim_{x\to a}\frac{x^2 - a^2}{x^{4/3} - a^{4/3}} \ \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to a}\frac{2x}{\frac{4}{3}x^{1/3}}$$

$$= \lim_{x\to a}\frac{3}{2}x^{2/3} = \frac{3}{2}a^{2/3}$$

Example 3

Evaluate : $\lim\limits_{x\to 0}\dfrac{x - \sin x}{x^3}$

Solution:

$$\lim_{x\to 0}\frac{x - \sin x}{x^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{1 - \cos x}{3x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\sin x}{6x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\cos x}{6}$$

$$= \frac{1}{6}$$

Example 4

Evaluate : $\lim\limits_{x\to 0}\dfrac{x\,e^x - \log(x + 1)}{x^2}$

Solution:

$$\lim_{x\to 0}\frac{x\,e^x - \log(x + 1)}{x^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{x.e^x + e^x - \frac{1}{x+1}}{2x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{x\, e^x + e^x + e^x + \frac{1}{(x+1)^2}}{2}$$

$$= \frac{1+1+1}{2} = \frac{3}{2}$$

Example 5

Evaluate : $\lim_{x\to\infty} \frac{3x^2 + 4x - 7}{2x^2 + 5x + 8}$

Solution:

$$\lim_{x\to\infty} \frac{3x^2 + 4x - 7}{2x^2 + 5x + 8} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6x + 4}{4x + 5} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{6}{4} = \frac{3}{2}$$

Example 6

Evaluate : $\lim_{x\to\pi/2} \frac{\sec 3x}{\sec x}$

Solution:

$$\lim_{x\to\pi/2} \frac{\sec 3x}{\sec x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\pi/2} \frac{\cos x}{\cos 3x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to\pi/2} \frac{-\sin x}{-3\sin 3x} = -\frac{1}{3}$$

Example 7

Find the value of $\lim_{x\to 0} \frac{\log \tan x}{\log x}$

Solution:

$$\lim_{x\to 0} \frac{\log \tan x}{\log x} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x} \sec^2 x}{\frac{1}{x}}$$

$$= \lim_{x\to 0} \frac{x \sec^2 x}{\tan x} \quad (\text{form } \tfrac{0}{0})$$

$$= \lim_{x\to 0} \frac{x}{\sin x \cos x} \quad (\text{form } \tfrac{0}{0})$$

$$= \lim_{x\to 0} \frac{2x}{\sin 2x}$$

$$= \lim_{x\to 0} \frac{2}{2 \cos 2x} = 1$$

Example 8

Find the value of $\lim_{x\to 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x}$

Solution:

$$\lim_{x\to 0} \frac{\sqrt{1+3x} - \sqrt{1-3x}}{x} \quad (\text{form } \tfrac{0}{0})$$

$$= \lim_{x\to 0} \frac{\frac{3}{2\sqrt{1+3x}} + \frac{3}{2\sqrt{1-3x}}}{1}$$

$$= \frac{3}{2} + \frac{3}{2} = 3$$

EXERCISE

1. Using L Hospital's rule, evaluate the following

a) $\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$

b) $\lim_{x\to 1} \frac{x^4 - 3x^3 + 2}{2x^3 - 5x^2 + 3x}$

c) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

d) $\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$

e) $\lim_{x\to 1} \frac{1 - 2x + x^2}{1 + \log x - x}$

f) $\lim_{x\to 0} \frac{\tan ax}{\tan bx}$

g) $\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x \sin x}$

h) $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$

i) $\lim_{x\to 0} \frac{x - \sin x \cos x}{x^3}$

j) $\lim_{x\to 0} \frac{(e^x - 1)\tan x}{x^2}$

2. Find the limiting values of the following (Use L Hospital's rule)

a) $\lim_{x\to \infty} \frac{2x^2 + 3x}{1 + 5x^2}$

b) $\lim_{x\to \infty} \frac{3x^2 - 5}{2x^3 + 4x + 3}$

c) $\lim_{x\to \pi/2} \frac{\sec 7x}{\sec 5x}$

d) $\lim_{x\to 0} \frac{\log \sin x}{\cot x}$

e) $\lim_{x\to 0} \frac{\log x}{\log \cot x}$

f) $\lim_{x\to\infty} \frac{x^4}{e^x}$

g) $\lim_{x\to\pi/2} \frac{\tan 5x}{\tan x}$

A. Is it possible to adjust the values of a and b so that

a) $\lim_{x\to 0} \frac{a \sin x - \sin 2x}{\tan^3 x}$ is finite

b) $\lim_{\theta\to 0} \frac{\theta(1 + a\cos\theta) - b\sin\theta}{\theta^3} = 1$

If possible, what are the values of a and b?

Answers

1. a) 3 b) 5 c) $\frac{1}{2}$ d) 2 e) –2 f) $\frac{a}{b}$ g) 0 h) 2 i) $\frac{2}{3}$ j) 1
2. a) $\frac{2}{5}$ b) 0 c) $-\frac{5}{7}$ d) 0 e) –1 f) 0 g) $\frac{1}{5}$

Rolle's Theorem

If the function $f(x)$ is

a) continuous in the closed interval $[a, b]$

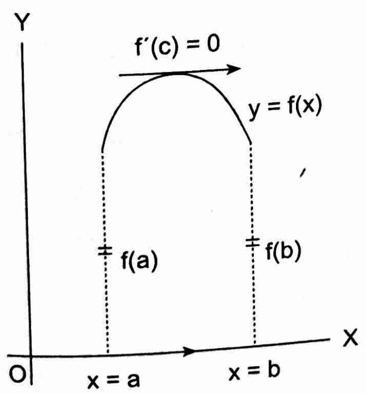
b) differentiable in the open interval (a, b)

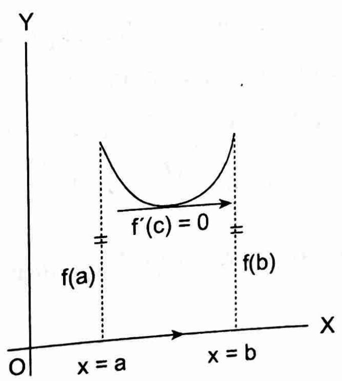
c) $f(a) = f(b)$

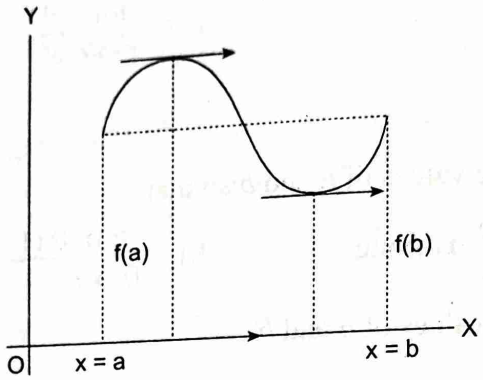
then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$

Geometrical Interpretation of Rolle's theorem

The first condition of Rolle's theorem says that the function $f(x)$ has a continuous graph in the interval $a \le x \le b$. By second condition, the graph has tangents at every point in $a < x < b$ and the last condition is $f(a) = f(b)$ i.e. the ordinates at $x = a$ and $x = b$ are equal. If all the conditions of Rolle's theorem are satisfied, then there is at least a point on the graph $(a < c < b)$ where the tangent is parallel to x-axis.







Failure of Rolle's theorem

If any one of the conditions of Rolle's theorem be not satisfied,then the conclusion of Rolle's theorem will not be true.

Example

Verify Rolle's theorem for $f(x) = x^2 - 4$ in $-3 \le x \le 3$.

Solution:

Since $f(x)$ is a polynomial function, so it is continuous in $-3 \le x \le 3$.

Again, $f'(x) = 2x$ which exists for all $x \in (-3, 3)$

Also, $f(-3) = (-3)^2 - 4 = 5$

$f(3) = (3)^2 - 4 = 5$

$\therefore$ $f(-3) = f(3)$

$\therefore$ all conditions of Rolle's theorem are satisfied.

Hence, there exists atleast a point $c \in (-3, 3)$ such that $f'(c) = 0$.

i.e. $2c = 0 \Rightarrow c = 0 \in (-3, 3)$

Mean Value Theorem (MVT)

If a function $f(x)$ is

a) continuous in the closed interval $[a, b]$

b) differential in the open interval (a, b)

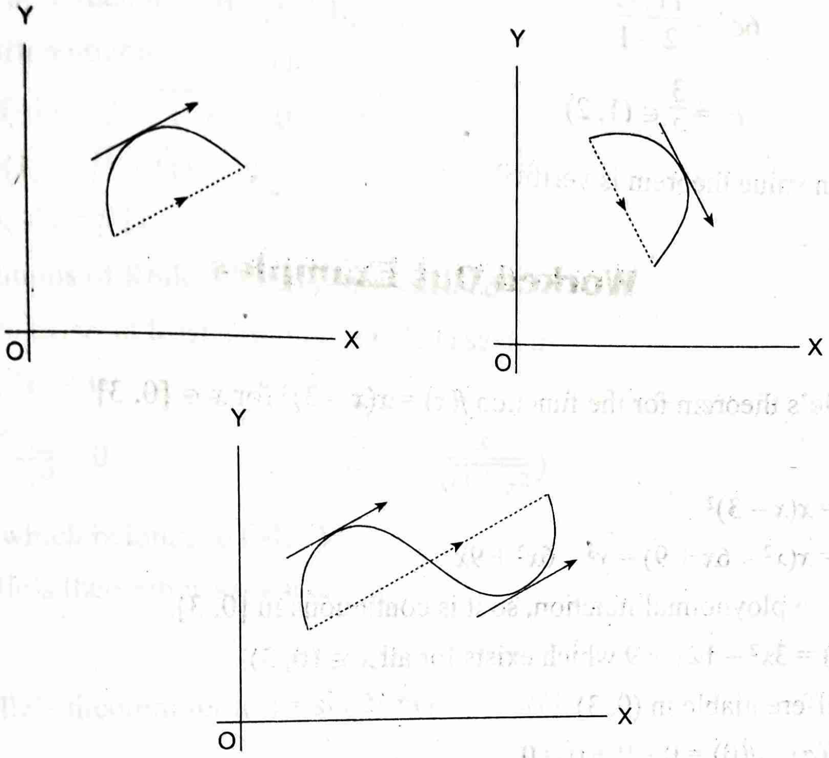
then there exists atleast one value $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

This is known as *Lagrange's mean value theorem.*

Geometrical Interpretation of Mean Value Theorem

Geometrically, Lagrange's mean value theorem says that in a continuous curve, in which tangent can be drawn at every point, there is atleast one point where the tangent is parallel to the secant joining the end point as shown in the figure given below.



Note: Putting $b = a + h$ so that $b - a = h$ is the length of the interval, the form of Lagrange's mean value theorem can be put in the form

$$f(a + h) = f(a) + h f'(a + \theta h)$$

where θ is a positive number less than 1 i.e. $0 < \theta < 1$.

Example

Verify mean value theorem for $f(x) = 3x^2 - 1$ in $[1, 2]$.

Solution:

$f(x)$ is a polynomial function, so it is continuous in $[1, 2]$. Again $f'(x) = 6x$ which exists for all $x \in (1, 2)$.

$\therefore f(x)$ is differentiable in $(1, 2)$.

$\therefore$ the conditions of mean value theorem are satisfies.

Hence there exists at least a point $c \in (1, 2)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Here $a = 1, b = 2$

$f(a) = f(1) = 3 \times 1 - 1 = 2$ $\qquad$ $f(b) = f(2) = 3 \times 4 - 1 = 11$

and $f'(c) = 6c$

Now, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$\Rightarrow \quad 6c = \dfrac{11 - 2}{2 - 1}$

$\Rightarrow \quad c = \dfrac{3}{2} \in (1, 2)$

Hence mean value theorem is verified.

Worked Out Examples

Example 1

Verify Rolle's theorem for the function $f(x) = x(x - 3)^2$ for $x \in [0, 3]$

Solution:

$f(x) = x(x - 3)^2$

$= x(x^2 - 6x + 9) = x^3 - 6x^2 + 9x$

Since $f(x)$ is a ploynomial function, so it is continuous in $[0, 3]$.

Again, $f'(x) = 3x^2 - 12x + 9$ which exists for all $x \in (0, 3)$

$\therefore f(x)$ is differentiable in $(0, 3)$

Also $\quad f(a) = f(0) = 0 - 0 + 0 = 0$

$f(b) = f(3) = (3)^3 - 6 \times 3^2 + 9 \times 3 = 0$

$f(0) = f(3)$

$\therefore$ all conditions of Rolle's theorem are satisfied.

Hence there exists at least a point $c \in (0, 3)$ such that

$f'(c) = 0$

$\Rightarrow \quad 3c^2 - 12c + 9 = 0 \quad (\because f'(x) = 3x^2 - 12x + 9)$

$\Rightarrow \quad c^2 - 4c + 3 = 0$

$\Rightarrow \quad (c - 1)(c - 3) = 0$

$\therefore \quad c = 1, 3$

But $c = 3 \notin (0, 3)$ and $c = 1 \in (0, 3)$

Hence Rolle's theorem is verified.

Example 2

Verify Rolle's theorem for the function $f(x) = \sqrt{1 - x^2}$ in the interval $-1 \leq x \leq 1$.

Solution:

$f(x) = \sqrt{1 - x^2}$

For every value of x such that $-1 \le x \le 1$, $f(x)$ has a definite value, so $f(x)$ is continuous for $-1 \le x \le 1$.

Again, $f'(x) = \frac{1}{\sqrt{1-x^2}}(-2x) = -\frac{x}{\sqrt{1-x^2}}$

which exists for all x such that $-1 < x < 1$.

$\therefore f(x)$ is differentiable in $(-1, 1)$.

Also $f(-1) = \sqrt{1-(-1)^2} = 0$

$f(1) = \sqrt{1-(1)^2} = 0$

$f(-1) = f(1)$

$\therefore$ all conditions of Rolle's theorem are satisfied.

Hence there exists at least a point $c \in (-1, 1)$ such that

$f'(c) = 0$

$\Rightarrow -\frac{c}{\sqrt{1-c^2}} = 0$ $\quad (\because f'(x) = -\frac{x}{\sqrt{1-x^2}})$

$\Rightarrow c = 0$ which belongs to $(-1, 1)$

Hence Rolle's theorem is satisfied.

Example 3

Verify Rolle's theorem for $f(x) = \sin 2x$ in $[-\pi/2, \pi/2]$

Solution:

$f(x) = \sin 2x$

For all $x \in [-\pi/2, \pi/2]$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[-\pi/2, \pi/2]$

Again, $f'(x) = 2\cos 2x$ which exists for all $x \in (-\pi/2, \pi/2)$

$\therefore f(x)$ is differentiable in $(-\pi/2, \pi/2)$.

Also, $f(a) = f(-\pi/2) = \sin(-\pi) = 0$

and $f(b) = f(\pi/2) = \sin \pi = 0$

$\therefore f(-\pi/2) = f(\pi/2)$

$\therefore$ all conditions of Rolle's theorem are satisfied.

Hence there exists at least a point $c \in (-\pi/2, \pi/2)$ such that $f'(c) = 0$

$f'(c) = 0$

$\Rightarrow 2\cos 2c = 0$

$\Rightarrow \cos 2c = \cos(\pm\pi/2)$

$\Rightarrow 2c = \pm\pi/2$

$\therefore c = -\pi/4, \pi/4 \in (-\pi/2, \pi/2)$

Hence Rolle's theorem is verified.

Example 4

Discuss the applicability of Rolle's theorem for $f(x) = \frac{1}{x^2 - 1}$ in [–2, 2].

Solution:

$$f(x) = \frac{1}{x^2 - 1}$$

The function $f(x)$ is not continuous at the points $x = -1$ and 1, both being interior points of [–2, 2]. Since the function does not satisfy the condition of continuity, so Rolle's theorem cannot be applied.

Example 5

Verify Lagrange's mean value theorem for the function

$$f(x) = (x - 1)(x - 2)(x - 3) \quad \text{in } [1, 4].$$

Solution:

$$f(x) = (x - 1)(x - 2)(x - 3)$$
$$= x^3 - 6x^2 + 11x - 6$$

$f(x)$ is a polynomial function, so it is continuous in [1, 4].

Again, $f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (1, 4)$ so, it is differentiable in (1, 4).

Hence the condition of mean value theorem are satisfied. So, there exists at least one value c in (1, 4) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But $f(b) = f(4) = (4)^3 - 6(4)^2 + 11(4) - 6 = 6$

and $f(a) = f(1) = (1)^3 - 6(1)^2 + 11(1) - 6 = 0$

$\therefore \quad f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow \quad 3c^2 - 12c + 11 = \frac{6 - 0}{4 - 1}$

$\Rightarrow \quad 3c^2 - 12c + 11 = 2$

$\Rightarrow \quad 3c^2 - 12c + 9 = 0$

$\Rightarrow \quad c^2 - 4c + 3 = 0$

$\Rightarrow \quad (c - 1)(c - 3) = 0$

$\therefore \quad c = 1, 3$

But $c = 1 \notin (1, 4)$ and $c = 3 \in (1, 4)$

Hence mean value theorem is satisfied.

Example 6

Using mean value theorem, find a point on the parabola $y = (x-3)^2$ where the tangent is parallel to the chord joining the points (3, 0) and (4, 1).

Solution:

Here the value of x ranges from 3 to 4.

$$y = f(x) = (x-3)^2, \qquad x \in [3, 4]$$

$f(x)$ is continuous in [3, 4] as it is a polynomial function.

Again, $f'(x) = 2x - 6$ which exists for all $x \in (3, 4)$.

So, it is differentiable in (3, 4)

Both conditions of mean value theorem are satisfied.

Hence mean value theorem can be applied

$$f(a) = f(3) = (3-3)^2 = 0$$

$$f(b) = f(4) = (4-3)^2 = 1$$

Now, by mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad c \in (3, 4)$$

$$\Rightarrow \quad 2c - 6 = \frac{1-0}{4-3}$$

$$\Rightarrow \quad 2c - 6 = 1$$

$$\therefore \quad c = \frac{7}{2} \in (3, 4)$$

$c = \frac{7}{2}$ is the x-coordinate of the point at which tangent drawn is parallel to the chord joining the points (3, 0) and (4, 1)

Put $x = \frac{7}{2}$ in $y = (x-3)^2$, we have

$$y = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$$

$\therefore$ the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$

Example 7

Using mean value theorem for the function $f(x) = \sin x$ in $[0, x]$ $(0 < x < \pi/2)$ prove that $\sin x < x$.

Solution:

$f(x) = \sin x$ is continuous in $[0, x]$

Again $f'(x) = \cos x$ exists for all x in $(0, x)$, hence $f(x)$ is differentiable in $(0, x)$

Since, $f(x) = \sin x$ satisfies both conditions of mean value theorem, so there exists at least one value $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \quad \cos c = \frac{\sin x - 0}{x - 0}$$

$$\Rightarrow \quad \cos c = \frac{\sin x}{x}$$

$$\Rightarrow \quad \frac{\sin x}{x} < 1$$

$$\Rightarrow \quad \sin x < x$$

EXERCISE

1. Verify Rolle's theorem for the following functions
   a) $f(x) = 3x^2 - 4$ in $[-1, 1]$
   b) $f(x) = 2x^2 - 3x + 1$ in $[\frac{1}{2}, 1]$
   c) $f(x) = (x + 1)(x - 2)$ in $[-1, 2]$
   d) $f(x) = x(x - 1)^2$ in $[0, 1]$
   e) $f(x) = (x - 1)(x - 2)(x - 3)$ in $[1, 3]$
   f) $f(x) = \sin x, x \in [0, \pi]$
   g) $f(x) = \cos 2x, x \in [-\pi, \pi]$
   h) $f(x) = \sqrt{25 - x^2}, x \in [-5, 5]$
2. Using Rolle's theorem, find a point on each of the curves represented by the following functions, where the tangent is parallel to x-axis.
   a) $f(x) = x^2 - 1$ in $[-2, 2]$
   b) $f(x) = 6x - x^2$ in $[0, 6]$
   c) $f(x) = \sin x, x \in [0, \pi]$
3. Verify Lagrange's mean value theorem for the following functions
   a) $f(x) = 3x^2 - 2x$ in $[1, 3]$
   b) $f(x) = x^2 - 2x + 4$ in $[1, 5]$
   c) $f(x) = 2x^2 - 10x + 29$ in $[2, 7]$
   d) $f(x) = x^3 + x^2 - 6x$ in $[-1, 4]$
   e) $f(x) = x(x - 1)^2$ in $[0, 2]$
   f) $f(x) = \sqrt{x^2 - 4}, x \in [2, 4]$
   g) $f(x) = e^x, x \in [0, 1]$
4. a) Using Lagrange's mean value theorem, find the point on the curve $f(x) = x(x - 2)$, the tangent at which is parallel to the chord joining the points $(1, -1)$ and $(4, 8)$
   b) Examine whether the function $f(x) = x^2 - 6x + 1$ satisfies Lagrange's mean value theorem. If it satisfies find the coordinates of the point at which the tangent is parallel to the chord joining the points A$(1, -4)$ and B$(3, -8)$.
5. Show that Rolle's theorem can not be applied for the function

$$f(x) = 1 - x^{2/3} \text{ in } [-1, 1].$$

A. If $f(x) = \tan x$, show that $f(0) = f(\pi)$. Is Rolle's theorem applicable to $f(x)$ in $(0, \pi)$? Give reason.

Answers

2. a) (0, –1) b) (3, 9) c) (π/2, 1)
4. a) (5/2, 5/4) b) (2, –7)

Chapter 16
Antiderivative

Antiderivative

The subject of integration can be treated in two different points of view. Actually it has developed in course of evaluation of the area under a plane curve and has been interpreted as a *limit of a sum* when the number of terms in the sum tends to infinity and each term tends to zero. The other interpretation of integration as the *inverse of differentiation* came some time later. Both points of view are identical. The equivalence of these two view-points has been established in what is known as the **Fundamental Theorem of Integral Calculus**.

At the elementary stage, it is better to introduce the subject of integration as the inverse of differentiation, keeping in reserve the other one to use in some application. For a given function $f(x)$, if there exists a function F(x) such that

$$\frac{d\,F(x)}{dx} = f(x),$$

then F(x) is said to be an *integral* of $f(x)$ with respect to x. Symbolically, we write

$$\int f(x)\,dx = F(x)$$

***Example* 1.**

$$\frac{d\,(\sin x)}{dx} = \cos x$$

$$\therefore \quad \int \cos x\,dx = \sin x$$

***Example* 2.**

$$\frac{d\,(\log(ax+b))}{dx} = \frac{a}{ax+b}$$

$$\therefore \quad \int \frac{a}{ax+b}\,dx = \log(ax+b)$$

The process of finding the integral of a function $f(x)$ is called the *integration* and the function $f(x)$ which has been integrated is called the *integrand*.

The integral of a function $f(x)$ is *not unique*, because if F(x) is an integral of $f(x)$, then F(x) + c is also an integral for any real number c. We know

$$\frac{d\,\{F(x)+c\}}{dx} = \frac{d\,F(x)}{dx} = f(x)$$

which implies $\int f(x)\,dx = F(x) + c$

For this reason, the integral is called the *indefinite integral.*

These ideas are there in what we have done in the first volume. For further knowledge of integration we now start with following fundamental results, known us **Standard Integrals**.

The standard integrals have been divided into different groups according to the different techniques to be used during the process of integration.

Standard Integrals (I)

In this section, we discuss some integrals which are directly related to standard differentiation formulae and so may be considered as some of fundamental integrals.

(a) $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Put $x = a\tan\theta$

$dx = a\sec^2\theta\,d\theta$

$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta)$

$= a^2\sec^2\theta$

$$\therefore \int \frac{1}{a^2 + x^2}\,dx = \int \frac{1}{a^2\sec^2\theta}\,.\,a\sec^2\theta\,d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

(b) $\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\log\frac{a + x}{a - x} + c$

$$\int \frac{1}{a^2 - x^2}\,dx = \int \frac{1}{(a + x)(a - x)}\,dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{a + x} + \frac{1}{a - x}\right)dx$$

$$= \frac{1}{2a}\{\log(a + x) - \log(a - x)\} + c$$

$$= \frac{1}{2a}\log\frac{a + x}{a - x} + c$$

(c) $\int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\log\frac{x - a}{x + a} + c$

$$\int \frac{1}{x^2 - a^2}\,dx = \int \frac{1}{(x + a)(x - a)}\,dx$$

$$= \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a}\{\log(x-a) - \log(x+a)\} + c$$

$$= \frac{1}{2a}\log\frac{x-a}{x+a} + c$$

(d) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

Put $x = a\sin\theta$ $\quad \therefore dx = a\cos\theta\, d\theta$

$$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$$

$$\therefore \int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{a\cos\theta}{a\cos\theta}d\theta$$

$$= \int d\theta = \theta + c$$

$$= \sin^{-1}\frac{x}{a} + c$$

(e) $\int\frac{dx}{\sqrt{x^2+a^2}} = \log(x + \sqrt{x^2+a^2}) + c = \sinh^{-1}\frac{x}{a} + c$

Put $x = a\tan\theta$ $\quad \therefore dx = a\sec^2\theta\, d\theta$

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta$$

$$\therefore \int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\sec^2\theta\, d\theta}{a\sec\theta}$$

$$= \int\sec\theta\, d\theta$$

$$= \log(\sec\theta + \tan\theta) + c'$$

$$= \log\left(\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right) + c'$$

$$= \log\left(\frac{x+\sqrt{a^2+x^2}}{a}\right) + c'$$

$$= \log(x + \sqrt{a^2+x^2}) - \log a + c'$$

$$= \log(x + \sqrt{a^2+x^2}) + c$$

Again, if we put $x = a\sinh y$ so that $dx = a\cosh y\, dy$

Then, $\int\frac{1}{\sqrt{x^2+a^2}}dx = \int\frac{a\cosh y\, dy}{\sqrt{a^2\sinh^2 y + a^2}}$

$$= \int\frac{a\cosh y}{a\cosh y}dy = \int dy$$

$$= y + c = \sinh^{-1}\frac{x}{a} + c$$

(f) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log(x + \sqrt{x^2 - a^2}) + c = \cosh^{-1}\frac{x}{a} + c$

Put $x = a \sec\theta$ $\quad\therefore dx = a \sec\theta \tan\theta \, d\theta$

$$\sqrt{x^2 - a^2} = \sqrt{a^2 \sec^2\theta - a^2} = a \tan\theta$$

$$\therefore \int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a \sec\theta \tan\theta \, d\theta}{a \tan\theta}$$

$$= \int \sec\theta \, d\theta$$

$$= \log(\sec\theta + \tan\theta) + c'$$

$$= \log\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) + c'$$

$$= \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) + c'$$

$$= \log(x + \sqrt{x^2 - a^2}) - \log a + c'$$

$$= \log(x + \sqrt{x^2 - a^2}) + c$$

Again if we put $x = a \cosh y$

so that $dx = a \sinh y \, dy$

Then, $\int \frac{1}{\sqrt{x^2 - a^2}} dx = \int \frac{a \sinh y \, dy}{\sqrt{a^2 \cosh^2 y - a^2}}$

$$= \int \frac{a \sinh y}{a \sinh y} dy = \int dy$$

$$= y + c = \cosh^{-1}\frac{x}{a} + c$$

Some Integrals Reducible to Standard Forms

The integrals of the forms $\int \frac{1}{ax^2 + bx + c} dx$ and $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can now be easily evaluated by converting them into above standard integrals as illustrated in the worked out examples below.

The other forms of integrals which can be reduced to the standard integrals are

$$\int \frac{mx + e}{ax^2 + bx + c} dx \quad \text{and} \quad \int \frac{mx + e}{\sqrt{ax^2 + bx + c}} dx$$

The integral $\int \frac{mx + e}{ax^2 + bx + c} dx$ ...... (i)

can be written as

$$p \int \frac{2ax + b}{ax^2 + bx + c} dx + q \int \frac{1}{ax^2 + bx + c} dx \quad \text{...... (ii)}$$

in which the numerator of the first integral is the differential coefficient of the denominator and the constants p and q are to be adjusted so as to make the sum (ii) equal to the given integral (i). We can easily see that

$$2ap = m \quad \text{and} \quad bp + q = e$$

which can be solved for p and q.

The integrals in (ii) can be easily integrated.

Similarly, $\int \frac{mx+e}{\sqrt{ax^2+bx+c}}\,dx$ can be expressed as the sum

$$p\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + q\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

and the values of p and q can be obtained from

$$2ap = m \quad \text{and} \quad bp + q = e$$

Worked Out Examples

Example 1

Integrate $\int \frac{dx}{4x^2+24x+45}$

Solution:

Let $\quad I = \int \frac{dx}{4x^2+24x+45} = \int \frac{dx}{(2x+6)^2+3^2}$

Put $\quad y = 2x+6$

$\therefore \quad dy = 2dx$

Now, $\quad I = \frac{1}{2}\int \frac{dy}{y^2+3^2} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\frac{y}{3} + c$

$$= \frac{1}{6}\tan^{-1}\frac{2x+6}{3} + c$$

Example 2

Integrate $\int \frac{dx}{x^2+10x-11}$

Solution :

$$\int \frac{dx}{x^2+10x-11} = \int \frac{dx}{(x+5)^2-6^2}$$

$$= \frac{1}{2\,.\,6}\log\frac{(x+5)-6}{(x+5)+6} + c$$

$$= \frac{1}{12}\log\frac{x-1}{x+11} + c$$

Example 3

Evaluate $\int \frac{3x+5}{x^2+4x+20}\,dx$

Solution:

Let $\int \frac{3x+5}{x^2+4x+20}\,dx$

$$= p\int \frac{2x+4}{x^2+4x+20}\,dx + q\int \frac{dx}{x^2+4x+20}$$

$\therefore \quad 2p = 3 \quad$ and $\quad 4p + q = 5$

or $\quad p = \frac{3}{2} \quad$ and $\quad q = -1$

Now $\int \frac{3x+5}{x^2+4x+20}\,dx$

$$= \frac{3}{2}\int \frac{(2x+4)\,dx}{x^2+4x+20} - \int \frac{dx}{x^2+4x+20}$$

$$= \frac{3}{2}\log(x^2+4x+20) - \int \frac{dx}{(x+2)^2+4^2}$$

$$= \frac{3}{2}\log(x^2+4x+20) - \frac{1}{4}\tan^{-1}\frac{x+2}{4} + c$$

Example 4

Evaluate $\int \frac{dx}{\sqrt{x^2+6x+34}}$

Solution:

$$\int \frac{dx}{\sqrt{x^2+6x+34}} = \int \frac{dx}{\sqrt{(x+3)^2+5^2}}$$

$$= \log\{(x+3) + \sqrt{(x+3)^2+5^2}\} + c$$

$$= \log\{(x+3) + \sqrt{x^2+6x+34}\} + c$$

Example 5

Evaluate $\int \frac{dx}{\sqrt{4x-x^2}}$

Solution:

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(4-4x+x^2)}}$$

$$= \int \frac{dx}{\sqrt{(2)^2-(x-2)^2}}$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + c$$

Example 6

Evaluate : $\int \frac{dx}{\sqrt{1 + e^{-2x}}}$

Solution:

Let $I = \int \frac{dx}{\sqrt{1 + e^{-2x}}} = \int \frac{e^x \, dx}{\sqrt{e^{2x} + 1}}$

Put $e^x = y$ so that $e^x \, dx = dy$

$I = \int \frac{dy}{\sqrt{y^2 + 1}} \quad = \log (y + \sqrt{y^2 + 1}) + c$

$= \log ((e^x + \sqrt{e^{2x} + 1}) + c$

Example 7

Evaluate : $\int \frac{dx}{\sqrt{x^2 - 8x + 15}}$

Solution :

Let $I = \int \frac{dx}{\sqrt{x^2 - 8x + 15}} = \int \frac{dx}{\sqrt{(x-3)(x-5)}}$

Put $x - 3 = y^2$ so that $dx = 2y \, dy$ and $x = y^2 + 3$

Now, $\quad I \quad = \int \frac{2y \, dy}{y\sqrt{y^2 + 3 - 5}}$

$= 2 \int \frac{dy}{\sqrt{y^2 - 2}}$

$= 2 \log (y + \sqrt{y^2 - 2}) + c$

$= 2 \log (\sqrt{x-3} + \sqrt{x-5}) + c$

Example 8

Integrate $\int \frac{3x + 2}{\sqrt{2x^2 + 5x + 4}} dx$

Solution:

Let $\int \frac{3x+2}{\sqrt{2x^2 + 5x + 4}} dx = p \int \frac{4x+5}{\sqrt{2x^2 + 5x + 4}} dx + q \int \frac{dx}{\sqrt{2x^2 + 5x + 4}}$

$\therefore \quad 4p = 3 \quad$ and $\quad 5p + q = 2$

or $\quad p = \frac{3}{4} \quad$ and $\quad q = -\frac{7}{4}$

$\therefore \int \frac{3x+2}{\sqrt{2x^2 + 5x + 4}} dx = \frac{3}{4} \int \frac{4x+5}{\sqrt{2x^2 + 5x + 4}} dx - \frac{7}{4} \int \frac{dx}{\sqrt{2x^2 + 5x + 4}}$

$$= \frac{3}{4} \cdot \frac{2}{1}\sqrt{2x^2 + 5x + 4} - \frac{7}{4} \cdot \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{x^2 + \frac{5}{2}x + 2}}$$

$$= \frac{3}{2}\sqrt{2x^2 + 5x + 4} - \frac{7}{4\sqrt{2}} \int \frac{1}{\sqrt{\left(x + \frac{5}{4}\right)^2 + \frac{7}{16}}} dx$$

$$= \frac{3}{2}\sqrt{2x^2 + 5x + 4} - \frac{7}{4\sqrt{2}} \cdot \log\left(x + \frac{5}{4} + \sqrt{x^2 + \frac{5}{2}x + 2}\right)$$

Example 9

Integrate $\int \frac{dx}{(11 + x)\sqrt{2 + x}}$

Solution:

Let $I = \int \frac{dx}{(11 + x)\sqrt{2 + x}}$

put $z^2 = 2 + x \qquad 2zdz = dx$

Then, $I = \int \frac{2z\, dz}{(9 + z^2)z} = 2\int \frac{dz}{z^2 + 3^2}$

$$= \frac{2}{3}\tan^{-1}\frac{z}{3} + c$$

$$= \frac{2}{3}\tan^{-1}\frac{2 + x}{3} + c$$

EXERCISE

Integrate

1. (i) $\int \frac{dx}{x^2 + 49}$ (ii) $\int \frac{dx}{4x^2 + 25}$

(iii) $\int \frac{x\, dx}{x^4 + 3}$ (iv) $\int \frac{2x + 3}{4x^2 + 1} dx$

(v) $\int \frac{6x + 1}{x^2 + 9} dx$ (vi) $\int \frac{dx}{e^x + e^{-x}}$

(vii) $\int \frac{dx}{4x^2 - 9}$ (viii) $\int \frac{dx}{5x^2 - 4}$

(ix) $\int \frac{dx}{x^2 + 6x + 8}$ (x) $\int \frac{dx}{1 + x + x^2}$

(xi) $\int \frac{dx}{1 + x - x^2}$ (xii) $\int \frac{\cos x\, dx}{\sin^2 x + 4\sin x + 5}$

(xiii) $\int \frac{x\,dx}{x^2+4x-12}$

(xiv) $\int \frac{x\,dx}{x^2+4x+40}$

(xv) $\int \frac{2x+3}{x^2+4x+5}\,dx$

(xvi) $\int \frac{(2x+2)\,dx}{3+2x-x^2}$

(xvii) $\int \frac{4x+3}{4x^2+12x+5}\,dx$

(xviii) $\int \frac{6x+2}{9x^2+6x+26}\,dx$

2. (i) $\int \frac{dx}{\sqrt{x^2-16}}$

(ii) $\int \frac{dx}{\sqrt{4x^2+9}}$

(iii) $\int \frac{dx}{\sqrt{x^2+4x-5}}$

(iv) $\int \frac{dx}{\sqrt{x^2+6x+10}}$

(v) $\int \frac{dx}{\sqrt{2ax+x^2}}$

(vi) $\int \frac{dx}{\sqrt{2ax-x^2}}$

(vii) $\int \frac{dx}{\sqrt{2x^2+3x+4}}$

(viii) $\int \frac{x\,dx}{\sqrt{a^4+x^4}}$

(ix) $\int \frac{x\,dx}{\sqrt{x^4+2x^2+10}}$

(x) $\int \frac{x\,dx}{\sqrt{x^2+4x+5}}$

(xi) $\int \frac{2x+3}{\sqrt{x^2+4x+20}}\,dx$

(xii) $\int \frac{x+2}{\sqrt{x^2+x+1}}\,dx$

(xiii) $\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx$

(xiv) $\int \frac{dx}{\sqrt{x^2-7x+12}}$

(xv) $\int \frac{dx}{\sqrt{8-2x-x^2}}$

(xvi) $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}\ (\beta > \alpha)$

(xvii) $\int \frac{dx}{(4x+3)\sqrt{x+3}}$

(xviii) $\int \frac{dx}{(2x+1)\sqrt{4x+3}}$

(xix) $\int \sqrt{\frac{1+x}{1-x}}\,dx$

Answer

1. (i) $\frac{1}{7}\tan^{-1}\frac{x}{7}+c$ (ii) $\frac{1}{10}\tan^{-1}\frac{2x}{5}+c$ (iii) $\frac{1}{2\sqrt3}\tan^{-1}\frac{x^2}{\sqrt3}+c$

(iv) $\frac{1}{4}\log(4x^2+1)+\frac{3}{2}\tan^{-1}2x+c$ (v) $3\log(x^2+9)+\frac{1}{3}\tan^{-1}\frac{x}{3}+c$ (vi) $\tan^{-1}(e^x)+c$

(vii) $\frac{1}{12}\log\frac{2x-3}{2x+3}+c$ (viii) $\frac{1}{4\sqrt5}\log\frac{\sqrt5x-2}{\sqrt5x+2}+c$ (ix) $\frac{1}{2}\log\frac{x+2}{x+4}+c$

(x) $\frac{2}{\sqrt3}\tan^{-1}\left(\frac{2x+1}{\sqrt3}\right)+c$ (xi) $\frac{1}{\sqrt5}\log\frac{\sqrt5-1+2x}{\sqrt5+1-2x}+c$ (xii) $\tan^{-1}(\sin x+2)+c$

(xiii) $\frac{1}{2}\log(x^2+4x-12)-\frac{1}{4}\log\frac{x-2}{x+6}+c$ (xiv) $\frac{1}{2}\log(x^2+4x+40)-\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right)+c$

(xv) $\log(x^2+4x+5)-\tan^{-1}(x+2)+c$ (xvi) $\log\frac{1+x}{3-x}-\log(3+2x-x^2)+c$

(xvii) $\frac{1}{2}\log(4x^2+12x+5)-\frac{3}{8}\log\frac{2x+1}{2x+5}+c$ (xviii) $\frac{1}{3}\log(9x^2+6x+26)+c$

2. (i) $\log\left(x+\sqrt{x^2-16}\right)+c$

(ii) $\frac{1}{2}\log\left(2x+\sqrt{4x^2+9}\right)+c$

(iii) $\log\left(x+2+\sqrt{x^2+4x-5}\right)+c$

(iv) $\log\left(x+3+\sqrt{x^2+6x+10}\right)+c$

(v) $\log\left(x+a+\sqrt{2ax+x^2}\right)+c$

(vi) $\sin^{-1}\left(\frac{x-a}{a}\right)+c$

(vii) $\frac{1}{\sqrt{2}}\log\left(x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right)+c$

(viii) $\frac{1}{2}\log\left(x^2+\sqrt{x^4+a^4}\right)+c$

(ix) $\frac{1}{2}\log\left(x^2+1+\sqrt{x^4+2x^2+10}\right)+c$

(x) $\sqrt{x^2+4x+5}-2\log\left(x+2+\sqrt{x^2+4x+5}\right)+c$

(xi) $2\sqrt{x^2+4x+20}-\log\left(x+2+\sqrt{x^2+4x+20}\right)+c$

(xii) $\sqrt{x^2+x+1}+\frac{3}{2}\log\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+c$

(xiii) $\frac{1}{2}\sqrt{2x^2-8x+5}+c$

(xiv) $\log\left(x-\frac{7}{2}+\sqrt{x^2-7x+12}\right)+c$

(xv) $\sin^{-1}\left(\frac{x+1}{3}\right)+c$

(xvi) $2\log\left(\sqrt{x-\alpha}+\sqrt{x-\beta}\right)+c$

(xvii) $\frac{1}{6}\log\frac{2\sqrt{x+3}-3}{2\sqrt{x+3}+3}+c$

(xviii) $\frac{1}{2}\log\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}+c$

(xix) $\sin^{-1}x-\sqrt{1-x^2}+c$

Standard Integrals (II)

In this section we consider the second set of standard integrals which can be evaluated by a technique known as *integration by parts*. This is a formula which can be deduced as given below.

We know $\quad \frac{d}{dx}(u\,v_1)=\frac{du}{dx}v_1+u\frac{dv_1}{dx}$

for any two differentiable functions u and v_1 of x. Integrating both sides, we get

$$uv_1=\int\left(\frac{du}{dx}v_1\right)dx+\int\left(u\frac{dv_1}{dx}\right)dx$$

or $\quad \int\left(u\frac{dv_1}{dx}\right)dx=uv_1-\int\left(\frac{du}{dx}v_1\right)dx \quad \ldots\ldots\text{(i)}$

Let $\frac{dv_1}{dx}=v$ so that $v_1=\int v\,dx$. Then the relation (i) becomes

$$\int(uv)\,dx=u\int v\,dx-\int\left(\frac{du}{dx}\int v\,dx\right)dx$$

which is the required formula.

With the application of this formula, we can evaluate the following standard integrals.

(a) $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

Integrating by parts, we get

$$\int e^{ax} \cos bx\, dx = e^{ax} \int \cos bx\, dx - \int \left(\frac{d\, e^{ax}}{dx} \int \cos bx\, dx\right) dx$$

$$= \frac{e^{ax} \sin bx}{b} - \frac{a}{b} \int e^{ax} \sin bx\, dx$$

$$= \frac{e^{ax} \sin bx}{b} - \frac{a}{b}\left[e^{ax} \int \sin bx\, dx - \int \left(\frac{de^{ax}}{dx} \int \sin bx\, dx\right) dx \right]$$

$$= \frac{e^{ax} \sin bx}{b} + \frac{a}{b^2} e^{ax} \cos bx - \frac{a^2}{b^2} \int e^{ax} \cos bx\, dx$$

or $\dfrac{a^2 + b^2}{b^2} \int e^{ax} \cos bx\, dx = e^{ax} \dfrac{(b \sin bx + a \cos bx)}{b^2}$

or $\int e^{ax} \cos bx\, dx = \dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$

Similarly, we can deduce

(b) $\int e^{ax} \sin bx\, dx = \dfrac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$

(c) $\int \sqrt{x^2 + a^2}\, dx = \dfrac{x\sqrt{x^2 + a^2}}{2} + \dfrac{a^2}{2} \log (x + \sqrt{x^2 + a^2})$

Integrating by parts, we get

$$\int \sqrt{x^2 + a^2}\, dx = \sqrt{x^2 + a^2} \int 1\, dx - \int \left(\frac{d\sqrt{x^2 + a^2}}{dx} \int 1\, dx\right) dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{2x}{2\sqrt{x^2 + a^2}}\, x\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \frac{(x^2 + a^2) - a^2}{\sqrt{x^2 + a^2}}\, dx$$

$$= x\sqrt{x^2 + a^2} - \int \sqrt{x^2 + a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2 + a^2}}$$

or $2\int \sqrt{x^2 + a^2}\, dx = x\sqrt{x^2 + a^2} + a^2 \log (x + \sqrt{x^2 + a^2})$

or $\int \sqrt{x^2 + a^2}\, dx = \dfrac{x\sqrt{x^2 + a^2}}{2} + \dfrac{a^2}{2} \log (x + \sqrt{x^2 + a^2})$

Similarly, we can deduce

(d) $\int \sqrt{x^2 - a^2}\, dx = \dfrac{1}{2} x\sqrt{x^2 - a^2} - \dfrac{1}{2} a^2 \log (x + \sqrt{x^2 - a^2})$

(e) $\int \sqrt{a^2 - x^2}\, dx = \dfrac{1}{2} x\sqrt{a^2 - x^2} + \dfrac{1}{2} a^2 \sin^{-1} \dfrac{x}{a}$.

More Integrals Reducible to Standard Forms

The integrals of the form $\int \sqrt{ax^2 + bx + c}\, dx$

can now be easily integrated by converting them into one of the above standard integrals as the case may be.

The other form of the integral which can be reduced to the sum of two integrable forms is

$$\int (mx + e) \sqrt{ax^2 + bx + c}\, dx$$

The integral $\int (mx + e) \sqrt{ax^2 + bx + c}\, dx$ ...... (i)

can be written as the sum of two integrals

$$p \int (2ax + b) \sqrt{ax^2 + bx + c}\, dx + q \int \sqrt{ax^2 + bx + c}\, dx$$

in which the constants p and q are to be adjusted so as to make the sum equal to the given integral (i). It should be noted that the integrand of the first integral is the product of $2ax + b$ and $\sqrt{ax^2 + bx + c}$ and $2ax + b$ is the differential coefficient of $ax^2 + bx + c$. The values of p and q can be obtained from $2ap = m$ and $bp + q = e$

Worked Out Examples

Example 1

Integrate $\int \sqrt{9x^2 + 24x + 25}\, dx$

Solution:

Let $I = \int \sqrt{9x^2 + 24x + 25}\, dx$

$= \int \sqrt{(3x + 4)^2 + 3^2}\, dx$

Put $y = 3x + 4$ $\quad \therefore dy = 3dx$

$\therefore \quad I = \frac{1}{3} \int \sqrt{y^2 + 3^2}\, dy$

$= \frac{1}{3} [\frac{1}{2} y \sqrt{y^2 + 3^2} + \frac{1}{2} . 3^2 \log (y + \sqrt{y^2 + 3^2}\,] + c$

$= \frac{1}{6} (3x + 4) \sqrt{9x^2 + 24x + 25} + \frac{3}{2} \log (3x + 4 + \sqrt{9x^2 + 24x + 25}\,) + c$

Example 2

Integrate $\int (2x + 5) \sqrt{4x^2 + 12x + 45}\, dx$

Solution:

Let $I = \int (2x + 5) \sqrt{4x^2 + 12x + 45}\, dx$

$= p \int (8x + 12) \sqrt{4x^2 + 12x + 45}\, dx + q \int \sqrt{4x^2 + 12x + 45}\, dx$

so that $8p = 2$ and $12p + q = 5$

or $p = \frac{1}{4}$ and $q = 2$

$$\therefore \quad I = \frac{1}{4}\int (8x + 12)\sqrt{4x^2 + 12x + 45}\, dx + 2\int \sqrt{4x^2 + 12x + 45}\, dx$$

$$= \frac{1}{4}\cdot\frac{2}{3}(4x^2 + 12x + 45)^{3/2} + 2\int \sqrt{(2x + 3)^2 + 6^2}\, dx$$

$$= \frac{1}{6}(4x^2 + 12x + 45)^{3/2} + 2\cdot\frac{1}{2}\int \sqrt{y^2 + 6^2}\, dy \quad \text{where } y = 2x + 3$$

$$= \frac{1}{6}(4x^2 + 12x + 45)^{3/2} + \frac{1}{2}y\sqrt{y^2 + 6^2} + \frac{6^2}{2}\log\left(y + \sqrt{y^2 + 6^2}\right) + c$$

$$= \frac{1}{6}(4x^2 + 12x + 45)^{3/2} + \frac{1}{2}(2x + 3)\sqrt{4x^2 + 12x + 45}$$
$$+ \ 18\log\left(2x + 3 + \sqrt{4x^2 + 12x + 45}\right) + c$$

EXERCISE

Integrate :

1. $\int \sqrt{25 - 9x^2}\, dx$
2. $\int \sqrt{(x - \alpha)(\beta - x)}\, dx$
3. $\int \sqrt{5 - 2x + x^2}\, dx$
4. $\int \sqrt{x^2 - 36}\, dx$
5. $\int \sqrt{3x^2 + 5}\, dx$
6. $\int \sqrt{3 - 2x - x^2}\, dx$
7. $\int \sqrt{4x^2 - 4x + 5}\, dx$
8. $\int \sqrt{2ax - x^2}\, dx$
9. $\int \frac{dx}{x + \sqrt{x^2 - 1}}$
10. $\int (x - 1)\sqrt{5 - 4x - x^2}\, dx$
11. $\int (2x + 1)\sqrt{4x^2 + 20x + 21}\, dx$
12. $\int (x + 2)\sqrt{x^2 + 10x - 11}\, dx$
13. $\int (2 - x)\sqrt{16 - 6x - x^2}\, dx$
14. $\int (x + 3)\sqrt{9x^2 + 12x + 13}\, dx$
15. $\int e^{3x}\sin 5x\, dx$
16. $\int e^{4x}\cos 2x\, dx$

Answer

1. $\frac{1}{6}\left(3x\sqrt{25 - 9x^2} + 25\sin^{-1}\frac{3x}{5}\right) + c$
2. $\frac{1}{4}(2x - (\alpha + \beta))\sqrt{(x - \alpha)(\beta - x)} + \frac{1}{8}(\beta - \alpha)^2\sin^{-1}\left(\frac{2x - \alpha - \beta}{\beta - \alpha}\right) + c$

3. $\frac{1}{2}(x-1)\sqrt{5-2x+x^2} + 2\log\left(x-1+\sqrt{5-2x+x^2}\right) + c$

4. $\frac{1}{2}x\sqrt{x^2-36} - 18\log\left(x+\sqrt{x^2-36}\right) + c$

5. $\frac{1}{2}x\sqrt{3x^2+5} + \frac{5}{2\sqrt{3}}\log\left(\sqrt{3}x+\sqrt{3x^2+5}\right) + c$

6. $\frac{1}{2}(x+1)\sqrt{3-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{2}\right) + c$

7. $\frac{1}{4}(2x-1)\sqrt{4x^2-4x+5} + \log\left(2x-1+\sqrt{4x^2-4x+5}\right) + c$

8. $\frac{1}{2}(x-a)\sqrt{2ax-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x-a}{a}\right) + c$

9. $\frac{x^2}{2} - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + c$

10. $-\frac{1}{3}\left(5-4x-x^2\right)^{3/2} - \frac{3}{2}(x+2)\sqrt{5-4x-x^2} - \frac{27}{2}\sin^{-1}\left(\frac{x+2}{3}\right) + c$

11. $\frac{1}{6}(4x^2+20x+21)^{3/2} - (2x+5)\sqrt{4x^2+20x+21} + 4\log\left(2x+5+\sqrt{4x^2+20x+21}\right) + c$

12. $\frac{1}{3}(x^2+10x-11)^{3/2} - \frac{3}{2}(x+5)\sqrt{x^2+10-11} + 54\log\left(x+5+\sqrt{x^2+10x-11}\right) + c$

13. $\frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + c$

14. $\frac{1}{27}(9x^2+12x+13)^{3/2} + \frac{7}{18}(3x+2)\sqrt{9x^2+12x+13} + \frac{7}{2}\log\left(3x+2+\sqrt{9x^2+12x+13}\right) + c$

15. $\frac{1}{34}e^{3x}(3\sin 5x - 5\cos 5x)$ 16. $\frac{1}{20}e^{4x}(4\cos 2x + 2\sin 2x)$

Standard Integrals of Some Special Trigonometrical Functions

Various techniques that may be applied for integrating some special trigonometric functions are briefly sketched in this section.

(a) $\int \operatorname{cosec} x\, dx = \log\left(\tan\frac{x}{2}\right) + c$

$$\int \operatorname{cosec} x\, dx = \int \frac{1}{\sin x}dx = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

$$= \frac{1}{2}\int \frac{1}{\sin\frac{1}{2}x\cos\frac{1}{2}x} \times \frac{\sec^2\frac{1}{2}x}{\sec^2\frac{1}{2}x}dx = \frac{1}{2}\int \frac{\sec^2\frac{1}{2}x}{\tan\frac{1}{2}x}dx$$

$$= \log\left(\tan\frac{1}{2}x\right) + c$$

(b) $\int \sec x\, dx = \log\left(\tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right)\right) + c$

$$\int \sec x\, dx = \int \frac{dx}{\cos x}$$

$$= \int \frac{1}{\sin\left(\frac{1}{2}\pi + x\right)} dx$$

$$= \frac{1}{2}\int \frac{dx}{\sin\left(\frac{1}{4}\pi + \frac{1}{2}x\right)\cos\left(\frac{1}{4}\pi + \frac{1}{2}x\right)}$$

$$= \frac{1}{2}\int \frac{\sec^2\left(\frac{1}{4}\pi + \frac{1}{2}x\right)}{\tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right)} dx$$

$$= \log\left(\tan\left(\frac{1}{4}\pi + \frac{1}{2}x\right)\right) + c$$

(c) $\int \frac{dx}{a + b\cos x}$

$$= \int \frac{dx}{a\left(\cos^2\frac{1}{2}x + \sin^2\frac{1}{2}x\right) + b\left(\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x\right)}$$

$$= \int \frac{1}{(a+b)\cos^2\frac{1}{2}x + (a-b)\sin^2\frac{1}{2}x} \times \frac{\sec^2\frac{1}{2}x}{\sec^2\frac{1}{2}x} dx$$

$$= \int \frac{\sec^2\frac{1}{2}x}{(a+b) + (a-b)\tan^2\frac{1}{2}x} dx$$

Case I. $a > b$

Put $\sqrt{a-b}\,.\tan\frac{1}{2}x = z$ $\quad\therefore\quad \frac{1}{2}\sqrt{a-b}\sec^2\frac{1}{2}x\, dx = dz$

Then the given integral transforms into

$$\frac{2}{\sqrt{a-b}}\int \frac{dz}{(a+b) + z^2}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{z}{\sqrt{a+b}} + c$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

Case II. $a < b$.

The given integral can be written as $\int \dfrac{\sec^2 \frac{1}{2} x}{(a+b) - (b-a)\tan^2 \frac{1}{2} x} dx$ ...... (i)

Put $\sqrt{b-a} \tan \frac{1}{2} x = z$

$\therefore \quad \frac{1}{2}\sqrt{b-a} \sec^2 \frac{1}{2} x \, dx = dz$

Then the integral (i) becomes

$$\frac{2}{\sqrt{b-a}} \int \frac{dz}{(a+b) - z^2} = \frac{2}{\sqrt{b-a}} \cdot \frac{1}{2\sqrt{b+a}} \log \frac{\sqrt{b+a} + z}{\sqrt{b+a} - z} + c$$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \left(\frac{\sqrt{b+a} + \sqrt{b-a} \tan \frac{1}{2} x}{\sqrt{b+a} - \sqrt{b-a} \tan \frac{1}{2} x} \right) + c$$

The integral $\int \dfrac{dx}{a + b \sin x}$ can also be similarly evaluated.

(d) Hyperbolic Functions

(i) $\int \sinh x \, dx = \int \frac{1}{2}(e^x - e^{-x}) \, dx$

$$= \frac{1}{2}(e^x + e^{-x}) + c = \cosh x + c$$

(ii) $\int \cosh x \, dx = \int \frac{1}{2}(e^x + e^{-x}) \, dx$

$$= \frac{1}{2}(e^x - e^{-x}) + c = \sinh x + c$$

(iii) $\int \tanh x \, dx = \int \dfrac{\sinh x}{\cosh x} dx$

$$= \int \frac{dz}{z} \qquad \text{where } z = \cosh x$$

$$= \log z + c = \log(\cosh x) + c$$

(iv) $\int \coth x \, dx = \int \dfrac{\cosh x}{\sinh x} dx = \log(\sinh x) + c$

(v) $\int \operatorname{cosech} x \, dx = \int \dfrac{1}{\sinh x} dx = \int \dfrac{2}{e^x - e^{-x}} dx$

$$= \int \frac{2e^x}{e^{2x} - 1} dx = 2\int \frac{1}{y^2 - 1} dy \qquad \text{where } y = e^x$$

$$= \log \frac{y-1}{y+1} + c = \log \frac{e^x - 1}{e^x + 1} + c$$

$$= \log \left(\tanh \frac{x}{2}\right) + c$$

(vi) $\int \text{sech}\, x\, dx \quad = \int \frac{1}{\cosh x} dx = \int \frac{dx}{\cosh^2 \frac{x}{2} + \sinh^2 \frac{x}{2}}$

$$= \int \frac{\text{sech}^2 \frac{x}{2}}{1 + \tanh^2 \frac{x}{2}} dx$$

$$= 2\int \frac{dz}{1+z^2}, \quad \text{where } z = \tanh \frac{x}{2}$$

$$= 2 \tan^{-1} z + c = 2 \tan^{-1}\left(\tanh \frac{x}{2}\right) + c$$

(vii) $\int \text{sech}^2 x\, dx = \tanh x$

(viii) $\int \text{cosech}^2 x\, dx = -\coth x$

(ix) $\int \text{sech}\, x \tanh x\, dx = -\text{sech}\, x$

(x) $\int \text{cosech}\, x \coth x\, dx = -\text{cosech}\, x$

Standard Integral of the Form (III)

An integral of the form $\int \frac{dx}{a \sin x + b \cos x}$ can be obtained in the following way.

Put $a = r\cos\alpha$ and $b = r \sin \alpha$

so that $r = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\frac{b}{a}$

$$\therefore \int \frac{dx}{a \sin x + b \cos x} = \frac{1}{r}\int \frac{dx}{\cos\alpha \sin x + \sin\alpha \cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)}$$

$$= \frac{1}{r}\int \text{cosec}\,(x+\alpha)\, dx$$

$$= \frac{1}{r}\log\left(\tan \frac{1}{2}(x+\alpha)\right) + c$$

$$= \frac{1}{\sqrt{a^2+b^2}} \log\left(\tan\frac{1}{2}\left(x + \tan^{-1}\frac{b}{a}\right)\right) + c$$

Worked Out Examples

Example 1

Integrate $\int \frac{dx}{1 - 3\sin x}$

Solution:

$$\int \frac{dx}{1-3\sin x} = \int \frac{dx}{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) - 6\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2} - 6\tan\frac{x}{2} + 1} \quad \ldots\ldots \text{(i)}$$

Put $z = \tan\frac{x}{2}$ $\quad \therefore dz = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

Then the integral (i) becomes

$$2\int \frac{dz}{z^2 - 6z + 1} = 2\int \frac{dz}{(z-3)^2 - (2\sqrt{2})^2}$$

$$= 2\,\frac{1}{2\,.\,2\sqrt{2}}\cdot\log\frac{z-3-2\sqrt{2}}{z-3+2\sqrt{2}} + c$$

$$= \frac{1}{2\sqrt{2}}\log\frac{\tan\frac{1}{2}x - 3 - 2\sqrt{2}}{\tan\frac{1}{2}x - 3 + 2\sqrt{2}} + c$$

Example 2

Evaluate : $\int \frac{dx}{3 - \cos x}$

Solution:

Let $\quad I = \int \frac{dx}{3 - \cos x}$

$$= \int \frac{dx}{3(\cos^2 x/2 + \sin^2 x/2) - (\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{dx}{2\cos^2 x/2 + 4\sin^2 x/2}$$

$$= \frac{1}{2}\int \frac{\sec^2 x/2\,dx}{1 + 2\tan^2 x/2}$$

Put $\sqrt{2}\tan\frac{x}{2} = y$ so that $\sqrt{2}\sec^2\frac{x}{2}\cdot\frac{1}{2}\,dx = dy$

$\therefore \quad dx = \sqrt{2}\,dy$

Now, $I = \sqrt{2}\int \frac{dy}{1+y^2} = \sqrt{2}\tan^{-1}y + c$

$= \sqrt{2}\tan^{-1}\left(\sqrt{2}\tan\frac{x}{2}\right) + c$

Example 3

Evaluate : $\int \frac{dx}{2\sin x + 3\cos x}$

Solution:

Let $I = \int \frac{dx}{2.2\sin\frac{x}{2}\cos\frac{x}{2} + 3\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$

$= \int \frac{\sec^2\frac{x}{2}\,dx}{4\tan\frac{x}{2} + 3 - 3\tan^2\frac{x}{2}}$

$= \frac{1}{3}\int \frac{\sec^2\frac{x}{2}\,dx}{1 + \frac{4}{3}\tan\frac{x}{2} - \tan^2\frac{x}{2}}$

Put $\tan\frac{x}{2} = y$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy$

$\therefore \sec^2\frac{x}{2}\,dx = 2\,dy$

Now, $I = \frac{2}{3}\int \frac{dy}{1 + \frac{4}{3}y - y^2} = \frac{2}{3}\int \frac{dy}{\frac{13}{9} - \left(y - \frac{2}{3}\right)^2}$

$= \frac{2}{3}\int \frac{dy}{\left(\frac{\sqrt{13}}{3}\right)^2 - \left(y - \frac{2}{3}\right)^2}$

$= \frac{2}{3}\cdot\frac{1}{\frac{2\sqrt{13}}{3}}\log\frac{\frac{\sqrt{13}}{3} - \left(y - \frac{2}{3}\right)}{\frac{\sqrt{13}}{3} + y - \frac{2}{3}} + c$

$= \frac{1}{\sqrt{13}}\log\frac{\sqrt{13} + 2 - y}{\sqrt{13} - 2 + y} + c$

$= \frac{1}{\sqrt{13}}\log\frac{\sqrt{13} + 2 - \tan\frac{x}{2}}{\sqrt{13} - 2 + \tan\frac{x}{2}} + c$

Example 4

Integrate $\int \frac{dx}{3 + 4\cosh x}$

Solution:

$$\int \frac{dx}{3 + 4\cosh x}$$

$$= \int \frac{dx}{3\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + 4\left(\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{7\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}} = \int \frac{\operatorname{sech}^2\frac{x}{2}\,dx}{7 + \tanh^2\frac{x}{2}} \quad \ldots\ldots \text{(i)}$$

Put $\tanh\frac{x}{2} = y \Rightarrow \operatorname{sech}^2\frac{x}{2}\,dx = 2\,dy$

Now, from (i)

$$\int \frac{dx}{3 + 4\cosh x}$$

$$= 2\int \frac{dy}{7 + y^2} = 2.\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{y}{\sqrt{7}}\right) + c$$

$$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{\tanh\frac{x}{2}}{\sqrt{7}}\right) + c$$

EXERCISE

Integrate

1. $\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$
2. $\int \frac{dx}{(\sin x + \cos x)^2}$
3. $\int \frac{dx}{4 - 5\sin^2 x}$
4. $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}\,dx$
5. $\int \frac{dx}{3\sin x - 4\cos x}$
6. $\int \frac{\sin 2x}{(\sin x + \cos x)^2}\,dx$
7. $\int \frac{dx}{2 + \cos x}$
8. $\int \frac{dx}{1 - 2\cos x}$
9. $\int \frac{dx}{1 + \sin x + \cos x}$
10. $\int \frac{dx}{2 + 3\cos x}$

11. $\int \frac{dx}{1 + 2\sin x}$

12. $\int \frac{dx}{2 + \sin x}$

13. $\int \frac{dx}{\sin x + \cos x}$

14. $\int \frac{dx}{3\sin x + 4\cos x}$

15. $\int \frac{\tanh x\, dx}{\cosh x + 64\,\text{sech}\, x}$

16. $\int \frac{\sinh x\, dx}{4\tanh x - \text{cosech}\, x\, \text{sech}\, x}$

17. $\int \frac{dx}{4 + 3\cosh x}$

18. $\int \frac{dx}{4 + 3\sinh x}$

Answer

1. $\frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right) + c$
2. $-\frac{1}{1 + \tan x} + c$
3. $\frac{1}{4}\log\frac{2 + \tan x}{2 - \tan x} + c$
4. $\log\left(\cos x + \sin x + \sqrt{\sin 2x}\right) + c$
5. $\frac{1}{5}\log\frac{2\tan\frac{x}{2} - 1}{2\tan\frac{x}{2} + 4} + c$
6. $x + \frac{1}{1 + \tan x} + c$
7. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c$
8. $\frac{1}{\sqrt{3}}\log\frac{\sqrt{3}\tan\frac{x}{2} - 1}{\sqrt{3}\tan\frac{x}{2} + 1} + c$
9. $\log\left(1 + \tan\frac{x}{2}\right) + c$
10. $\frac{1}{\sqrt{5}}\log\frac{\sqrt{5} + \tan\frac{x}{2}}{\sqrt{5} - \tan\frac{x}{2}} + c$
11. $\frac{1}{\sqrt{3}}\log\frac{\tan\frac{x}{2} + 2 - \sqrt{3}}{\tan\frac{x}{2} + 2 + \sqrt{3}} + c$
12. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2} + 1}{\sqrt{3}}\right) + c$
13. $\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2} + \frac{\pi}{8}\right) + c$
14. $\frac{1}{5}\log\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{4}{3}\right) + c$
15. $\frac{1}{8}\tan^{-1}\left(\frac{\cosh x}{8}\right) + c$
16. $\frac{1}{4}\sinh x + \frac{1}{16}\log\frac{2\sinh x - 1}{2\sinh x + 1} + c$
17. $\frac{1}{\sqrt{7}}\log\frac{\sqrt{7} + \tanh\frac{x}{2}}{\sqrt{7} - \tanh\frac{x}{2}} + c$
18. $\frac{1}{5}\log\frac{1 + 2\tanh\frac{x}{2}}{4 - 2\tanh\frac{x}{2}} + c$

Integration of Rational Fractions

The integration of an algebraic rational fraction is generally not possible as it is. To make its integration possible, the rational function must be resolved into *partial fractions*. Often, the denominator of the given rational fraction can be resolved into real linear and quadratic factors which may be single or repeated. According to the nature of the factors of the denominator, we have to adopt different procedures for the breaking up of the rational fraction into partial fractions. To some extent, we have given some simple methods of breaking some algebraic rational fractions into partial fractions. Now we use this knowledge to integrate some algebraic rational fractions.

Worked Out Examples

Example 1

Integrate $\int \frac{2x}{(2x+3)(3x+5)}\,dx$

Solution:

Let $\frac{2x}{(2x+3)(3x+5)} = \frac{A}{2x+3} + \frac{B}{3x+5}$

$\therefore\quad 2x = A(3x+5) + B(2x+3)$

Equating the coefficient of x and constant terms we get

$$3A + 2B = 2$$

$$5A + 3B = 0$$

$\therefore\quad A = -6$ and $B = 10$

$\therefore\quad \frac{2x}{(2x+3)(3x+5)} = -\frac{6}{2x+3} + \frac{10}{3x+5}$

So we have by integration

$$\int \frac{dx}{(2x+3)(3x+5)} = -3\log(2x+3) + \frac{10}{3}\log(3x+5) + c$$

Example 2

Integrate $\int \frac{2x^2+3}{x^3+3x^2+2x}\,dx$

Solution:

The denominator is

$$x^3 + 3x^2 + 2x = x(x^2+3x+2) = x(x+1)(x+2)$$

Let $\frac{2x^2+3}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

$\therefore\quad 2x^2 + 3 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$

Put $x = 0, -1, -2$ in succession in the above identity. The we get

$$A = \frac{3}{2},\quad B = -5,\quad C = \frac{11}{2}.$$

$\therefore$ The given integral becomes

$$\frac{3}{2}\int \frac{1}{x}\,dx - 5\int \frac{1}{x+1}\,dx + \frac{11}{2}\int \frac{1}{x+2}\,dx$$

$$= \frac{3}{2}\log x - 5\log(x+1) + \frac{11}{2}\log(x+2) + c$$

Example 3

Integrate $\int \frac{x^2}{(x+2)(x+3)^2} dx$

Solution:

Let $\frac{x^2}{(x+2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

Multiplying both sides by $(x+2)(x+3)^2$, we get

$$x^2 = A(x+3)^2 + B(x+2)(x+3) + C(x+2)$$

Put $x = -2, -3$ in succession. Then we get $A = 4$ and $C = -9$

Equating the constant terms, we get

$$9A + 6B + 2C = 0$$

or $\quad B = -3$

$$\therefore \int \frac{x^2}{(x+2)(x+3)^2} dx = 4\int \frac{1}{x+2} dx - 3\int \frac{1}{x+3} dx - 9\int \frac{1}{(x+3)^2} dx$$

$$= 4\log(x+2) - 3\log(x+3) + \frac{9}{x+3} + c$$

Example 4

Integrate $\int \frac{5x^2}{(x+1)(2x^2+3)} dx$

Solution:

Let $\frac{5x^2}{(x+1)(2x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+3}$

or $\quad 5x^2 = A(2x^2+3) + (Bx+C)(x+1)$

Equating the coefficient of like powers of x and constant terms, we get

$$2A + B = 5, \quad B + C = 0, \quad 3A + C = 0$$

$$\therefore A = 1, \quad B = 3, \quad C = -3$$

$$\therefore \int \frac{5x^2}{(x+1)(2x^2+3)} dx = \int \frac{1}{x+1} dx + \int \frac{3x-3}{2x^2+3} dx$$

$$= \int \frac{1}{x+1} dx + \frac{3}{4}\int \frac{4x}{2x^2+3} dx - \frac{3}{2}\int \frac{1}{x^2+\frac{3}{2}} dx$$

$$= \log(x+1) + \frac{3}{4}\log(2x^2+3) - \frac{3}{2}\sqrt{\frac{2}{3}}\tan^{-1}\sqrt{\frac{2}{3}}x + c$$

$$= \log(x+1) + \frac{3}{4}\log(2x^2+3) - \sqrt{\frac{3}{2}}\tan^{-1}\sqrt{\frac{2}{3}}x + c$$

Some special cases

(A) Here the integration of the case of a rational fraction in which the numerator and the denominator consist of only even powers of x will be illustrated.

Example 1

Integrate $\int \frac{5x^2}{3x^4 + x^2 - 2} dx$

Solution:

Put $x^2 = y$.

Then we get

$$\frac{5x^2}{3x^4 + x^2 - 2} = \frac{5y}{3y^2 + y - 2} = \frac{5y}{(3y-2)(y+1)}$$

$$= \frac{A}{3y-2} + \frac{B}{y+1}$$

$$\therefore \quad 5y = A(y+1) + B(3y-2)$$

Equating the coefficient of y and constant terms, we get

$$A + 3B = 5 \quad \text{and} \quad A - 2B = 0$$

or $\quad B = 1 \quad$ and $\quad A = 2$

$$\therefore \frac{5x^2}{3x^4 + x^2 - 3} = \frac{2}{3y-2} + \frac{1}{y+1}$$

$$= \frac{2}{3x^2-2} + \frac{1}{x^2+1}$$

Now,

$$\int \frac{5x^2}{3x^4 + x^2 - 3} dx = \frac{2}{3}\int \frac{1}{x^2 - \frac{2}{3}} dx + \int \frac{1}{x^2+1} dx$$

$$= \frac{2}{3} \cdot \frac{\sqrt{3}}{2\sqrt{2}} \cdot \log \frac{x - \sqrt{\frac{2}{3}}}{x + \sqrt{\frac{2}{3}}} + \tan^{-1}x + c$$

$$= \frac{1}{\sqrt{6}} \log \frac{\sqrt{3}x - \sqrt{2}}{\sqrt{3}x + \sqrt{2}} + \tan^{-1}x + c$$

(B) Here the integration of the case of a rational fraction in which the numerator consists of only odd powers of x and the denominator only even powers of x will be illustrated.

Example 1.

Integrate $\int \frac{(2x^3 + 8x)\, dx}{2x^4 - 5x^2 + 3}$

Solution:

Put $x^2 = z$ $\therefore 2x\,dx = dz$ then

$$\int \frac{2x\,(x^2+4)\,dx}{2x^4 - 5x^2 + 3} = \int \frac{(z+4)\,dz}{2z^2 - 5z + 3}$$

$$= \int \frac{(z+4)\,dz}{(2z-3)\,(z-1)}$$

Let $\dfrac{z+4}{(2z-3)\,(z-1)} = \dfrac{A}{2z-3} + \dfrac{B}{z-1}$

$\therefore$ $z + 4 = A\,(z-1) + B\,(2z-3)$

Equating the coefficients of z and constant terms, we get

$A + 2B = 1$ and $-A - 3B = 4$

$\therefore$ $B = -5$ and $A = 11$

Now,

$$\int \frac{(2x^3+8x)\,dx}{2x^4 - 5x^2 + 3} = 11\int \frac{1}{2z-3}\,dz - 5\int \frac{1}{z-1}\,dz$$

$$= \frac{11}{2}\log(2z-3) - 5\log(z-1) + c$$

$$= \frac{11}{2}\log(2x^2-3) - 5\log(x^2-1) + c$$

(C) For any real numbers a and b $(a \neq b)$ and positive integers m and n,

$$\int \frac{dx}{(x-a)^m\,(x-b)^n}$$

is the other case which can be integrated as illustrated below.

Example 1.

Integrate $\displaystyle\int \frac{dx}{(x-2)^2\,(x-3)^3}$

Solution:

Put $x - 2 = z\,(x-3)$

$\therefore$ $x = \dfrac{3z-2}{z-1}$

or $dx = \dfrac{3\,(z-1) - (3z-2)}{(z-1)^2}\,dz$

$= -\dfrac{1}{(z-1)^2}\,dz$

Also, $\dfrac{1}{(x-2)^2\,(x-3)^3} = \dfrac{1}{z^2\,(x-3)^5}$

$$= \frac{1}{z^2\left(\frac{3z-2}{z-1}-3\right)^5}$$

$$= \frac{(z-1)^5}{z^2}$$

$$\therefore I = \int \frac{(z-1)^5}{z^2} \times -\frac{1}{(z-1)^2}\, dz$$

$$= -\int \frac{(z-1)^3}{z^2}\, dz$$

$$= -\int \frac{z^3 - 3z^2 + 3z - 1}{z^2}\, dz$$

$$= -\int \left(z - 3 + \frac{1}{z} - \frac{1}{z^2}\right) dz$$

$$= -\frac{z^2}{2} + 3z - \log z - \frac{1}{z} + c$$

$$= -\frac{1}{2}\cdot\frac{(x-2)^2}{(x-3)^2} + \frac{3(x-2)}{x-3} - \log\frac{x-2}{x-3} - \frac{x-3}{x-2} + c$$

EXERCISE

Integrate

1. $\int \frac{9}{(2x+1)(x+5)}\, dx$
2. $\int \frac{13}{(3x+4)(4x+1)}\, dx$
3. $\int \frac{1}{(x+2)(x+3)^2}\, dx$
4. $\int \frac{3x}{(x-a)(x-b)}\, dx$
5. $\int \frac{x+1}{x^2-2x-8}\, dx$
6. $\int \frac{5}{(x+5)(2x^2+5)}\, dx$
7. $\int \frac{1}{(a^2+x^2)(b^2+x^2)}\, dx$
8. $\int \frac{1-x}{x^2+x^3}\, dx$
9. $\int \frac{x^2-1}{x^4+x^2+1}\, dx$
10. $\int \frac{1}{x^4-1}\, dx$
11. $\int \frac{x^2}{x^4-2x^2-15}\, dx$
12. $\int \frac{x^3\, dx}{2x^4-3x^2-5}$
13. $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\, dx$
14. $\int \frac{x^3\, dx}{(x^2+a^2)(x^2+b^2)}$
15. $\int \frac{dx}{(x-1)^2(x-2)^3}$

Answer

1. $\log(2x+1) - \log(x+5) + c$

2. $\log(4x+1) - \log(3x+4) + c$

3. $\log\dfrac{x+2}{x+3} + \dfrac{1}{x+3} + c$

4. $\dfrac{3}{a-b}\{a\log(x-a) - b\log(x-b)\} + c$

5. $\dfrac{1}{6}\{5\log(x-4) + \log(x+2)\} + c$

6. $\dfrac{1}{11}\left(\log(x+5) - \dfrac{1}{2}\log(2x^2+5) + \sqrt{10}\tan^{-1}\sqrt{\dfrac{2}{5}}\,x\right) + c$

7. $\dfrac{1}{a^2-b^2}\left(\dfrac{1}{b}\tan^{-1}\dfrac{x}{b} - \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}\right) + c$

8. $2\log\left(\dfrac{1+x}{x}\right) - \dfrac{1}{x} + c$

9. $\dfrac{1}{2}\log\dfrac{x^2-x+1}{x^2+x+1} + c$

10. $\dfrac{1}{4}\log\dfrac{x-1}{x+1} - \dfrac{1}{2}\tan^{-1}x + c$

11. $\dfrac{\sqrt{5}}{16}\log\dfrac{x-\sqrt{5}}{x+\sqrt{5}} + \dfrac{\sqrt{3}}{8}\tan^{-1}\dfrac{x}{\sqrt{3}} + c$

12. $\dfrac{1}{14}\log(x^2+1) + \dfrac{5}{28}\log(2x^2-5) + c$

13. $\dfrac{a}{a^2-b^2}\tan^{-1}\dfrac{x}{a} - \dfrac{b}{a^2-b^2}\tan^{-1}\dfrac{x}{b} + c$

14. $\dfrac{1}{2(a^2-b^2)}\{a^2\log(x^2+a^2) - b^2\log(x^2+b^2)\} + c$

15. $-\dfrac{1}{2}\left(\dfrac{x-1}{x-2}\right)^2 + 3\left(\dfrac{x-1}{x-2}\right) - 3\log\left(\dfrac{x-1}{x-2}\right) - \dfrac{x-2}{x-1} + c$

Chapter 17
Differential Equations

Ordinary Differential Equations

A real-valued function of a real variable, i.e., $f: \mathrm{R} \to \mathrm{R}$, is usually expressed in the form

$$y = f(x). \qquad \text{...... (1)}$$

If this function is differentiable at a point $x = c \in \mathrm{R}$, we often write

$$\left.\frac{dy}{dx}\right|_{x=c} = f'(c). \qquad \text{...... (2)}$$

If it is differentiable for all values of $x \in \mathrm{R}$, we usually write

$$\frac{dy}{dx} = f'(x) = g(x), \quad \text{say,} \qquad \text{...... (3)}$$

In particular, if we have simple function defined by

$$y = c/x \quad \text{or} \quad xy = c, \qquad \text{...... (4)}$$

where c is a constant, simple differentiation yields

$$\frac{dy}{dx} = -\,c/x^2 \qquad \text{...... (5)}$$

This is an equation involving the first derivative of the function defined by $y = c/x$, the independent variable and the constant $-c$. We may put the equation (5) in the following forms :

...... (6)

a) $dy = (-c/x^2)\, dx$,

in which the variables x and y are separated

b) $\dfrac{dy}{dx} = -\left(\dfrac{c/x}{x}\right) = -\dfrac{y}{x}$, ... (7)

in which x and y appear in the form y/x only,

c) $\dfrac{dy}{dx} + \dfrac{1}{x}.y = 0$, ... (8)

in which the coefficient of y is a function of x alone,

d) $ydx + xdy = 0$, ... (9)

in which the left-hand side could be written as a single (exact) differential $d(xy)$.

Looking superficially at the above different forms of the same equation (5), we may roughly classify them into the following standard forms :

I. Variables separated form :

An equation of the form $Y\,dy = X\,dx$

where Y is a function of y alone and X that of x alone.

II. Homogeneous form :

This is an equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$,

where $f(y/x)$ is a function of $\frac{y}{x}$

III. Linear form :

$$\frac{dy}{dx} + Py = Q,$$

where P and Q are functions of x only.

IV. Exact form :

$$M(x,y)\,dx + N(x,y)\,dy = 0,$$

where M and N are functions of x and y, such that the left hand side of this equation can be expressed as a single (or *perfect* or *exact*) differential of the form d $f(x, y)$, where $f(x, y)$ is a function of x and y.

We thus observe that an equation may contain the differential coefficient (dy/dx), or differentials dx and dy, with or without the dependent variable y or independent variable x or both. In each of the examples considered, the highest derivative of y is 'one'. An equation like this is called an *ordinary differential equation of the first order*. We further notice that the highest power of the highest (here first) derivative is also '*one*'. We call such an equation, *an ordinary* (having only one independent variable x) *differential equation of the first order and first degree.*

The power of the first derivative may be greater than one. For example, the power of the differential equations

$$\left(\frac{dy}{dx}\right)^2 - 3\left(\frac{dy}{dx}\right) + 2 = 0 \qquad \text{or} \qquad y + x\left(\frac{dy}{dx}\right) = x^4\left(\frac{dy}{dx}\right)^2$$

are '*two*'. There are examples of differential equation of the *first order* and *second degree.* Obviously, there exist first order *higher degree* differential equations.

We may likewise consider differential equations in which the order of the highest derivative is more than one. The differential equations

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \qquad \text{and} \qquad \frac{d^2y}{dx^2} + y = \sin 2x$$

are of the *second order* but of the first degree.

A *general second order differential equation* is generally written in the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

P, Q and R are functions of x.

If an equation could be written in the form

$$\frac{dy}{dx} = f(y) \qquad \text{or,} \qquad y' = f(y),$$

where $f(y)$ is a function of y alone, it is called an *autonomous differential equation*. The equation $\frac{dy}{dx} = y$ is autonomous.

We now give some formal definitions which we often need.

a) A *differential equation* is an equation which involves the derivatives or differentials with or without the dependent variable or independent variable or both. For example,

$$\frac{dy}{dx} = -\frac{y}{x} \qquad \text{or} \qquad x\,dy + y\,dx = 0,$$

$$\frac{d^2y}{dx^2} + y = \sin x.$$

b) A differential equation is said to be *ordinary* if there is *no partial derivative*.

c) A differential equation is said to be of *order one* if the order of the highest derivative is one. It is said to be of *order n* if the order of the highest derivative appearing in it is n.

The differential equations $\frac{dy}{dx} = -\frac{y}{x}$ and $\frac{d^2y}{dx^2} + y = \sin x$ are respectively of order one and two.

d) The *degree* of a differential equation is the power to which the highest derivative in it is raised. The degree of the differential equations

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{y}{x}\right)^2 \qquad \text{and} \qquad \frac{d^2y}{dx^2} + y = x^n$$

are two and one respectively.

e) A *solution* of a differential equation is any relation between the variables, which is free from derivatives or differentials and which satisfy the equation identically.

The equation $$\frac{dy}{dx} = -\frac{y}{x} \qquad \text{...... (A)}$$

is satisfied by $$y = \frac{1}{x} \qquad \text{...... (B)}$$

So, it is a solution of (A).

One can easily verify that it is also satisfied by

$$y = \frac{c}{x} \qquad \text{...... (C)}$$

where c is an *arbitrary constant*.

The solution (B) of (A) is called a *particular solution* or *particular integral* (P.I.) of (A). The solution (B) is obtained by substituting $y = 1$ when $x = 1$, hence it is known as the particular solution. The solution (C) of (A) containing one arbitrary constant is called the *general solution* of the differential equation of order one. We, therefore, define a general solution of a differential equation of order n is that solution which contains n arbitrary constants. A general solution is also known as a *complete solution* or *complete primitive*.

We have adopted a simplistic approach in introducing the basic terminology related to a differential equation. An alternative but interesting approach could be to start with mathematical models of problems in real life situations such as growth and decay problems in biological sciences, business and economics, physical sciences, engeneering, etc. We do deal with some such problems as applications only at the end.

Equations of the First Order and First Degree

An ordinary differential equation of the *first order* and *first degree* may be written in the form

$$\frac{dy}{dx} = f(x, y) \qquad \ldots\ldots (1)$$

where $f(x, y)$ is a function of x and y. If

$$f(x, y) = -\frac{M(x,y)}{N(x,y)},$$

where M and N are functions of x and y, we may write (1) in the form

$$M(x, y)\, dx + N(x, y)\, dy = 0, \qquad \ldots\ldots (2)$$

For a general equation like this, there is no good theorem which guarantees a solution which can be expressed as a '*nice formula*' in terms of known functions. We shall, therefore, restrict ourselves to cases in which we need not have to face such difficulties.

a) Standard Form I

Variables Separated form

If the equation $\frac{dy}{dx} = f(x,y)$ could be written in the form

$$\frac{dy}{dx} = \frac{X}{Y} \qquad \ldots\ldots (3)$$

where X is a function of x alone and Y is a function of y alone, it can be written as

$$Y\, dy = X\, dx, \qquad \ldots\ldots (4)$$

in which the variables are separated.

Then, by direct integration, we arrive at the general solution

$$\int Y\, dy = \int X\, dx + c \qquad \ldots\ldots (5)$$

where c is an arbitrary constant of integration.

Worked Out Examples

Example 1.

Solve, by separation of variables, the equation $\frac{dy}{dx} = -\frac{y}{x}$

Solution:

Clearly, $\frac{dy}{y} = -\frac{dx}{x}$

Integrating both sides, we get

$$\int \frac{dy}{y} = -\int \frac{dx}{x} + A,$$ where A is an arbitrary constant.

so, $\log y = -\log x + \log c$

by putting A = log c so as to make the simplification easy and nice

Thus $\log y = \log c/x$

or $y = \frac{c}{x}$

or $xy = c$ is the required solution.

Example 2.

Solve, by separation of variables, the equation $x\,dy + y\,dx = 0$

Solution:

Dividing both sides of the equation by xy, we have

$$\frac{dy}{y} + \frac{dx}{x} = 0,$$ in which the variables are separated.

Integrating, we have

$$\int \frac{dy}{y} + \int \frac{dx}{x} = \log c$$ where c is an arbitrary constant.

So, $\log y + \log x = \log c$

or, $\log xy = \log c$

or, $xy = c$ is the required solution.

Example 3

Solve $(1 + x^2)\, dy = (1 + y^2)\, dx$

Solution:

We rewrite the given equation in the form

$$(1 + x^2)\, dy - (1 + y^2)\, dx = 0$$

or, $$\frac{dy}{1 + y^2} - \frac{dx}{1 + x^2} = 0,$$

we have

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} c$$

or, $\tan^{-1}\dfrac{y-x}{1+xy} = \tan^{-1} c$

or, $\dfrac{y-x}{1+xy} = c,$ as the required solution.

Example 4

Solve : $\dfrac{dy}{dx} = \dfrac{xy+y}{xy+x}$

Solution:

$$\frac{dy}{dx} = \frac{xy+y}{xy+x}$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y(x+1)}{x(y+1)}$

$\Rightarrow \dfrac{y+1}{y}\,dy = \dfrac{x+1}{x}\,dx$

$\Rightarrow \left(1+\dfrac{1}{y}\right)dy = \left(1+\dfrac{1}{x}\right)dx$

Integrating,

$$y + \log y = x + \log x + c$$

$\Rightarrow \log y - \log x + y - x = c$

$\Rightarrow \log\dfrac{y}{x} + y - x = c$

Example 5

Solve : $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Solution:

Dividing both sides by $\tan x \tan y$,

We get $\dfrac{\sec^2 x}{\tan x}dx + \dfrac{\sec^2 y}{\tan y}dy = 0$

Integrate both sides to find

$$\int \frac{\sec^2 x}{\tan x}dx + \int \frac{\sec^2 y}{\tan y}dy = \log c.$$

or, $\log \tan x + \log \tan y = \log c$ $\left[\because \int \dfrac{\sec^2\theta}{\tan\theta}\,d\theta = \int \dfrac{dz}{z} = \log z\right]$

So, $\tan x \tan y = c,$ where c is an arbitrary constant, is the required solution

Example 6

Solve : $e^{x-y}\,dx + e^{y-x}\,dy = 0$

Solution:

Multiplying both sides by e^{x+y}

We get

$$e^{2x}\,dx + e^{2y}\,dy = 0$$

Integrating, we have

$$\int e^{2x}\,dx + \int e^{2y}\,dy = \frac{c}{2}\,(c = \text{constants})$$

Integrating, we have

or $\quad \dfrac{e^{2x}}{2} + \dfrac{e^{2y}}{2} = \dfrac{c}{2}$

i.e. $\quad e^{2x} + e^{2y} = c \quad$ is the required solution.

EXERCISE

Solve, by separation of variables, the following differential equations.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$
2. $x\,dx + y\,dy = 0$
3. $\dfrac{dy}{dx} = \dfrac{x^2+1}{y^2+1}$
4. $x^2\,dy - y^2\,dx = 0$
5. $(1+x^2)\dfrac{dy}{dx} = 1$
6. $\dfrac{dy}{dx} = \dfrac{e^x+1}{y}$
7. $\dfrac{dy}{dx} + 4x = 2.e^{2x}$
8. $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0$
9. $(1+x)\,y\,dx + (1+y)x\,dy = 0$
10. $(xy^2+x)\,dx + (yx^2+y)\,dy = 0$
11. $x\dfrac{dy}{dx} + y - 1 = 0$
12. $\dfrac{dy}{dx} = e^{x-y} + x^3.e^{-y}$
13. $\tan x\,dy + \tan y\,dx = 0$
14. $\dfrac{dy}{dx} = -\dfrac{1+\cos 2y}{1-\cos 2x}$
15. $x\dfrac{dy}{dx} + y = 2$

Answer

1. $x^2 - y^2 = c$
2. $x^2 + y^2 = c$
3. $\dfrac{x^3}{3} + x = \dfrac{y^3}{3} + y + c$
4. $x - y = cxy$
5. $y = \tan^{-1} x + c$
6. $y^2 = 2e^x + 2x + c$

7. $y = e^{2x} - 2x^2 + c$
8. $x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$
9. $\log xy + x + y = c$
10. $(x^2+1)(y^2+1) = c$
11. $x(y-1) = c$
12. $e^y = e^x + \frac{1}{4}x^4 + c$
13. $\sin x \sin y = c$
14. $\cot x - \tan y = c$
15. $x(y-2) = c$

b) Standard Form II

Homogenous Equations

A differential equation of the first order and first degree is said to be *homogeneous* if it can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

A simple test for this is to put $\frac{y}{x} = v$ or $y = vx$, and see whether all x's cancel out or not. If so, it is homogeneous, otherwise not.

To solve such an equation, we put $y = vx$ and

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

in the given equation then we have a differential equation with variables v and x, i.e. we get

$$v + x\frac{dv}{dx} = f(v)$$

or, $$\frac{dv}{f(v) - v} = \frac{dx}{x}$$

in which the variable are separated. Now by integration we get the required solution.

Worked Out Examples

Example 1

Solve: $\frac{dy}{dx} = \frac{y+x}{x}$

Solution:

The given equation is obviously homogeneous since $\frac{dy}{dx} = \frac{y}{x} + 1$

Put $y = vx$, so that

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

and $$v + x\frac{dv}{dx} = v + 1$$

or, $$x\frac{dv}{dx} = 1$$

or, $$dv = \frac{dx}{x} \quad \text{(Variables separated)}$$

Integrating, we get

$$v = \log x + \log c$$

where c is an arbitrary constant. Rewriting $v = \frac{y}{x}$, and simplifying, we get

$$y = x \log c\, x$$

as the required solution

Example 2

Solve : $$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Solution:

The given equation is homogeneous, since

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{y}{x}\right)^{-1}$$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

to find

$$v + x\frac{dv}{dx} = \frac{1}{2}\left[v - \frac{1}{v}\right]$$

or, $$x\frac{dv}{dx} = -\frac{1}{2}\left[v + \frac{1}{v}\right]$$

or, $$\frac{2v}{v^2+1}\,dv + \frac{dx}{x} = 0$$

Since $d\,(v^2 + 1) = 2\,v\,dv$, we have

$$\frac{d\,(v^2+1)}{v^2+1} + \frac{dx}{x} = 0$$

Integrating, we get

$$\log (v^2 + 1) + \log x = \log c,$$ where c is an arbitrary constant.

So, $$(v^2 + 1)\,x = c$$

Rewriting $v = \frac{y}{x}$, we get

$$\left(\frac{y^2}{x^2} + 1\right)x = c$$

or, $$(x^2 + y^2) = c\,x$$ as the required solution

Example 3

Solve : $\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$

Solution:

Put $\frac{y}{x} = v$ so that $y = vx$

then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, the given equation is

$$v + x\frac{dv}{dx} = v - \sin^2 v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\sin^2 v$$

$$\Rightarrow \quad -\frac{dv}{\sin^2 v} = \frac{dx}{x}$$

Integrating, we have

$$\cot v = \log x + c$$

$$\Rightarrow \quad \cot\left(\frac{y}{x}\right) = \log x + c$$

Example 4

Solve $\frac{dy}{dx} = \frac{y+1}{y+1+x}$

Solution:

This equation is not homogeneous. Putting $y + 1 = z$,

we have $\frac{dy}{dx} = \frac{dz}{dx}$ and $\frac{dz}{dx} = \frac{z}{z+x} = \frac{\left(\frac{z}{x}\right)}{\left(\frac{z}{x}\right)+1}$ which is homogeneous.

Put $\frac{z}{x} = v$ or $z = vx$,

so that $\frac{dz}{dx} = v + x\frac{dv}{dx}$

and $v + x\frac{dv}{dx} = \frac{v}{v+1}$

or, $x\frac{dv}{dx} = \frac{v}{v+1} - v = \frac{-v^2}{v+1}$

or, $\frac{v+1}{v^2}dv = -\frac{dx}{x}$

or, $\frac{dv}{v}+\frac{dv}{v^2}=-\frac{dx}{x}$

Integrating term by term, we get

$$\log v - v^{-1} = -\log x + \log c$$

$$\log v + \log x - \log c = \frac{1}{v}$$

or, $\log \frac{vx}{c} = \frac{1}{v}$

or, $\frac{vx}{c} = e^{1/v}$

or, $\frac{z}{c} = e^{x/z}$

$$y+1 = c\, e^{\frac{x}{y+1}}$$

as the required solution.

EXERCISE

Solve the following equations

1. $\frac{dy}{dx}=\frac{y-2x}{x}$
2. $xy\frac{dy}{dx}=x^2+y^2$
3. $x^2\frac{dy}{dx}+y^2=xy$
4. $\frac{dy}{dx}=\frac{y}{x}+\tan\frac{y}{x}$
5. $x(x-y)\,dy=y(x+y)\,dx$
6. $(x^2-y^2)\frac{dy}{dx}=xy$
7. $(x^2+y^2)\,dy=xy\,dx$
8. $\frac{dy}{dx}=\frac{x^2+y^2}{2x^2}$
9. $x\,dy-y\,dx=\sqrt{x^2+y^2}\,dx$
10. $2\frac{dy}{dx}=\frac{y}{x}+\frac{y^2}{x^2}$
11. $\frac{dy}{dx}=\frac{x-y}{x+y}$, $y=0$ when $x=2$
12. $(x^3+y^3)dy-x^2y\,dx=0, f(0)=1$

Answer

1. $x+2x\log x=cx$
2. $y^2=2x^2(\log x+c)$
3. $x=y(\log x+c)$
4. $\sin\left(\frac{y}{x}\right)=cx$
5. $x+y\log cxy=0$
6. $x^2+2y^2\log cy=0$
7. $x^2=2y^2\log cy$
8. $2x=(x-y)\log cx$
9. $y+\sqrt{x^2+y^2}=cx^2$
10. $(y-x)^2=cxy^2$
11. $x^2-2xy-y^2=4$
12. $x^3=3y^3\log y$

c) Standard form III

Exact Equations

A differential equation written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

where M and N are functions of x or y or both, is said to be *exact* if there exists a function $f(x,y)$ of x and y such that

$$M\,dx + N\,dy = d f(x,y),$$

i.e. when $M\,dx + N\,dy$ is an exact or a perfect differential.

The differential equation $y\,dx + x\,dy = 0$

is exact, since $y\,dx + x\,dy = d(xy) = 0$ which gives

$$xy = c$$

where c is an arbitrary constant. But, the differential equation

$$x\,dy - y\,dx = 0$$

is *not exact* as it stands.

It, however, becomes exact if we *multiply* both sides of it by $\frac{1}{x^2}$, since

$$\frac{x\,dy - y\,dx}{x^2} = 0 \quad \text{becomes} \quad d\left(\frac{y}{x}\right) = 0$$

On integration, we have

$$\frac{y}{x} = c \quad \text{or} \quad y = cx,$$

where c is an arbitrary constant as its solution.

An expression or factor such as $\frac{1}{x^2}$ is called an *integrating factor* (**I.F.**). Integrating factors may be found in several ways. But we shall focus our interest mainly on those cases in which I. F. can be found by simple observations or inspection. One should not forget that an equation may be exact by simply regrouping various terms of the equation. For example the equation

$$(5x + 2y - 4)\,dy + (12x + 5y - 9)\,dx = 0$$

may be regrouped as

$$5(x\,dy + y\,dx) + (2y - 4)\,dy + (12x - 9)\,dx = 0$$

which is equivalent to

$$d(5xy) + d(y^2) - d(4y) + d(6x^2) - d(9x) = 0$$

or, $d[5xy + y^2 - 4y + 6x^2 - 9x] = 0$ which is exact.

On integration, we get the required solution

$$5xy + y^2 - 4y + 6x^2 - 9x = c,$$

where c is an arbitrary constant.

Worked Out Examples

Example 1

Solve $y\,dx - x\,dy = 0$

Solution:

Dividing both sides of the given equation by y^2, we have

$$\frac{y\,dx - x\,dy}{y^2} = 0 \quad \text{or,} \quad d\left(\frac{x}{y}\right) = 0$$

This gives, on integration,

$$\frac{x}{y} = c \quad \text{or} \quad x = cy,$$

where c is an arbitrary constant, is the required solution.

Example 2

Solve $x\,dy + (x + y)\,dx = 0.$

Solution:

Regrouping the terms of the given equation we have

$$(x\,dy + y\,dx) + x\,dx = 0.$$

Equivalently, we have

$$d\,(xy) + d\left(\frac{1}{2}x^2\right) = 0$$

or, $$d\,(x\,y + \frac{1}{2}x^2) = 0$$

On integration, we have

$$xy + \frac{1}{2}x^2 = c$$ where c is an arbitrary constant, as the required solution.

Example 3

Solve : $2xy\,dx - x^2\,dy = 0$

Solution:

$2xy\,dx - x^2\,dy = 0$

Dividing both sides by y^2

$$\frac{2xy\,dx - x^2\,dy}{y^2} = 0 \Rightarrow d\left(\frac{x^2}{y}\right) = 0$$

Integrating, $\frac{x^2}{y} = c$

$\therefore \quad x^2 = cy$

Example 4

Solve: $y(1+xy)\,dx - x\,dy = 0$

Solution:

$$y(1+xy)dx - x\,dy = 0$$

$$\Rightarrow\ y\,dx + xy^2\,dx - x\,dy = 0$$

$$\Rightarrow\ (y\,dx - x\,dy) + xy^2\,dx = 0$$

$$\Rightarrow\ \frac{y\,dx - x\,dy}{y^2} + x\,dx = 0$$

$$\Rightarrow\ d\left(\frac{x}{y}\right) + \frac{1}{2}d(x^2) = 0$$

$$\Rightarrow\ d\left(\frac{x}{y} + \frac{1}{2}x^2\right) = 0$$

Integrating, $\frac{x}{y} + \frac{1}{2}x^2 = c$

Example 5

Solve : $\frac{dy}{dx} = \frac{x-y+1}{x+y+1}$

Solution:

$$\frac{dy}{dx} = \frac{x-y+1}{x+y+1}$$

$$\Rightarrow\ x\,dy + y\,dy + dy = x\,dx - y\,dx + dx$$

$$\Rightarrow\ (x\,dy + y\,dx) + y\,dy + dy - x\,dx - dx = 0$$

$$\Rightarrow\ d(xy) + \frac{1}{2}d(y^2) + dy - \frac{1}{2}d(x^2) - dx = 0$$

$$\Rightarrow\ d\left(xy + \frac{1}{2}y^2 + y - \frac{1}{2}x^2 - x\right) = 0$$

Integrating,

$$xy + \frac{1}{2}y^2 + y - \frac{1}{2}x^2 - x = c$$

$$\Rightarrow\ 2xy + y^2 + 2y - x^2 - 2x = 2c$$

$$\Rightarrow\ 2xy + y^2 + 2y - x^2 - 2x = c' \text{ where } c' = 2c$$

EXERCISE

Solve, by reducing to exact form, the following equations

1. $y\,dx - x\,dy = xy\,dy$
2. $2\,xy\,dx + x^2\,dy = 0$

3. $2xy\,dy - y^2\,dx = 0$
4. $(x + y)\,dy + (y - x)\,dx = 0$
5. $x\,dy + (x + 1)\,y\,dx = 0$
6. $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$
7. $\sin x \cos x\,dx + \sin y \cos y\,dy = 0$
8. $\frac{dy}{dx} = \frac{\cos^2 y}{\sin^2 x}$
9. $(x + 2y - 3)\,dy - (2x - y + 1)\,dx = 0$
10. $\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$
11. $(x^2 + xy^2)\,dx + (x^2y + y^2)\,dy = 0$
12. $\frac{dy}{dx} = \frac{2x + 3y + 5}{2y - 3x + 6}$
13. $2xy^3\,dx - 3x^2y^2\,dy = 0$

Answer

1. $\log\frac{x}{y} = y + c$
2. $x^2y = c$
3. $y^2 = cx$
4. $2xy + y^2 - x^2 = c$
5. $x + \log xy = c$
6. $x^3 - 3axy + y^3 = c$
7. $\sin^2 x + \sin^2 y = 2y + c$
8. $\cot x + \tan y = c$
9. $xy + y^2 - x^2 - 3y - x = c$
10. $x^2 + y^2 - 2xy + 10y - 2x = c$
11. $2x^3 + 2y^2 + 3x^2y^2 = c$
12. $x^2 - y^2 + 3xy + 5x - 6y = c$
13. $x^2 = c\,y^3$

d) Standard form IV

Linear Equations

A first order and first degree differential equation is said to be a *linear equation* if it can be written in the form

$$\frac{dy}{dx} + Py = Q \qquad \ldots\ldots\ldots \text{(i)}$$

where P and Q are functions of x or constants (but not of y).

For example, the equation $\frac{dy}{dx} + xy = x^2$ is linear.

The linear equation

$$\frac{dy}{dx} + Py = Q$$

becomes easily integrable if both sides of it is multiplied by $e^{\int P\,dx}$

This quantity or factor is called the *Integrating Factor* (I.F.) of the linear equation.

Sometimes, we may come across a differential equation of the form

$$\frac{dy}{dx} + Py = Q\,y^2 \qquad \ldots\ldots\ldots \text{(ii)}$$

The equation (ii) can be changed into the form

$$y^{-2}\frac{dy}{dx} + Py^{-1} = Q$$

Substituting $y^{-1} = z$ and $y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$ in (ii)

The equation (ii) reduces to

$$\frac{dz}{dx} - Pz = -Q$$

which is the linear differential equation of the form (i).

Worked Out Examples

Example 1

Solve: $\frac{dy}{dx} + \frac{1}{x}y = x^2$ given that $y = 1$ when $x = 1$.

Solution:

Here $P = \frac{1}{x}$, $Q = x^2$ and so

$$\int P\,dx = \int \frac{1}{x}\,dx = \log x$$

and, $\quad$ I. F. $= e^{\log x} = x$

Multiplying both sides of the given equation by x, we have

$$x\frac{dy}{dx} + y = x^3$$

or, $\quad x\,dy + y\,dx = x^3\,dx$

or, $\quad d(xy) = x^3\,dx$

Integrating , we have

$$xy = \frac{x^4}{4} + c,$$

where c is an arbitrary constant, as the required solution.

But when $x = 1, y = 1$

$$1 = \frac{1}{4} + c$$

$$\therefore \quad c = \frac{3}{4}$$

Now, the solution is

$$xy = \frac{x^4}{4} + \frac{3}{4}$$

$$\Rightarrow \quad 4xy = x^4 + 3$$

Example 2

Solve: $\sin x \frac{dy}{dx} + \cos x\, y = x \sin x$

Solution:

Dividing both sides by $\sin x$, we have

$$\frac{dy}{dx} + \frac{\cos x}{\sin x} y = x, \qquad \text{which is linear.}$$

Here $P = \frac{\cos x}{\sin x}$, and so ,if $u = \sin x$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \log u.$$

So, I. F. $= e^{\int P\, dx} = e^{\log u} = u = \sin x.$

Multiplying both sides by the I. F., we get

$$\sin x \frac{dy}{dx} + \cos x . y = x \sin x$$

or, $\sin x\, dy + y \cos x .\, dx = x \sin x\, dx$

or, $d\,(y \sin x) = x \sin x\, dx$

Integrating, we get

$$y \sin x = x \int \sin x\, dx - \int \frac{dx}{dx} \left(\int \sin x\, dx \right) dx + c,$$

where c is an arbitrary constant.

So, the required solution is $y \sin x = -x \cos x + \sin x + c.$

Example 3

Solve: $\cos^2 x \frac{dy}{dx} + y = 1.$

Solution:

Dividing both sides by $\cos^2 x$, we have

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \sec^2 x$$

So, I.F. $= e^{\int P\, dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Multiplying both sides by $e^{\tan x}$, we get

$$e^{\tan x} \frac{dy}{dx} + y\, e^{\tan x} \sec^2 x = e^{\tan x} \sec^2 x$$

or, $$\frac{d\, e^{\tan x} y}{dx} = e^{\tan x} \sec^2 x$$

or, $$d\, e^{\tan x} y = e^{\tan x} \sec^2 x\, dx$$

Integrating, we have

$$e^{\tan x}y = \int d\,(e^{\tan x}) + c$$

$$e^{\tan x}\,y = e^{\tan x} + c$$

or, $\quad y = 1 + c\,e^{-\tan x}$

as the required solution.

Example 4

Solve : $(1 + x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1}x}$

Solution:

$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{1 + x^2}\cdot y = \frac{e^{\tan^{-1}x}}{1 + x^2}$$

$$\text{I.F.} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

Multiplying both sides by $e^{\tan^{-1}x}$, we have

$$e^{\tan^{-1}x}\frac{dy}{dx} + \frac{e^{\tan^{-1}x}}{1 + x^2}\cdot y = \frac{\left(e^{\tan^{-1}x}\right)^2}{1 + x^2}$$

$$\Rightarrow \quad d\,(e^{\tan^{-1}x}.y) = \frac{\left(e^{\tan^{-1}x}\right)^2}{1 + x^2}\,dx$$

Integrating,

$$e^{\tan^{-1}x}.\,y = \int \frac{(e^{\tan^{-1}x})^2}{1 + x^2}\,dx$$

Put $\tan^{-1}x = u \Rightarrow \dfrac{1}{1 + x^2}\,dx = du$

$$\int \frac{(e^{\tan^{-1}x})^2}{1 + x^2}\,dx = \int e^{2u}\,du$$

$$= \frac{1}{2}e^{2u} + c$$

$$= \frac{1}{2}(e^{\tan^{-1}x})^2 + c$$

$$\therefore \quad e^{\tan^{-1}x}.y = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$$

$$y = \frac{1}{2}e^{\tan^{-1}x} + c.e^{-\tan^{-1}x}$$

Example 5

Reduce the equation $\frac{dy}{dx}+\frac{1}{x}y=y^2$ to linear form and solve.

Solution:

Dividing each term of the equation by y^2, we have

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=1$$

Put $y^{-1}=z$, then $\quad -y^{-2}\frac{dy}{dx}=\frac{dz}{dx}$

Using this substitution in the equation, we get

or, $$\frac{dz}{dx}-\frac{1}{x}z=-1$$

This is linear. Here

$$\text{I. F.}=e^{\int \frac{1}{x}dx}=e^{-\log x}=e^{\log x^{-1}}=\frac{1}{x}$$

Multiplying the above equation by $\frac{1}{x}$ we have

$$\frac{1}{x}\frac{dz}{dx}-\frac{1}{x^2}.z=-\frac{1}{x}$$

$$\Rightarrow \quad \frac{x\,dz-z\,dx}{x^2}=-\frac{dx}{x}$$

or, $$d\left(\frac{z}{x}\right)=-\frac{dx}{x}$$

Integrating, we have

$$\frac{z}{x}=-\log x+c, \qquad \text{where } c \text{ is arbitrary constant.}$$

So, the required solution is

$$\frac{z}{x}=(-\log x+c) \qquad \text{or,} \qquad \frac{1}{xy}=(-\log x+c)$$

EXERCISE

Solve the following equations

1. $\frac{dy}{dx}+3y=e^{-x}$
2. $\frac{dy}{dx}+\frac{3}{x}y=9x^5$
3. $x\frac{dy}{dx}-y=2x^3$
4. $(1+x^2)\frac{dy}{dx}+2xy=4x^2$

5. $(x+1)\dfrac{dy}{dx} + 2y = \dfrac{e^x}{x+1}$

6. $\sin x \dfrac{dy}{dx} + \cos x \,.\, y = \sin x \cos x$

7. $\tan x \dfrac{dy}{dx} + y = \sec x$

8. $\dfrac{dy}{dx} + 2 \tan x \, y = \sin x$

9. $(1 - x^2)\dfrac{dy}{dx} - xy = 1$

10. $\dfrac{dy}{dx} + \dfrac{2x}{1+x^2} y = \dfrac{1}{(1+x^2)^2}$

11. $\dfrac{dy}{dx} - 2xy = x$

12. $\dfrac{dy}{dx} + \dfrac{y}{x^2} = \dfrac{1}{x^2}$

13. $(1 + x)\dfrac{dy}{dx} - xy = 1 - x$

14. $x\dfrac{dy}{dx} + 2y = x^2 \log x$

15. $\dfrac{dy}{dx} + y = xy^2$

16. $\dfrac{dy}{dx} + y \tan x = y^3 \sec x$

Answer

1. $y = \dfrac{1}{2} e^{-x} + c\, e^{-3x}$
2. $y = x^6 + \dfrac{c}{x^3}$
3. $y = x^3 + cx$
4. $(1 + x^2).y = \dfrac{4}{3}x^3 + c$
5. $(x+1)^2.y = e^x + c$
6. $y \sin x + \dfrac{1}{4}\cos 2x = c$
7. $y \sin x = x + c$
8. $y = \cos x + c \cos^2 x$
9. $y\sqrt{1 - x^2} = \sin^{-1}x + c$
10. $(1 + x^2)y = \tan^{-1}x + c$
11. $y = -1/2 + c\, e^{x^2}$
12. $y = 1 + ce^{1/x}$
13. $(1 + x)y = x + ce^x$
14. $y = \dfrac{1}{4}x^2 \log x - \dfrac{1}{16}x^2 + \dfrac{c}{x^2}$
15. $xy + y + cye^x = 1$
16. $(c - 2 \sin x)y^2 = \cos^2 x$

Chapter 18
Computational Methods

Introduction : Linear Programming

Various types of problems may have to face in business and economic activities. These problems happen because of the limited resources. Here we are concerned with the objective of getting maximum profit with limited investment or maximum production with limited resources etc. Such type of problems are referred to the problems of optimization. These types of problems are tackled by the mathematical method known as the linear programming. Thus linear programming is the mathematical method of getting the optimal solution of the desired objective under certain conditions. The linear programming problem is abbreviated by LP problem or LPP.

Mathematical Model of Linear Programming Problem

The purpose of the LP problem is to maximize or minimize the objective function for which it is formed. So, mathematically, the LP problem is stated in the following way.

Optimize $z = ax_1 + bx_2$

Satisfying the condition

$$\left.\begin{aligned} a_1x_1 + b_1x_2 \ (\leq \text{ or } \geq)\ c_1 \\ a_2x_1 + b_2x_2 \ (\leq \text{ or } \geq)\ c_2 \end{aligned}\right\} \quad \ldots\ldots(i)$$

and $\quad x_1, x_2 \geq 0$

The function $z = ax_1 + bx_2$ which is to be maximized or minimized is known as the objective function. The conditions (i) which the objective function has to satisfy are known as the **constraints**. The variables x_1 and x_2 are known as the **decision variables**. $x_1 \geq 0$, $x_2 \geq 0$ are the conditions known as non-negative condition.

The values of x_1 and x_2 satisfying the constraints are known as the solution of the LP problem. This solution known as the feasible solution is known as the optimal solution if it makes the objective function maximum.

Standard form of a LP problem

The LP problem may be of maximizing or minimizing the problem according as the objective function is to maximize or minimize respectively. The standard form of both types of problems are given below.

A standard maximizing LP problem is LP problem of maximizing its objective function. All the given constraints are in the form ≤ and the decision variables are non-negative.

For example: Maximize $z = 3x + 5y$ subject to (s.t.)

$$3x + 2y \leq 10$$

$$2x + 3y \leq 12$$

$$x, y \geq 0$$

A standard minimizing LP problem is the LP problem of minimizing its objective function. All the given constraints will be in the form ≥ and the decision variables are non-negative.

For example: Minimize $F = 12x + 15y$ s.t.

$$5x + 8y \geq 20$$

$$3x + 7y \geq 15$$

$$x, y \geq 0$$

Simplex Method

The LP problem consisting of two decision variables can be solved even by graphical method but when the number of decision variables increases, it will not be convinient to to solve by graphical method. Then for such a problem, there is another method of solving the LP problem known as the **simplex method**. The method can be used for two or more decision variables. This is more effective and most commonly used iteration method to get the optimal solution of the LP problem.

Initially this method gives the zero value of the objective function and in each of its iteration, the value of the objective function changes upto the value where no iteration is needed.

For the use of simplex method, we need the following terms.

Slack variable:

If the LP problem has the constraint of the form $ax + by \leq c$, then the non-negative variable by the addition of which on the left hand side of the inequality, changes the inequality into equality, is known as the **slack variable**.

Let the inequality be $3x + 2y \leq 6$.

If r, the non-negative variable makes $3x - 2y + r = 6$ then r is known as the slack variable.

Surplus variable:

If the LP problem has the constraint of the form $ax + by \geq c$, then the non-negative variable by the subtraction of which on the left side of the inequality, changes the inequality into equality form, is known as the **surplus variable**.

Let the inequality be $5x + 8y \geq 20$

If s be the non-negative variable, then $5x + 8y - s = 20$. Here s is known as the **surplus variable**.

Basic and Non-basic variable:

The decision variables are known as basic variables when their values are non-zero. But the decision variables having zero values are known as the non-basic variables.

Basic feasible solution:

A feasible solution to the system ax + by = b is known as the basic feasible solution when all the basic variables are non-negative.

A table in which the coefficients of all the variables of the equations formed from the given constraints and that of objective function are presented is known as the simplex tableau.

For example:

Max Z = 4x + 5y subject to

$x + y \le 4$

$3x + 4y \le 10$

$x \ge 0, y \ge 0$

Let r and s be the non-negative slack variables. Then

$x + y + r = 4$

$3x + 4y + s = 10$

$\Rightarrow$ $x + y + r + 0.s + 0.Z = 4$

$3x + y + 0.r + s + 0.Z = 10$

$-4x - 5y + 0.r + 0.s + Z = 0$

Now, presenting the coefficients of the equations, we have the following table

Basic variables	x	y	r	s	F	RHS
r	1	1	1	0	0	4
s	3	4	0	1	0	10
	–4	–5	0	0	1	0

This presentation is the initial simplex tableau in the solution of LP problem by simplex method.

Method of solving LP problem by Simplex method

The steps to be used in solving the LP problem by simplex method are as follows:

a) Formulate the given LP problem if necessary.

b) Change all constraints that are in inequality form into equality form by the addition of slack or surplus variable as the case may be.

c) Initially the decision variables in the constraints will be zero whereas the slack or surplus variable are not.

d) Express the given LP problem in the standard form known as the **standard form**.

e) Express all the coefficients of the variables involved in the LP problem in the initial simplex tableau as shown above.

f) **Optimality Test:**

For the test of the optimality of the objective function, we see the bottom row of the simplex tableau. If all the elements (entries) of the bottom row are either zero or positive then the value of the objective function is optimal. No more simplex tableau is to be formed.

But if any of the entries be negative, then the value of the objective function is not optimal. In such a case, we pass on to the another improved simplex tableau to get the optimal value of the objective function.

g) If improved or revised simplex tableau is to be formed, first we find one entering variable and one outgoing variable. The column which contains the most negative value at the bottom row is known as the pivot column. The example presented in the above initial simplex tableau, the most negative value is –5 which is the column containing the variable y. So, y-column is the pivot column and y is the entering variable.

Next is to find outgoing variable. To find the outgoing variable, we divide the number in the column containing RHS by the corresponding number in the pivot column

i.e. $\frac{4}{1} = 4$ and $\frac{10}{4} = 2.5$

Since 2.5 < 4, so the row containing the least positive ratio i.e. row containing the ratio 2.5 is known as pivot row.

Now the entry (i.e. element) in the intersection of the pivot row and pivot column is the outgoing variable i.e the element is 4 which lies in the row containing the basic variable 's'. So, the basic variable s is the outgoing variable and 's' is to be replaced by y. Here 4 is said to be the pivot element.

Basic variables	x	y	r	s	Z	RHS	Ratio
r	1	1	1	0	0	4	$\frac{4}{1} = 4$
s	3	4	0	1	0	10	$\frac{10}{4} = 2.5$
	–4	–5	0	0	1	0	

g) Change the pivot element 4 into 1 and use row operation to make the rest elements of the pivot column zero.

h) Now, if there is no more negative entry in the last row, then no next improved or revised simplex tableau is to be formed and get the optimized value of Z which will be in the last entry of the Z-column.

i) If still there remains negative entry, repeat the above process until all the elements in the last row are zero or positive. After getting all the entries in the last row non-negative, the last entry in the Z column gives the optimal value of Z.

This can also be checked by putting the values of the decision variables x and y in the objective function. The above method is generally for the solution of maximizing LP problem.

Worked Out Examples

Example 1.

Find all possible basic solutions of the following system of equations:

$$3x_1 + 2x_2 + 2x_3 = 12$$

$$x_1 + 2x_2 + x_3 = 10$$

Also, select the basic feasible solution.

Solution:

As the given system of equation has only two equations but 3 variables so there will be only two basic variables and the rest is non-basic.

Case I: When $x_3 = 0$; we have

$$3x_1 + 2x_2 = 12$$

$$\underset{-}{}x_1 \underset{-}{+} 2x_2 = \underset{-}{}10$$

Subtracting $\quad 2x_1 = 2 \qquad \therefore x_1 = 1$

Substituting the value of x_1, we have

$$3 \times 1 + 2x_2 = 12$$

$$\Rightarrow \quad 2x_2 = 9 \qquad \therefore x_2 = \frac{9}{2}$$

$\therefore x_1 = 1,\ x_2 = \frac{9}{2}$ (basic variables) and $x_3 = 0$ (non-basic)

Case II: When $x_2 = 0$, we have

$$3x_1 + 2x_3 = 12$$

$$x_1 + x_3 = 10$$

Multiplying 2nd equation by 3 and then subtracting from 1st equation,

$$-x_3 = -18 \qquad \therefore x_3 = 18$$

Substituting the value of x_3 in 2nd equation

$$x_1 + 18 = 10 \qquad \therefore x_1 = -8$$

$\therefore x_1 = -8,\quad x_3 = 18$ (basic); $\quad x_2 = 0$ (non-basic)

Case III: When $x_1 = 0$, we have

$$2x_2 + 2x_3 = 12$$

$$2x_2 + x_3 = 10$$

Subtracting, $x_3 = 2$ $\therefore x_3 = 2, x_2 = 4$

$\therefore x_2 = 4,$ $x_3 = 2$ (basic); $x_1 = 0$ (non-basic)

Case I and III give basic feasible solution because the basic variables in these cases are non-negative.

Example 2

Reformulate the following LP problem into its standard form:

Maximize $Z = 3x_1 + 4x_2$ s.t.

$$6x_1 + 4x_2 \le 60$$
$$x_1 + 2x_2 \le 22$$
$$x_1, x_2 \ge 0$$

Solution:

Let r and s be the non-negative slack variables, then

$$6x_1 + 4x_2 + r = 60$$
$$x_1 + 2x_2 + s = 22$$

Now, the reformulation of the given LP problem in its standard form (i.e. canonical form) is

Max. $Z = 3x_1 + 4x_2 + 0.r + 0.s$ subject to

$$6x_1 + 4x_2 + 1.r + 0.s = 60$$
$$x_1 + 2x_2 + 0.r + 1.s = 22$$
$$x_1, x_2, r, s \ge 0$$

Example 3

Using simplex method, solve the following LP problem

Maximize $Z = 5x - 3y$ subject to

$$3x + 2y \le 6$$
$$-x + 3y \ge -4$$
$$x, y \ge 0$$

Solution:

$5x - 3y = Z$

The given constraints are

$$\left.\begin{aligned} 3x + 2y &\le 6 \\ -x + 3y &\ge -4 \\ x, y &\ge 0 \end{aligned}\right\} \Rightarrow \begin{aligned} 3x + 2y &\le 6 \\ x - 3y &\le 4 \\ x, y &\ge 0 \end{aligned}$$

(Making the constant term positive)

Now, the given LP problem in its standard form is

$3x + 2y + 1.r + 0.s + 0.Z = 6$

$x - 3y + 0.r + 1.s + 0.Z = 4$

$-5x + 3y + 0.r + 0.s + 1.Z = 0$

The initial simplex tableau with the coefficients of the objective function in the last row is

Basic variables	x	y	r	s	Z	RHS	Ratio
r	[3]	2	1	0	0	6	$\frac{6}{3} = 2$
s	1	−3	0	1	0	4	$\frac{4}{1} = 4$
	−5 ↑	3	0	0	1	0	

Since the last row contains the negative entry, so Z = 0 is not the optimal solution.

The most negative entry is −5, so x-column is the pivot column. Also since $\frac{6}{3} = 2$, $\frac{4}{1} = 4$ and $2 < 4$, so r-row is the pivot row. Thus, 3, the entry which is the intersection of the x-column and r-row, is the pivot element.

Operating $\frac{1}{3}R_1$

Basic variables	x	y	r	s	Z	RHS
x	[1]	$\frac{2}{3}$	$\frac{1}{3}$	0	0	2
s	1	−3	0	1	0	4
	−5	3	0	0	1	0

Again operating $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 5R_1$, we have

Basic variables	x	y	r	s	Z	RHS
x	1	$\frac{2}{3}$	$\frac{1}{3}$	0	0	2
s	0	$-\frac{11}{3}$	$-\frac{1}{3}$	1	0	2
	0	$\frac{19}{3}$	$\frac{5}{3}$	0	1	10

Since all entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max Z = 10 when x = 2, y = 0.

The maximum value of Z can also be checked by substituting the values of x and y.

i.e. Max. $Z = 5x - 3y = 5 \times 2 - 3 \times 0 = 10$

Example 4

Solve the following LP problem, using simplex method

Maximize $P = 30x + 20y$ subject to

$$2x + y \le 24$$
$$x + 2y \le 15$$
$$x, y \ge 0$$

Solution:

Introducing the non-negative slack variables r and s, we have

$$2x + y + r = 24$$
$$x + 2y + s = 15$$

Now, the given LP problem in the standard form is

$$2x + y + 1.r + 0.s + 0.P = 24$$
$$x + 2y + 0.r + 1.s + 0.P = 15$$
$$-30x - 20y + 0.r + 0.s + 1.P = 0$$

These equations in **initial simplex tableau** are as follows

Basic variables	x	y	r	s	P	RHS	Ratio
r	[2]	1	1	0	0	24	$\frac{24}{2} = 12$
s	1	2	0	1	0	15	$\frac{15}{1} = 15$
	−30 ↑	−20	0	0	1	0	

Since the last row contains the negative entries, so P = 0 is not the optimal solution.

The most negative entry is −30, so x-column is the pivot column i.e. x is the entering variable.

Also, since $\frac{24}{2} = 12$, $\frac{15}{1} = 15$ and 12 < 15, so r-row is the pivot row i.e. r is the outgoing variable row i.e. r is the outgoing variable and hence the entry 2 is the pivot element.

Operating $\frac{1}{2} R_1$

Basic variables	x	y	r	s	P	RHS
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	12
s	1	2	0	1	0	15
	–30 ↑	–20	0	0	1	0

Performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 30R_1$

Basic variables	x	y	r	s	P	RHS	Ratio
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	12	$\frac{12}{1/2} = 24$
s	0	$\boxed{\frac{3}{2}}$	$-\frac{1}{2}$	1	0	3	$\frac{3}{3/2} = 2$
	0	–5	15	0	1	360	

Since the last row still contains the negative entry, so the solution is not optimal. The negative entry is –5, so y-column is the pivot column. Since $\frac{12}{1/2} = 24$, $\frac{3}{3/2} = 2$ and $2 < 24$, so s-row is the pivot row. Hence $\frac{3}{2}$ is the pivot element.

Peforming $R_2 \to \frac{2}{3} R_2$, we have

Basic variables	x	y	r	s	P	RHS
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	12
y	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	2
	0	–5	15	0	1	360

Again performing $R_1 \to R_1 - \frac{1}{2} R_2$, $R_3 \to R_3 + 5R_2$

Basic variables	x	y	r	s	P	RHS
x	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	11
y	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	2
	0	0	$\frac{40}{3}$	$\frac{10}{3}$	1	370

Since all the entries in the last row are non-negative, so the solution is optimal.

$\therefore$ max. P = 370, when x = 11, y = 2

The maximum value of P can also be checked by substituting the values of x and y.

$\therefore$ max. $P = 30x + 20y = 30 \times 11 + 20 \times 2 = 370$

Example 5

Two spare parts X and Y are to be produced. Each one has to go through two processes A and B. Each of X has to spend 3 hours in A and 9 hours in B. Also each of Y has to spend 4 hours in A and 4 hours in B. The time available for A and B are at most 36 hours and 60 hours respectively. If the profits per unit of X and Y are Rs. 50 and Rs. 60 respectively, formulate the above LP problem and solve it using simplex method to find the number of spare parts X and Y that are to be produced in order to maximize the profit. Also, find the maximum profit.

Solution:

The above informations are presented in the following table.

Process	Spare parts		Total time available
	X	Y	
A	3	4	36
B	9	4	60
Profit per unit	Rs. 50	Rs. 60	

Let x and y be the number of parts of the type X and Y respectively. Then the formulation of the above LP problem is given below

Max profit (Z) = 50x + 60y subject to

$$3x + 4y \le 36$$

$$9x + 4y \le 60$$

$$x, y \ge 0$$

Let r and s be the non-negative slack variables. Then,

$$3x + 4y + r = 36$$

$$9x + 4y + s = 60$$

Expressing the given LP problem in standard form,

$$3x + 4y + 1.r + 0.s + 0.Z = 36$$

$$9x + 4y + 0.r + 1.s + 0.Z = 60$$

$$-50x - 60y + 0.r + 0.s + 1.Z = 0$$

Presenting these equations into initial simplex tableau, we have

Basic variables	x	y	r	s	Z	RHS	Ratio
r	3	[4]	1	0	0	36	$\frac{36}{4} = 12$
s	9	4	0	1	0	60	$\frac{60}{4} = 15$
	–50	–60 ↑	0	0	1	0	

Since –60 is the most negative entry, so y-column is the pivot column. Again sice $\frac{36}{4} = 12$, $\frac{60}{4} = 15$ and $12 < 15$, so r-row is the pivot row, thus getting 4 (intersection of y-column and r-row) is the pivot entry.

Perform $R_1 \rightarrow \frac{1}{4} R_1$

Basic variables	x	y	r	s	Z	RHS
y	$\frac{3}{4}$	1	$\frac{1}{4}$	0	0	9
s	9	4	0	1	0	60
	–50	–60	0	0	1	0

Performing $R_2 \rightarrow R_2 - 4R_1$ and $R_3 \rightarrow R_3 + 60R_1$

Basic variables	x	y	r	s	Z	RHS	Ratio
y	$\frac{3}{4}$	1	$\frac{1}{4}$	0	0	9	$\frac{9}{3/4} = 12$
s	[6]	0	–1	1	0	24	$\frac{24}{6} = 4$
	–5 ↑	0	15	0	1	540	

–5 is the only negative entry, so x-column is the pivot column. Again since $\frac{9}{3/4} = 12$, $\frac{24}{6} = 4$ and $4 < 12$ so s-row is the pivot row thus getting 6 as the pivot entry.

Perform $R_2 \rightarrow \frac{1}{6} R_2$

Basic variables	x	y	r	s	Z	RHS
y	$\frac{3}{4}$	1	$\frac{1}{4}$	0	0	9
x	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	4
	–5	0	15	0	1	540

Performing $R_1 \to R_1 - \frac{3}{4}R_2$, $R_3 \to R_3 + 5R_2$

Basic variables	x	y	r	s	Z	RHS
y	0	1	$\frac{3}{8}$	$-\frac{1}{8}$	0	6
x	1	0	$-\frac{1}{6}$	$\frac{1}{6}$	0	4
	0	0	$\frac{85}{6}$	$\frac{5}{6}$	0	560

Since all entries in the last row are non-negative, so optimal solution is obtained.

$\therefore$ maximum value of Z = 560 when x = 4, y = 6.

Checking: Max. $Z = 50x + 60y = 50 \times 4 + 60 \times 6 = 560$

EXERCISE

1. Reformulate the following LP problem into standard form.

a) Maximize $P = 20x + 30y$ s.t.
$3x + y \le 15$
$x + 3y \le 12$
$x, y \ge 0$

b) Maximize $F = 7x + 5y$ s.t.
$4x + 3y \le 48$
$2x + y \le 20$
$x, y \ge 0$

2. Using simplex method, find the optimal solutions of the following LP problems

a) Maximize $Z = 2x + y$ s.t.
$x + 2y \le 10$
$x + y \le 6$
$x, y \ge 0$

b) Maximize $Z = x + 3y$ s.t.
$x + y \le 5$
$3x + y \le 15$
$x, y \ge 0$

c) Maximize $F = 5x + 12y$ s.t.
$3x + y \le 12$
$x + 2y \le 12$
$x, y \ge 0$

d) Max. $Z = 4x - 6y$ s.t.
$2x - 3y \le 8$
$x + y \le 24$
$x, y \ge 0$

e) Maximize $U = 25x + 45y$ s.t.
$x + 3y \le 21$
$2x + 3y \le 24$
$x \ge 0, y \ge 0$

f) Maximize $P = 8x + 10y$ s.t.
$x + 2y \le 30$
$2x + 2y \le 40$
$x \ge 0, y \ge 0$

g) Maximize $f = 5x_1 + 3x_2$ s.t.
$2x_1 + x_2 \le 40$
$x_1 + 2x_2 \le 50$
$x_1, x_2 \ge 0$

h) Maximize $C = 7x + 5y$ s.t.
$4x + 3y \le 48$
$2x + y \le 20$
$x, y \ge 0$

i) Maximize $Z = 5x + 5y$ s.t.
$2x + y \le 20$
$2x + 3y \le 24$
$x, y \ge 0$

j) Maximize $F = 5x_1 + 7x_2$ s.t.
$2x_1 + 3x_2 \le 13$
$3x_1 + 2x_2 \le 12$
$x_1, x_2 \ge 0$

k) Maximize $g = 15x + 12y$ s.t.
$2x + 3y \le 21$
$3x + 2y \le 24$
$x, y \ge 0$

l) Maximize $z = 10x_1 + 12x_2$ s.t.
$3x_1 + x_2 \le 12$
$x_1 + 2x_2 \le 14$
$x_1, x_2 \ge 0$

3. A small scale dealer deals in rice and wheat. He has Rs. 1500 for investment. A bag of rice costs him Rs. 180 and a bag of wheat Rs. 120. He has a storage capacity of 10 bags only. He sells a bag of rice at a profit of Rs. 11 and a bag of wheat at a profit of Rs. 8. How many bags of each must he buy to make a maximum profit? Also find the maximum profit.

A. The students in the class be divided into two groups. One group is to solve the following LP problem

$$\text{Max. } Z = ax + by \text{ s.t.}$$
$$a_1x + b_1y \le c_1$$
$$a_2x + b_2y \le c_2$$
$$x, y \ge 0$$

Giving different values of the constants a, b, a_1, b_1, ... and p, q, α, β, ..., obtain the optimal value by each group. Present the results in the class by both groups. Use simplex method.

Answers

1. a) Max. $P = 20x + 30y + 0.r + 0.s$ s.t.
$3x + y + r + 0.s = 15,$
$x + 3y + 0.r + s = 12,$
$x, y, r, s \ge 0$
where r and s are the slack variables.

b) Max $F = 7x + 5y + 0.r + 0.s$ s.t.
$4x + 3y + r + 0.s = 48,$
$2x + y + 0.r + s = 20,$
$x, y, r, s \ge 0$

2. a) Max. Z = 12, x = 6, y = 0
b) Max. Z = 15, x = 0, y = 5
c) Max. F = 84, x = 0, y = 7
d) Max. Z = 16, x = 4, y = 0
e) Max. U = 345, x = 3, y = 6
f) Max. P = 180, x = 10, y = 10
g) Max. f = 110, $x_1 = 10$, $x_2 = 20$
h) Max. C = 82, x = 6, y = 8
i) Max. Z = 55, x = 9, y = 2
j) Max. F = 31, $x_1 = 2$, $x_2 = 3$
k) Max. g = 126, x = 6, y = 3
l) Max. Z = 92, $x_1 = 2$, $x_2 = 6$

3. 5 kg of rice and 5 kg of wheat; Max profit = Rs. 95

System of Linear Equations

An equation of the form

$$ax + by + cz = k$$

where x, y and z are variables and a, b, c and k are known numbers, is called a **linear equation**. If $k = 0$, the linear equation is said to be **homogeneous**.

A set of linear equations taken together is said to form a linear system; and a set of homogeneous linear equations forms a homogeneous linear system. For instances, of the following two systems of linear equations:

a) $x + 2y = 3$
$x - y = 2$

b) $x + y + z = 0$
$x + 2y - z = 0$
$2x + y + 3z = 0$

the first one is non-homogeneous and the second one is homogeneous.

A system consisting of 3 linear equations in 3 unknowns is often written in the form

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3$$

where a_{ij} and c_i are real numbers. If $c_i = 0$, for $i, j = 1, 2, 3$; then the system of equations are known as homogeneous system of equations.

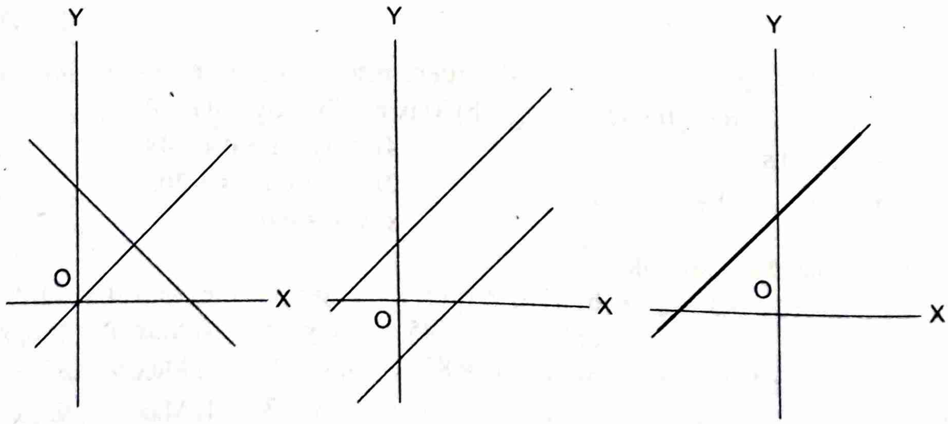
A set of values for the variables or unknowns, say

$$x_1 = k_1, \quad x_2 = k_2, \quad x_3 = k_3$$

is called a solution of the system of linear equations if they satisfy the equations.

A system of linear equations may have one solution, no solution or infinitely many solutions.

One Solution	No Solution	Infinitely many solutions
$x - y = 0$ $x + y = 1$	$x - y = -1$ $x - y = 1$	$x - y = 1$ $3x - 3y = 3$



If a system of linear equations has a solution they can be found in several ways. We shall consider the following methods only:

a) Gaussian elimination method

b) Gauss-Seidel method

and c) Matrix inversion method

a) Gaussian Elimination Method

One of the methods for solving a system of simultaneous linear equations is the elimination method known as Gaussian elimination method.

In this process, the given system is reduced to a simpler system in which

i) the first unknown or variable x_1 in the first equation will have a non-zero coefficient, and

ii) the first unknown or variable x_1 in each of the remaining equations will have zero coefficient but the second unknown or variable x_2 will have a non-zero coefficient.

We then consider the subsystem of equations obtained by excluding the first equation and proceed as before to eliminate the second variable x_2. This process is known as the **forward elimination**. Continuing the above process, we arrive at an equivalent system that looks like an inverted steps or a triangle. The step-like form or echelon form has the leading terms with non-zero coefficients further to the right in each succeeding equation. One may make each of those non-zero coefficients equal to 1 by division. Four different cases arises. These four cases are given in the forms of examples 1, 2, 3 and 4.

Coefficient of the first variable of the first equation ≠ 0:

Suppose the given system is

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \ (a_{11} \neq 0) \qquad \text{......(i)}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \qquad \text{......(ii)}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 \qquad \text{......(iii)}$$

Here, the coefficient of the first variable in the first equation is *not* zero as is required. Eliminate x_1 from equation (i) and (ii) getting equation with variables x_2 and x_3. Denote this equation by (iv).

Again, eliminate x_1 from equations (i) and (iii) (or (ii) and (iii)) getting equation with variables x_2 and x_3. Denoted the equation by (v). Further, eliminate x_2 from equation (iv) and (v) getting an equation with variable x_3 only. The process is known as the **forward elimination**. Denote this equation by (vi)

Now we have the three equation of the following type

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1$$

$$p_1x_2 + q_1x_3 = r_1$$

and $$q_2x_3 = r_3$$ where p_1, q_1, q_2, r_1 and r_3 are constants.

We solve the quations (i), (v) and (vi) for x_1, x_2, x_3. From (iii), x_3 is obtained. Use of x_3 in (v) gives the value of x_2. Again use of x_2 and x_3 in i) gives the value of x_1. This process is known as the **backward substitution**. Thus, we obtain the values of x_1, x_2 and x_3. This is Gaussian elimination method of solving the equations.

Note: If the coefficient of the variable in the first equation is zero, we interchange first and second or first and third equations. Now, we proceed the above method.

In the Gauss Elimination method, the coefficients of the variables of the equation a_{ij} where $i = j$ are known as the **pivot elements**.

The linear simultaneous equations are arranged so that the pivot elements are non-zero i.e. a_{11}, a_{22}, a_{33}, ... are non-zero. If in case, the pivot elements be zero, we interchange the equations making the pivot elements non-zero. The equations are arranged so that the absolute values of the pivot elements are large. Rest process is same as in Gauss Elimination method. This method of solving simultaneous equations is known as the **Gauss Elimination method with partial pivoting**.

Gauss Jordan method

A little bit different from Gauss Elimination method, there is another method of solving linear simultaneous equation known as Gauss Jordan method in which the coefficients a_{11}, a_{22}, a_{33} are made unity and all other coefficients zero. Now, the form of the following three linear simultaneous equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad ......(i)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad ......(ii)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad ......(iii)$$

in Gauss elimination and Gauss Jordan method will be as follows

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}'x_2 + a_{23}'x_3 &= b_2' \\ a_{33}'x_3 &= b_3' \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + 0.x_2 + 0.x_3 &= c_1 \\ 0.x_1 + 1.x_2 + 0.x_3 &= c_2 \\ 0.x_1 + 0.x_2 + 1.x_3 &= c_3 \end{aligned}$$

which in matrices form will be

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a'_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b'_2 \\ b'_3 \end{pmatrix} \quad \rightarrow \text{Gauss Elimination method}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \rightarrow \text{Gauss Jordan method}$$

The unit matrix form in the Gauss Jordan method can specially be used in inverting a matrix.

On solving the system of equations using Gauss Elimination method, we can draw the following conclusions from the equations (i), (iv) and (vi)

a) If the coefficients of x, y and z in equations (i), (iv) and (vi) are non-zero, then the system is **consistent** and has **a unque solution.**

b) If the last equation (vi) is of the form $0.z = k \neq 0$ (a constant), i.e. the coefficient of z is zero, no value of z will satisfy the equation (vi), hence the system is **inconsistent.**

c) If the last equation (vi) is of the form $0.z = 0$, then every value of z will satisfy the equation (vi), hence the system is **consistent** and has **infinitely many solutions.**

Worked Out Examples

Example 1

Solve the following equations using Gauss Elimination method:

$$5x - 3y = 19$$
$$2x + 5y = -11$$

Solution:

$$5x - 3y = 19 \qquad \ldots\ldots(i)$$
$$2x + 5y = -11 \qquad \ldots\ldots(ii)$$

Multiplying equation (i) by $\frac{2}{5}$ and then subtracting from (ii)

$$\frac{31y}{5} = -\frac{93}{5} \qquad \ldots\ldots(iii)$$

Now, we have the following equation,

$$5x - 3y = 19$$

and $$\frac{31y}{5} = -\frac{93}{5}$$

From equation (iii),

$$y = -3$$

Using the value of y in eq (i), we have

$$5x + 3 \times 3 = 19$$
$$\Rightarrow \quad 5x = 10$$
$$\therefore \quad x = 2$$

$\therefore$ the required solution is $x = 2, y = -3$.

Example 2

Solve the following equations using Gauss elimination method

$$x - 2y + 3z = 2$$
$$2x - 3y + z = 1$$
$$3x - y + 2z = 9$$

Solution:

$$x - 2y + 3z = 2 \quad \ldots\ldots(i)$$
$$2x - 3y + z = 1 \quad \ldots\ldots(ii)$$
$$3x - y + 2z = 9 \quad \ldots\ldots(iii)$$

Multiplying the equation (i) by 2 and then subtracting from equation (ii), we have

$$y - 5z = -3 \quad \ldots\ldots(iv)$$

Again, multiplying equation (i) by 3 and then subtracting from the equation (iii), we have

$$5y - 7z = 3 \quad \ldots\ldots(v)$$

Multiplying equation (iv) by 5 and then subtracting from (v),

$$18z = 18 \quad \ldots\ldots(vi)$$

Now, we have the following three equations

$$x - 2y + 3z = 2 \quad \ldots\ldots(i)$$
$$y - 5z = -3 \quad \ldots\ldots(iv)$$
$$18z = 18 \quad \ldots\ldots(vi)$$

From the equation (vi), we have $z = 1$,

Use of $z = 1$ in (iv), we have $y = 2$.

Using $y = 2$, and $z = 1$ in given equation (i), we have $x = 3$

$\therefore$ the required solution is $x = 3, y = 2, z = 1$.

Example 3

Solve the following equations using Gaussian elimination method

$$2x_2 + 3x_3 = 7$$
$$3x_1 - 2x_2 + 2x_3 = 1$$
$$2x_1 + 3x_2 - 3x_3 = 5$$

Solution:

The coefficient of first variable x_1 in first equation is zero. So, interchanging the first two equations, we have

$$3x_1 - 2x_2 + 2x_3 = 1 \quad \ldots\ldots(i)$$
$$2x_2 + 3x_3 = 7 \quad \ldots\ldots(ii)$$
$$2x_1 + 3x_2 - 3x_3 = 5 \quad \ldots\ldots(iii)$$

Multiplying equation (i) by $\frac{2}{3}$ then subtracting from equation (iii),

we have $\quad x_2 - x_3 = 1 \quad \ldots\ldots(iv)$

Multiplying equation (ii) by $\frac{1}{2}$ and then subtracting from (iv), we have

$$-\frac{5}{2}x_3 = -\frac{5}{2} \quad \ldots\ldots(v)$$

Now, we have the following three equations

$$3x_1 - 2x_2 + 2x_3 = 1 \quad \text{......(i)}$$

$$x_2 - x_3 = 1 \quad \text{......(iv)}$$

$$-\frac{5}{2}x_3 = -\frac{5}{2} \quad \text{......(v)}$$

From equation (v), $x_3 = 1$

Using $x_3 = 1$ in (iv), we have $x_2 = 2$

Substituting $x_2 = 2$ and $x_3 = 1$ in (i), we have $x_1 = 1$

$\therefore$ the required solution is $x_1 = 1, x_2 = 2$ and $x_3 = 1$.

Example 4

Solve the following system of equations (Use Gauss elimination method)

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_1 + x_2 - x_3 = 1$$

$$2x_1 + 3x_2 + 2x_3 = 3$$

Solution:

$$x_1 + 2x_2 + 3x_3 = 2 \quad \text{......(i)}$$

$$x_1 + x_2 - x_3 = 1 \quad \text{......(ii)}$$

$$2x_1 + 3x_2 + 2x_3 = 3 \quad \text{......(iii)}$$

Subtracting equation (i) from equation (ii), we have

$$x_2 + 4x_3 = 1 \quad \text{......(iv)}$$

Multiplying equation (i) by 2 and then subtracting from equation (iii), we have

$$x_2 + 4x_3 = 1 \quad \text{......(v)}$$

Subtracting (iv) from (v), we have

$$0 = 0$$

Now, we have the following three equations

$$x_1 + 2x_2 + 3x_3 = 2$$

$$x_2 + 4x_3 = 1$$

$$0.x_3 = 0$$

The third equation is true for all values of x_3. The variable x_3 is said to be the free variable. We can assign any number of values to this free variable and solve the first two equations for the rest two variables. Hence we get an *infinitely many solutions* in such a case. Thus if $x_3 = k$, then $x_2 = 1 - 4k$ and $x_1 = 5k$.

Example 5

Examine the consistency of the equation. Solve if possible

$$x_1 + x_2 - x_3 = 1$$
$$2x_1 + 3x_2 - 3x_3 = 3$$
$$x_1 - 3x_2 + 3x_3 = 2$$

Solution:

$$x_1 + x_2 - x_3 = 1 \quad \ldots\ldots(i)$$
$$2x_1 + 3x_2 - 3x_3 = 3 \quad \ldots\ldots(ii)$$
$$x_1 - 3x_2 + 3x_3 = 2 \quad \ldots\ldots(iii)$$

Multiplying equation (i) by 2 and then subtracting from equation (ii), we have

$$x_2 - x_3 = 1 \quad \ldots\ldots(iv)$$

Again subtracting equation (i) from equation (iii), we have

$$4x_2 - 4x_3 = -1 \quad \ldots\ldots(v)$$

Multiplying equation (iv) by 4 and then subtracting from (v), we have

$$0 = -5$$

Now, we have the following equations,

$$x_1 + x_2 - x_3 = 1$$
$$x_2 - x_3 = 1$$
$$0.x_3 = -5$$

Here no value of x_3 satisfies the third equation.

$\therefore$ the system has *no solution*.

That is, the system of equations is inconsistent.

Example 6

Solve the following system of equations using partial pivoting method

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 3x_3 = 4$$
$$x_1 - x_2 + x_3 = 0$$

Solution:

Since the largest absolute value of the coefficient of x_1 is 4, so we interchange the first two equations

$$4x_1 + 2x_2 + 3x_3 = 4 \quad \ldots\ldots(i)$$
$$2x_1 + 2x_2 + x_3 = 6 \quad \ldots\ldots(ii)$$
$$x_1 - x_2 + x_3 = 0 \quad \ldots\ldots(iii)$$

Multiplying equation (i) by $\frac{1}{2}$ and then subtracting from (ii) and multiplying equation (i) by $\frac{1}{4}$ and then subtracting from (iii), we have

$$x_2 - \frac{1}{2}x_3 = 4$$

$$-\frac{3}{2}x_2 + \frac{1}{4}x_3 = -1$$

Interchanging the two equations as the absolute value of coefficient of x_2 in second equation is large, we have

$$-\frac{3}{2}x_2 + \frac{1}{4}x_3 = -1 \quad \text{......(iv)}$$

$$x_2 - \frac{1}{2}x_3 = 4 \quad \text{......(v)}$$

Multiplying equation (iv) by $\frac{2}{3}$ and then adding with (v), we have

$$-\frac{1}{3}x_3 = \frac{10}{3} \quad \text{......(vi)}$$

Now, we have the following three equations,

$$4x_1 + 2x_2 + 3x_3 = 4 \quad \text{......(i)}$$

$$-\frac{3}{2}x_2 + \frac{1}{4}x_3 = -1 \quad \text{......(iv)}$$

$$-\frac{1}{3}x_3 = \frac{10}{3} \quad \text{......(vi)}$$

From (vi), $x_3 = -10$

Using the value of x_3 in (iv),

$$-\frac{3}{2}x_2 - \frac{10}{4} = -1$$

$$-\frac{3}{2}x_2 = -1 + \frac{10}{4}$$

$$\Rightarrow \quad -\frac{3}{2}x_2 = \frac{6}{4}$$

$$\therefore \quad x_2 = -1$$

Using the values of x_2 and x_3 in (i),

$$4x_1 - 2 - 30 = 4$$

$$\Rightarrow \quad 4x_1 = 36$$

$$\therefore \quad x_1 = 9$$

$\therefore$ the solution is $x_1 = 9, x_2 = -1, x_3 = -10$

EXERCISE

1. Solve the following system of equations using Gauss Elimination method:

a) $x + 2y = 5$, $5x - 3y = -1$

b) $3x - 2y = 5$, $4x - y = 10$

c) $2x + 7y = 3$, $6x - 5y = -17$

d) $5x - 8y = 28$, $3x + 7y = 5$

2. Solve the following system of equations using Gaussian elimination method

a) $x + 3y - 2z = 0$, $2x - 3y + z = 1$, $4x - 3y + z = 3$

b) $x + 3y - z = -2$, $3x + 2y - z = 3$, $-6x - 4y - 2z = 18$

c) $x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$

d) $2x - 3y + 3z = 27$, $4x + y - 2z = 0$, $-6x - 4y + 2z = 0$

e) $3x + 2y - z = 1$, $2x - 2y + 4z = -2$, $-x + \frac{1}{2}y - z = 0$

f) $3x_1 + 6x_2 + x_3 = 16$, $2x_1 + 4x_2 + 3x_3 = 13$, $x_1 + 3x_2 + 2x_3 = 9$

g) $x_1 - 2x_2 + 3x_3 = 10$, $2x_1 + 3x_2 - 2x_3 = 1$, $-x_1 - 2x_2 + 4x_3 = 13$

h) $3x_1 - x_2 + x_3 = 2$, $-15x_1 + 6x_2 - 5x_3 = 5$, $5x_1 - 2x_2 + 2x_3 = 1$

3. Solve, using Gaussian elimination method with partial pivot, the following equations:

a) $5x + 4y = 9$, $x - 5y = 25$

b) $x - 8y = -26$, $4x + 3y = 1$

c) $3x + y - 2z = 10$, $2x + 4y - 5z = 24$, $x - 2y + 3z = -11$

d) $x - y + z = 10$, $4x - 2y - 5z = 9$, $3x + 4y - 2z = 1$

4. Examine whether the following system of equations are consistent. If so, solve the following system of equations using Gaussian elimination method

a) $x_1 - x_2 + x_3 = 1$, $3x_1 + x_2 + 5x_3 = 11$, $4x_1 + 2x_2 + 7x_3 = 16$

b) $x + 3y + 4z = 8$, $2x + y + 2z = 5$, $5x + 2z = 7$

c) $x_1 + x_2 + x_3 = -3$, $3x_1 + x_2 - 2x_3 = -2$, $2x_1 + 4x_2 + 7x_3 = 7$

d) $x_1 + 2x_2 + 3x_3 = 4$, $4x_1 + 5x_2 + 6x_3 = -7$, $7x_1 + 8x_2 + 9x_3 = 10$

Answers

1. a) 1, 2 b) 3, 2 c) –2, 1 d) 4, –1
2. a) 1, 1, 2 b) 1, –3, –6 c) –15, 8, 2 d) 3, –2, 5
   e) 1, –2, –2 f) 1, 2, 1 g) 1, 3, 5 h) 3, 15, 8
3. a) 5, –4 b) –2, 3 c) 1, 3, –2 d) 5, –2, 3
4. a) Consistent; $\frac{6-3k}{2}, \frac{4-k}{2}$, k, infinitely many solutons

   b) Consistent; $\frac{7-2k}{5}, \frac{11-6k}{5}$, k, infinitely many solutions

   c) No solution; Inconsistent

   d) No solution; Inconsistent

Chapter 19
Mechanics

Introduction: Parallel Forces

Forces and their effects, both qualitative and quantitative aspects so far discussed, are restricted to the cases in which the forces are supposed to act at a point or on a particle. We now extend them to forces on a *rigid body*, i.e. *a definite portion of matter whose shape is not changed under the action of the forces*. In other words, we now deal with forces acting on bodies whose particles remain at invariable distances from one another. This is an *idealized* situation. No body is perfectly rigid and free from deformation (i.e. change in shape), however small it may be, under the action of forces. *Rigidity*, the ability of bodies to resist the change in their shape, is the opposite of what is known as *elasticity*. An important characterization of a rigid body is that a force acting at a point (or on a particle) of the body produces the same effect independently of its position in the body as long as it lies on its line of action. this characteristic is known as the *Principle of Transmissibility of Forces*, and is often stated as '*A force acting at a point in a rigid body may be supposed to act at any point of the body but lying on its line of action.*'

Forces acting on a body may or may not be concurrent. They may be coplanar or non-coplanar. We restrict our discussion to coplanar forces only. They may be,

a) all meeting at a point

b) all parallel and

c) some meeting at a point while others being parallel.

Concurrent forces have already been defined and discussed in detail. We now begin with the definition of parallel forces, see how they can be compounded and then pass onto the problems dealing with moments and couples.

Parallel Forces

If two forces act along parallel lines they are said to be *parallel forces*.

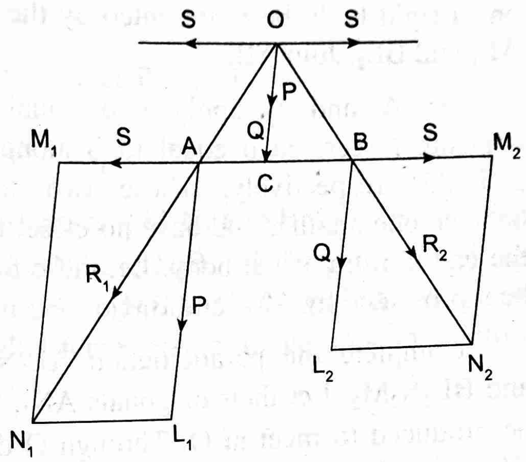
Two parallel forces are said to be *like* when they act in the same direction and they are said to be *unlike* when they act in opposite directions.

A single force whose effect is the same as that of two parallel forces is known to exist. Since they do not meet at a point, so a special device is needed to find their resultant. This is done here by considering like and unlike parallel forces separately :

a) Resultant of two like parallel forces

Let two like parallel forces P and Q acting at points A and B respectively of a rigid body be represented by the lines AL_1 and BL_2. Join AB.

At A and B apply two equal and opposite forces, each equal to S along BA and AB respectively. These two forces balance one another and have no effect upon the equilibrium of the body. Let these forces be represented by AM_1 and BM_2.



Complete the parallelogram $AL_1N_1M_1$ and $BL_2N_2M_2$. Let the diagonals N_1A and N_2B be produced to meet at O. Through O draw OC parallel to AL_1 or BL_2 to meet AB in C. Now the forces P and S at A have a resultant R_1 represented by AN_1. Let its point of application be transferred to O. Similarly the forces Q and S at B have the resultant R_2 represented by BN_2. Let its point of application be transferred to O.

The force R_1 at O may be resolved into two components S parallel to AM_1 and P in the direction OC. Similarly the force R_2 at O may be resolved into two components S parallel to BM_2 and Q in the direction OC.

Thus the given forces are equivalent to two forces P and Q along OC and two forces each equal to S acting in opposite directions. The first two forces are equivalent to a single force (P + Q) along OC and the last two forces balance one another. Hence the resultant of two like parallel forces P and Q is equivalent to a force (P + Q) acting along OC.

To find the point of application of the resultant, we proceed as follows :

Since the triangles OCA, AL_1N_1 are similar

so, $$\frac{OC}{CA} = \frac{AL_1}{L_1N_1} = \frac{AL_1}{AM_1} = \frac{P}{S}$$

or, $$P.CA = S.OC \qquad \text{...... (i)}$$

Again, since the triangles OCB, BL_2N_2 are similar

so, $$\frac{OC}{CB} = \frac{BL_2}{L_2N_2} = \frac{BL_2}{BM_2} = \frac{Q}{S}$$

or, $$Q.CB = S.OC \qquad \text{...... (ii)}$$

From (i) and (ii), it follows that

$$P.CA = Q.CB$$

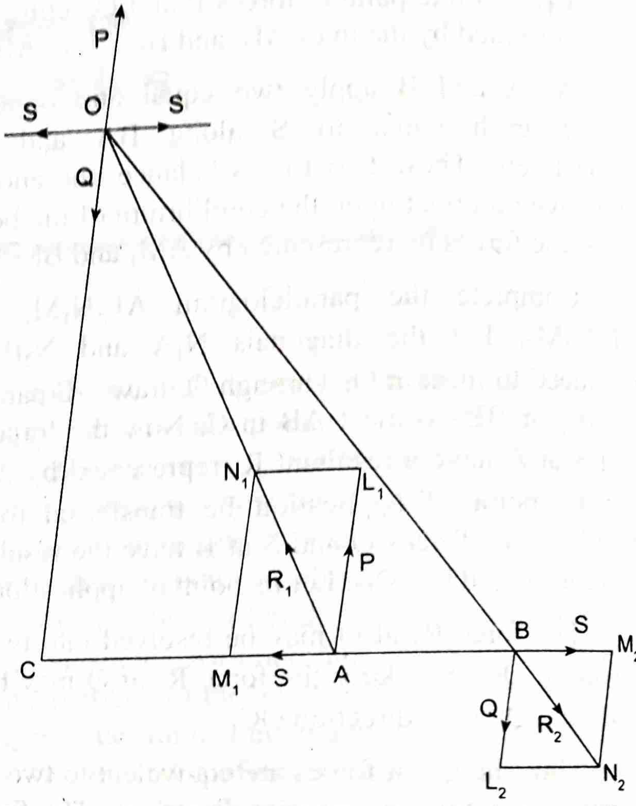
or, $$\frac{CA}{CB} = \frac{Q}{P}$$

i.e. C divides the line AB internally in the inverse ratio of the forces P and Q.

Cor. 1. $\frac{P}{CB} = \frac{Q}{CA} = \frac{P+Q}{CB+CA} = \frac{R}{AB}$

b) Resultant of two unlike parallel forces

Let two unlike parallel forces P, Q (P > Q) acting at points A and B respectively on a rigid body be represented by the lines AL_1 and BL_2. Join AB.



At A and B apply two equal and opposite forces, each equal to S along BA and AB respectively. These two forces balance one another and have no effect upon the equilibrium of the body. Let these forces be represented by AM_1 and BM_2.

Complete the parallelogram $AL_1N_1M_1$ and $BL_2N_2M_2$. Let their diagonals AN_1, N_2B be produced to meet at O. Through O draw OC parallel to AL_1 parallel to BL_2 meeting BA produced at C. Now the forces P and S at A have a resultant R_1 represented by AN_1. Let its point of application be transferred to O. Similarly the forces Q and S at B have a resultant R_2 represented by BN_2. Let its point of application be transferred to O.

The force R_1 at O may be resolved into two components S parallel to AM_1 and P in the direction CO. Similarly the force R_2 at O may be resolved into two components S parallel to BM_2 and Q in the direction of OC.

Thus the given forces are equivalent to two forces P along CO and Q along OC and two forces each equal to S acting in opposite directions. The first two forces are equivalent to a single force (P – Q) acting along the direction CO (i.e. acting at C in the direction parallel to that of the greater force P) and the last two forces balance one another.

Hence the resultant of two unlike parallel forces P and Q (P > Q) is equivalent to a force (P – Q) acting in the direction of the greater force.

To find the point of application of the resultant, we proceed as follows :

Since the triangles OCA, N_1M_1A are similar

so, $$\frac{OC}{CA} = \frac{N_1M_1}{M_1A} = \frac{AL_1}{AM_1} = \frac{P}{S}$$

or, $P.CA = S.OC$ ...... (i)

Again, since the triangles OCB, BL_2N_2 are similar

so, $$\frac{OC}{CB} = \frac{BL_2}{L_2N_2} = \frac{BL_2}{BM_2} = \frac{Q}{S}$$

or, $Q.CB = S.OC$ ...... (ii)

From (i) and (ii) if follows that

$$P.CA = Q.CB$$

or, $$\frac{CA}{CB} = \frac{Q}{P}$$

i.e. C divides AB externally in the inverse ratio of the forces P and Q.

Cor. 1. $\frac{P}{CB} = \frac{Q}{CA} = \frac{P-Q}{CB-CA} = \frac{R}{AB}$

Note : If the forces P and Q are unlike and equal, the triangles AM_1N_1 and BM_2N_2 being equal in all respects, $\angle M_1AN_1 = \angle M_2BN_2$. In this case AN_1 and N_2B will be parallel and not meeting at any such point O. Hence the geometrical condition for finding the resultant fails. Thus we see that two equal and unlike parallel forces cannot be compounded into a single force.

Resultant of Several Parallel Forces

In practice, we often come across a large number of parallel forces acting at the same time. They are found to combine together to form a single force. For instance, the weight of a body. A body is made up of several particles rigidly connected to one another. We know each particle is attracted towards the centre of the earth. That is, each material particle of the body has a weight. The line of action of the force, (i.e. weight of a particle of the body) is towards the centre of the earth. But the distance of each particle of the body from the centre of the earth is so great that for all pratical purposes, the lines of action of the weights of all particles may be considered parallel to one another and so have a resultant. The weight of the body as a whole is therefore, the resultant of all these weights acting parallelly.

In case our body is a thin straight rod, we may assume the particles of weights $w_1, w_2 \ldots w_n$ of the rod lie along a straight line at distance $x_1, x_2 \ldots$ from a fixed point O. Repeated application of the process of finding the resultant of two like parallel forces yields

i) the final resultant or weight of the rod to be

$$w = w_1 + w_2 + \ldots = \Sigma w_i$$

ii) The point of application of the weight is at a distance

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ldots}{w_1 + w_2 + \ldots} = \frac{\Sigma w_i x_i}{\Sigma w_i} \quad \text{from O.}$$

This is a fixed point on the rod and is called the centre of gravity of the rod.

If the rigid body is a plane it can similarly be shown that a point $(\bar{x}, \bar{y})$, through which the weight of the body acts is

$$\bar{x} = \frac{\Sigma w_i x_i}{\Sigma w_i} \quad \text{and} \quad \bar{y} = \frac{\Sigma w_i y_i}{\Sigma w_i},$$

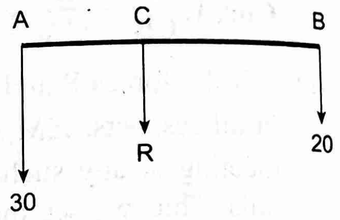
where (x_i, y_i) denote position of any one of the particles. This point is found to be unique, and is the centre of gravity of the body. In what follows, we remember that the *position of the centre of gravity of a uniform homogeneous body having standard gemoetric shape generally lies at its geometric centre.*

Worked Out Examples

Example 1

Two parallel forces of 30 kg wt and 20 kg wt are acting at a distance 40 cms apart. Find their resultant if forces are like.

Solution :



Let R, the resultant of two like parallel forces, pass through C.

Here, AB = 40 cms.

Let AC = x cm

Then, CB = $(40 - x)$ cms

Since the forces form like parallel forces, so

$$R = 30 + 20 = 50 \text{ kg wt}$$

and $30 \times AC = 20 \times CB$

or, $30 \times x = 20 \times (40 - x)$

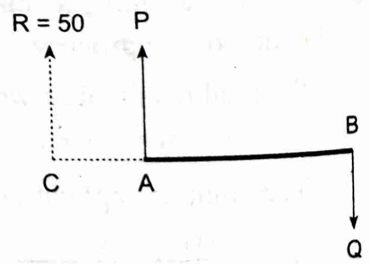
$\therefore \quad x = 16$ cms

So, the resultant is a force 50 kg. wt. parallel to the given forces and it acts at a distance of 16 cms from the point at which the weight of 30 kg acts.

Example 2

Replace a force of magnitude 50 kg wt by two unlike parallel forces, one at a distance of 2 m and other at 8 m from the given force.

Solution :



Let P and Q (P > Q) be two unlike parallel forces whose resultant is 50 kg wt. Let P, Q and R = 50 kg wt act at the points A, B and C respectively.

Here, CB = 8m and AC = 2m

Since the forces form unlike parallel forces, so

$$\frac{P}{CB} = \frac{Q}{CA} = \frac{R}{AB}$$

or, $\dfrac{P}{8} = \dfrac{Q}{2} = \dfrac{50}{6}$

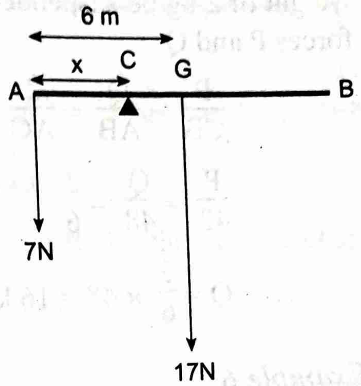
$\therefore \quad P = \dfrac{50}{6} \times 8 = 66\frac{2}{3}$ kg wt and $Q = \dfrac{50}{6} \times 2 = 16\frac{2}{3}$ kg wt

Example 3

A uniform rod, 12 metres long and weighing 17 N can turn freely about a point in it and the rod is in equilibrium when a weight of 7 N is hung at one end; how far from the end is the point about which it can turn ?

Solution :

Let AB be a uniform rod 12 m long and its weight 17 N acts through the middle point G. Let the rod turn freely about a point C which is x m from the end A when a weight of 7 N is hung.



$$\therefore \quad 7\,AC = 17\,CG$$

$$7x = 17(6 - x)$$

$$7x + 17x = 102$$

$$x = \frac{17}{4} = 4\frac{1}{4}$$

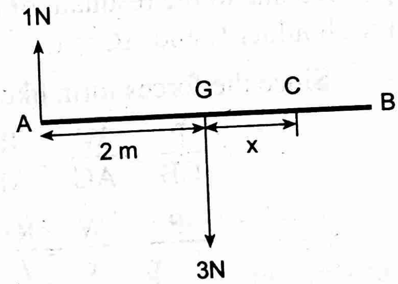
$\therefore$ required distance is $4\frac{1}{4}$ m.

Example 4

A uniform bar 4m long and weighing 3 N, passes over a prop and is supported in a horizontal position by a force of 1 N acting vertically upwards at the other end. Find the distance of the prop from the centre of the bar.

Solution :

Let AB be the uniform bar of the length 4m and weight 3 N acting through its centre G. Let it pass over a prop C. It is supported in a horizontal position by a force of 1 N acting vertically upwards at the other end A. Let x be the distance of the prop C from the centre G.



$$3 \times GC = 1 \times AC$$

or, $\quad 3 \times x = 1 \times (2 + x)$

or, $\quad 2x = 2$

or, $\quad x = 1$

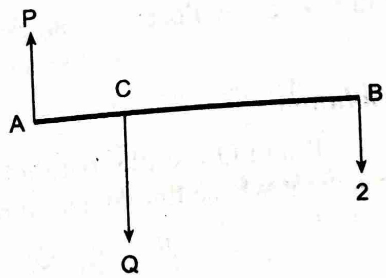
$\therefore$ distance of the prop from the centre of the bar is 1m.

Example 5

A straight weightless rod, 48 cms. in length, rests in a horizontal position between two pegs placed at a distance of 6 cms. apart, one peg being at one end of the rod, and a weight of 2 kg is suspended from the other end. Find the pressure on the pegs.

Solution :

Let AB be a straight weightless rod resting between two pegs at A and C so that



$$AB = 48 \text{ cms}$$

and $\quad AC = 6$ cms

Then, $\quad CB = AB - AC$

$$= 48 - 6$$

$$= 42 \text{ cms}$$

Let P and Q be the pressure on the pegs at A and C. The system is in equilibrium when a weight of 2 kg be suspended at the end B. So, here 2 kg wt is the resultant of two unlike parallel forces P and Q.

$$\therefore \quad \frac{P}{CB} = \frac{Q}{AB} = \frac{2}{AC}$$

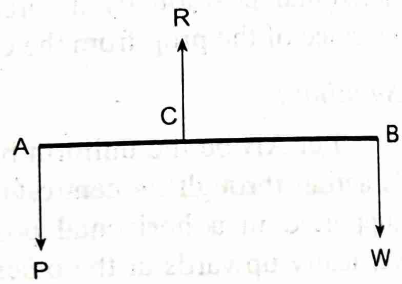
$$\frac{P}{42} = \frac{Q}{48} = \frac{2}{6}$$

$$\therefore \quad Q = \frac{2}{6} \times 48 = 16 \text{ kg wt} \quad , \quad P = \frac{2}{6} \times 42 = 14 \text{ kg wt}$$

Example 6

A man carries a bundle at the end of a stick which is placed over the shoulder. What is the distance between his hand and shoulder when the pressure on his shoulder is least ?

Solution :



Let AB be a stick of length l. Let C be the position of the shoulder. If W be the weight of the bundle suspended at the end B and P, the pressure due to the hand at A, then the shoulder is pressed due to the resultant of P and W. Let R be the reaction on the shoulder C and AC = x.

Since the forces form like parallel forces, so

$$\frac{P}{CB} = \frac{W}{AC} = \frac{R}{AB}$$

$$\frac{P}{l-x} = \frac{W}{x} = \frac{R}{l}$$

$$\therefore \quad R = \frac{Wl}{x}$$

Since Wl is constant so R depends upon x. R will be least when x will be greatest. But the greatest value of x is l, the length of the stick. Hence R will be least when the distance between the hand and the shoulder is equal to the length of the stick.

Example 7

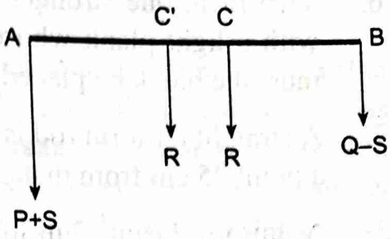
Two like parallel forces of magnitudes P and Q are acting at the end points A and B of a rod AB of length l. If two opposite forces each of magnitude S are added to P and Q, then prove that the line of action of the new resultant will be displaced through a distance

$$\frac{Sl}{P+Q}$$

Solution :

P and Q are like parallel forces acting at the points A and B respectively. Let R be their resultant whose line of action passes through C. Here, AB = l

$$R = P + Q$$



Also, $\dfrac{P}{CB} = \dfrac{Q}{AC} = \dfrac{R}{AB}$

or, $\dfrac{P}{CB} = \dfrac{Q}{AC} = \dfrac{P+Q}{l}$

$\therefore \quad CB = \dfrac{P.l}{P+Q}$

When the forces S and –S be added to P and Q respectively, let R_1 be the new resultant whose line of action passes through C′.

$$R_1 = (P + S) + (Q - S) = P + Q$$

Also, $\dfrac{P+S}{C'B} = \dfrac{Q-S}{AC'} = \dfrac{R_1}{AB}$

$\dfrac{P+S}{C'B} = \dfrac{Q-S}{AC'} = \dfrac{P+Q}{l}$

$\therefore \quad C'B = \dfrac{(P+S).l}{P+Q}$

Now, $\quad CC' = C'B - CB$

$= \dfrac{(P+S)l}{P+Q} - \dfrac{P.l}{P+Q} = \dfrac{S.l}{P+Q}$

EXERCISE

1. Find two like parallel forces acting at a distance of 2.5 m apart, which are equivalent to a given force of 30 N, the line of action of one being at a distance of 50 cm from the given force.
2. Find two unlike parallel forces acting at a distance of 12 cm which are equivalent to a force of 20 N, the greater of the two forces being at a distance of 6 cm from the given force.
3. a) Two like parallel forces P and Q act at points 18 m apart; if the resultant force be 9 N and acts at a distance 6m from P, find Q.
   b) Two like parallel forces act at the points whose distance apart is 4 m. If the resultant of the forces acts at a distance of 1.5 m from the greater force, find the ratio of the force.
4. a) Two unlike parallel forces, the greater of which is 75 N, have a resultant 25 N. Find the ratio of the distances of the resultant from the component forces.
   b) Two unlike paralle forces of magnitude 30 N and 75 N have the resultant acting at a distance of 12 cm from the greater force. Find the distance between the two forces.
5. The extremities of a straight bamboo pole 3m long rests on two smooth pegs at A and B in the same horizontal line. A heavy load hangs from a point C of the pole. If AC = 3BC and the pressure at B be 140 N more than that at A, find the weight of the load.

6. Two men, one stronger than the other have to remove a block of stone weighing 270 N with a light plank whose length is 6 m. If the stronger man is able to carry 180 N, how must the block be placed so as to allow him that share of the weight.

7. A straight uniform rod is 3m long when a load of 5 N is placed at one end it balances about a point 25 cm from that end. Find the weight of the rod.

8. A uniform beam, 7 m long, is supported in a horizontal position by two props which are 4m apart, so that the beam projects 1m beyond one of the props. If the beam weighs 80 N, find the pressure on the two pegs.

9. A uniform beam 4m long, is supported in a horizontal position by two props which are 3m apart, so that the beam projects one metre beyond one of the props. Show that the force on one of the prop is double that on the other.

10. A straight weightless rod 60 cm in length, rests in a horizontal position between two pegs placed at a distance of 6 cm apart, one peg being at one end of the rod, and a weight of 2 N is suspended from the other end; find the pressure on the pegs.

11. a) A man carries a bundle at the end of a stick 75 cm long which is placed on his shoulder. What should be the distance between his hand and shoulder so that the pressure on the shoulder may be three times the weight of the bundle.

    b) A man carries a bundle at the end of a stick which is placed over his shoulder; if the distance between his hand and shoulder be changed, how does the pressure on his shoulder change ?

12. Two parallel forces P and Q, act at a given points of a body; if Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces are simply interchanged.

13. P and Q (P > Q) are two like parallel forces acting at A and B. Show that if they interchange positions, the point of application of the resultant is displaced a distance $\frac{P-Q}{P+Q}$ AB.

14. P, Q are like parallel forces. If P is moved parallel to itself through a distance x, show that the resultant of P and Q moves a distance $\frac{Px}{P+Q}$.

15. Two unlike parallel forces P and Q (P > Q) are acting at a distance d. If each force be increased by S, show that the resultant will be unaltered in magnitude but its point of application will be moved through a distance $\frac{S.d}{P-Q}$.

Answer

1. 24 N and 6 N	2. 30 N and 10 N	3. a) 3 N b) 5 : 3	4. a) 2 : 3 b) 18 cm
5. 280 N	6. 2m away from the stronger man	7. 1 N	
8. 50 N, 30 N	10. 20 N and 18 N		
11. a) 25 cm	b) The force varies inversely as the distance between his hand and his shoulder		

Introduction : Newton's Laws of Motion

Matter and motion are two inseparable fundamental concepts. Matter in motion and the forces that bring about the motion are linked together by three celebrated laws known as Newton's Laws of Motion, after the great mathematician, **Sir Isaac Newton** (1642–1727). These laws form the basis of what is known as *Dynamics*. Moreover, these laws are of fundamental importance in classical physics. Newton's laws are basically true in the case of bodies moving with speeds less than that of light. If the speed is greater than that of light, Newton's laws need some modification. This modification was provided by **Albert Einstein** (1879–1955) in his theory of '*Special Theory of Relativity*'.

Newton's First Law of Motion

Newton's first law of motion is usually stated in the following form:

Every body continues in its state of rest or of uniform motion in a straight line unless compelled by some external force to change that state.

Obviously, the law embodies two aspects (i) Property of inertia and (ii) Definition of force

i) Property of inertia

Common experience tells us that objects at rest shows no tendency to move by itself; and, similarly an object, already in motion, continues to move in a straight line. The first tendency of a material object is known as the *inertia of rest* and the second is called the *inertia of moton* of the object. Inertia of rest and inertia of motion of a body, when taken together, give rise to the notion of '*inertia of motion of a body*'. The following examples would help us to feel the existence of inertial property of a material object.

a) Common experience of falling backwards when a car in which we are standing loosely, suddenly starts, exhibits the property of inertia of rest of a body at rest.

b) The property of inertia of motion is exhibited by our experience of being thrown forward when we alight, without precaution, from a moving bus.

Examples cited above convince us that material objects possess certain amounts of inertia. The question how large or how little is the amount of inertia in a given body gives rise to the important notion of '*mass*'. Before we come to this point, we briefly explain the second aspect of Newton's first law of motion.

ii) Definition of force

In physical sciences, it has now become a common practice to accept the term 'Force' as one of the *undefinables* (i.e. *primitive term*) and use it to define other concepts. We, however, for practical purpose give a workable definition of force on the basis of Newton's first law of motion.

As explained earlier, the property of inertia of a body has no tendency of its own to change its state of rest or uniform motion. If it is not acted upon by some *external force*, it will remain either at rest or in uniform motion. That is, its original or natural state remains undisturbed or unbroken. A force is always necessary to cause any change in the state or uniform motion of a body. This is why we may define a force in the following way :

A force is that which acting on a body,changes or tends to change the state of rest or of uniform motion of the body.

The fact that inertial property of a body lies at the foundation of physical sciences, gave rise to the new name '*Principle of inertia*' or '*Law of inertia*' to Newton's first law of motion.

Mass and Momentum

The definition of force derived from Newton's first law of motion tells us that a force acting on a given body affects its property of inertia. Attempts to describe by how much or how little a given force acting on a given body changes its inertia, gave rise to one of the fundamental notions of physical sciences, namely, the notion of '*Mass*' of the body, usually denoted by italic m. We now proceed to do this.

To measure or describe by how much or how little a given force acting on a given body changes its amount of inertia content, we may assign to each body a *numerical quantity* (or factor or multiplier) that describes how much or how little it changes its state of rest or uniform motion per unit time (i.e. it accelerates) under the action of a given force. This quantity is called the '*mass*' of the body,and is denoted by the symbol m (italic).

Suppose a given force act on two bodies of masses m_1 and m_2, the accelerations of the two bodies are also different. Experiments of various types show that a body of smaller mass acquires a greater acceleration as compared with a body of larger mass. If a_1 and a_2 are the accelerations of two bodies of masses m_1 and m_2 under the action of the same force, the ratio $\frac{a_2}{a_1}$ may be used to compare the two masses. For definitions, the convention used is to say that the larger mass, m_1 say, is $\frac{a_2}{a_1}$ times the mass m_2 and write $m_1 = \frac{a_2}{a_1} m_2$

This equation may be used to measure the mass of a given body by comparison with a standard unit of mass. A standard unit of mass is called the kilogramme kg. It is the mass of a solid cylinder made of platinum-iridium alloy kept at the International Bureau of Weights and Measures at Sivres, near Paris.

Given two bodies with masses m_1 and m_2, the mass of the composite body formed by fastening them together can be measured as described above. It is actually found to be $m_1 + m_2$ at all times. Thus, mass is found to be additive and is directly related to the quantity of matter. This explains the use of the term '*mass*' by Newton to mean '*quantity of matter*'.

Having acquired the notion of mass, we now proceed to define a new concept called '*momentum*'. We recall that a given force acting on two bodies with different masses builds up a higher velocity on a lighter body than on the heavy one. Secondly, a force applied on a greater mass is greater than a force on a smaller mass to acquire the same velocity after travelling the same distance from rest.

In the second case, we say that the heavier body has greater quantity of motion or momentum than the lighter one. **Newton**, for the first time, defined momentum as follows :

The momentum of a moving body is defined as the product of its mass and its velocity. If m is the mass of the body and v is its velocity, then

$$\text{momentum} = \text{mass} \times \text{velocity}$$

The unit of momentum in the **SI** system is, therefore, 1 kilogram multiplied by 1 metre per second (i.e. kg ms^{-1})

Newton's Second Law of motion

The importance of the connection between force and momentum was expressed by Newton in his second law of motion in the following form:

The rate of change of momentum of a body is proportional to the impressed force and takes place in the direction in which the force acts.

Let us now see how this law provides us a method of measuring force. Suppose a force F acts on a body of mass m for a time t, and changes its velocity from u to v. The change in momentum of the body in time t is obviously

$$mv - mu$$

and, therefore, the rate of change of momentum is $\frac{mv - mu}{t}$

Then, by Newton's second law of motion,

$$F \propto \frac{mv - mu}{t}$$

or, $$F \propto m\frac{(v-u)}{t}$$

But, $$\frac{v-u}{t} = \frac{\text{change in velocity}}{\text{time}} = \text{acceleration}$$

$$= a \text{ say}$$

Thefore, $F \propto ma$

or, $F = k \times ma$ where k is a constant

This is the equation which enables us to define an absolute unit of force. If we take $m = 1$ kg and $a = 1$ m/s² = 1 ms^{-2}, the numerical value of F becomes 1 if we take $k = 1$. It is with this fact in mind, we define the unit of force as that force which acting on a body of unit mass produces a unit acceleration. We, then have k = 1; and therefore

$$F = ma$$

The SI unit of force is the force which produces an acceleration of 1 m/s² (or 1 ms^{-2}) when it acts on a mass of 1 kg. This unit of force is called the newton (N). Thus, when m is in kilograms and a in metres per sec. per sec., F is in newtons; and

$$F = ma \text{ (N)}$$

or, $$\text{Force} = \text{mass} \times \text{acceleration}$$

The CGS unit of force is dyne which is a force acting on a mass of 1 g (1 gram) produces an acceleration of 1 cm/sec²

The magnitude of a force may be defined as the number of units in the given force. Its direction is the direction of the acceleration produced by it. The force is completely known if its i) *point of application*, i.e., the point at which the force acts, ii) *magnitude* and iii) *direction*, are known.

Once a force is completely known, it can be geometrically represented by a straight line segment

i) drawn through the point representing the point of application
ii) drawn in the direction pointing the direction of force, and
iii) drawn so that the length of the line segment is proportional to the magnitude of the force.

Weight and Mass

We all know that material objects, large or small, around us are constantly *pulled* or *attracted* by earth. This pull is called the *gravitational attraction or force due to graity*. The space round the earth where the mass of an object experiences a gravitational pull or force due to gravity, is called the *gravitational field* of the earth. The weight of an object is defined as the force acting on it due to the gravitational pull or gravity. As usual, the force of gravity acting on a body of mass produces an acceleration. In the absence of resistance due to air, all material objects, regardless of their masses, fall with the same acceleration at the same point near the surface of the earth. In such a situation the objects are said to be *falling freely* under the action of gravity. The motion itself is given the name 'free fall'.

The acceleration of a body falling freely under the action of gravity is called the '*acceleration due to gravity*' and is denoted by the italic g to distinguish itself from the Roman letter g used to denote mass in grams. At or near the surface, its value is taken to be 9.8 ms^{-2} or 980 cms^{-2} approximately. It is, however, to be noted that the value of 'g' varies from place to place and depends on latitude and elevation. The value of g is slightly greater at the poles than at the equator (explain why ?).

For a body of mass m falling freely under the action of gravity, the force acting on it or its weight w, is given by

$$w = mg, \qquad \text{where } g \text{ is the acceleration due to gravity.}$$

If the mass of the body is in kilogram and its acceleration of free fall is in metres per second squared, the weight of the body will be in newtons. So, a body of mass 1 kg with $g = 9.8\ ms^{-2}$, the weight

$$\begin{aligned} w &= mg \\ &= 1 \times 9.8 = 9.8\ \text{N} \end{aligned}$$

Since acceleration due to gravity varies from place to place on the surface of the earth, the weight of a body having a certain quantity of mass differs from place to place. In other words, mass of a body is constant all over the earth, but weight is not.

Our earth is nearly six times larger than the moon as regards the content of mass. On the moon, pull due to moon i.e. the gravitational force of the moon is comparably small. Consequently, the value of g on the moon is 1.67 ms^{-2}. Man would therefore tend to 'float' while walking on the surface of the moon.

Momentum change and mass change

If a body of mass *m* changes its velocity from *u* to *v* in time *t* under the action of a force F, Newton's second law of motion may be expressed in the following form

$$F = \frac{mv - mu}{t}$$

$$\therefore \quad F \times t = mv - mu = \text{momentum change}$$

The quantity F × *t* (force × time) is called *impulse* of the force on the body. The unit of impulse is obviously Ns or $Kgms^{-1}$.

We further note that

$$F = \frac{mv - mu}{t}$$

$$= \frac{m}{t}(v - u) = \text{mass per second} \times \text{velocity change}$$

This form of the equation of motion of the body is useful in dealing with moving bodies whose masses are uniformly increasing or decreasing as in the cases of rain-fall and rocket firing.

Newton's third law of motion

Isolated material object is *not* known in this universe. Of various objects around us, some are near one another while other quite far away. In any case, we are free to think of two objects regardless of the distance separating them. We know that each of the objects has some mass. But, mass being a measure of the inertia of the object, is inseparable with the force associated with it. Once we have two objects, the forces associated with the masses of the objects mutually interact. The two interacting forces are called '*action*' and '*reaction*'. We have no hard and fast rule to name one as the action and the other the 'reaction' to it. If we call one of them action, the other is called its reaction. One should simply remember that action and reaction act on different objects.

Newton, after observing many phenomena involving action and reaction, summed up his experience in the following way

'*To every action, there is an equal and opposite reaction*'.

This statement is known as Newton's third law of motion.

By the way of illustration, we give below some examples.

(i) Consider a book placed on a table. The book presses vertically downwards on the table with a force equal to its weight. If this weight is the only force, the book would have gone through the table. But this did not happen. Its motion must therefore have been checked by an equal and opposite force, called the reaction of the table acting vertically upwards along the same line of action.

(ii) When a bullet is fired from a rifle with a certain force (action), the rifle is pushed back by an equal and opposite force (reaction).

(iii) If a heavy block is suspended by a string (light and inextensible), the string exerts a force (action) on the block to hold it and the block exerts an equal and opposite force (reaction) making it tight and inextensible.

Conservation of Linear Momentum

A body at rest has no momentum. If the body starts moving with a given velocity, it will have momentum whose direction is same as that of its velocity. So, it has both magnitude and direction. It is therefore a vector quantity. Moreover, it is linear in nature. If the body has a mass m and changes its velocity from u to v in time t and in the direction of a force F acting on it. Newton's second law of motion tells

$$F = \frac{mv - mu}{t}$$

In case, F = 0, we have $0 = mv - mu$

$$\therefore \quad mu = mv$$

In other words, whenever F = 0, the body moves with a constant linear momentum.

Instead of a single body, if we have two bodies of masses m_1 and m_2 under mutual interaction but nothing else, each body exerts a force on the other. This brings a change in momentum of each of the two bodies. According to Newton's third law of motion, the forces on the two bodies are always equal and opposite in direction. If the velocity of the first body changes from u_1 to v_1 in time t, the force associated wth mass m_1 is $\frac{m_1(v_1 - u_1)}{t}$

Similarly, if the velocity of the second body changes from u_2 to v_2 during the same time t, the force associated with mass m_2 is $\frac{m_2(v_2 - u_2)}{t}$

Hence, $$-\frac{m_1(v_1 - u_1)}{t} = \frac{m_2(v_2 - u_2)}{t}$$

or, $$-m_1v_1 + m_1u_1 = m_2v_2 - m_2u_2$$

i.e. $$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Thus, when two bodies interact with each other and no external forces act on the system, the total momentum of the system remains constant both in magnitude and direction.

The above statement is called the *Principle of conservation of Linear Momentum*, and is one of the most important principles of mechanics.

The principle of conservation of linear momentum can be extended to the case of more than two bodies moving under similar conditions.

As a typical example, we may consider the motion of a shot and a gun. Suppose m and M are the masses of the shot and the gun respectively. Also, assume that v and V are the velocities with which the shot leaves the muzzle of the gun and recoil of the gun which is free to move.

Then, $m(v - 0) = M(V - 0)$ (why ?)

or, $mv = MV$

i.e. motion of a shot = motion of a gun in a straight line.

Collision

Consider two bodies moving on the same straight line on a smooth horizontal table.

If they start from a point and move in opposite directions they will not meet or join together. In the same way if they move from different points with the same velocity and in the same direction, they will never come together. But if the two bodies move from two different points towards each other or one following the other has a greater velocity than the latter, they will meet or come into contact or collide at some point. Then, two possibilities are obvious:

(i) They stick together (*coalesce*) and continue to move as a unit after the collision with a common velocity, and

(ii) They bounce off one from the other and continue to move separately.

It is to be noted that a collision may be between a moving body and fixed object also, i.e. a ball falling on a ground. In the first case, we say that the collision is perfectly *inelastic* and in the second case it is said to be *elastic*. In the case of perfectly elastic collision, the colliding bodies are first deformed as soon as they come in contact and then regain their original form.

In particular, suppose two perfectly inelastic bodies of masses m_1 and m_2 moving along the same line with velocities v_1 and v_2 respectively coalesces and moves together with a common velocity V. Then, by the principle of conservation of linear momentum, we have

$$m_1v_1 + m_2v_2 = (m_1 + m_2)\,V$$

or,
$$V = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$$

Worked Out Examples

Example 1

A heavy particle of mass 0.50 kg is hanging from a string fixed with the roof. Find the pull of the earth on the particle. Given that $g = 9.8\ m/s^2$.

Solution :

The pull of the earth on the particle = mg

$$= 0.50\ kg \times 9.8\ m/s^2$$

$$= 4.9\ N \text{ vertically downwards}$$

Example 2

The pull of the earth on a mass of 5 kg is 49 N. What is the acceleration due to gravity.

Solution :

The pull F of the earth on a mass of m kg is given by

$$F = mg$$ where g is the acceleration due to gravity.

So, in our case

$$49\ N = 5\ kg\ g$$

or, $g = \frac{49 \text{ N}}{5 \text{ kg}} = 9.8 \text{ N kg}^{-1}$

$= 9.8 \text{ (kg m s}^{-2}\text{) kg}^{-1}$

$= 9.8 \text{ ms}^{-2}$

Example 3

The pull of the earth on a body is 49 N. If the acceleration due to gravity is $g = 9.8$ m/sec², find the mass of the body.

Solution :

The pull F of the earth on a body of mass m is given by

$F = mg$ where g is the acceleration due to gravity.

Here, $49 \text{ N} = m\, 9.8 \text{ m/sec}^2$

or, $49 \text{ kg} \times \text{m/sec}^2 = m\, 9.8 \text{ m/sec}^2$

or, $49 \text{ kg} = 9.8\, m$

$\therefore$ $m = 5 \text{ kg}$

Example 4

A person of mass 50 kg jumped from a certain height and landed on a ground with a velocity of 10 ms⁻¹. He is brought to rest in one-tenth of a second. What is the force acting on the person ?

Solution :

If the force F acting on the mass m changes the velocity from u to v, then

$$F = ma = m\,\frac{v-u}{t}$$

Here, $F = 50 \text{ kg}\, \frac{0 - 10 \text{ ms}^{-1}}{1/10 \text{ s}}$

$= -5000 \text{ kg ms}^{-2} = -5000 \text{ N}$

The negative sign shows that the force is acting against the direction of the moving person.

Alternative method

Change in momentum of the man $= m \times v = 50 \times 10$

Impulse of a force $= F \times t = F \times \frac{1}{10}$

Since impulse of a force = Change in momentum

So, $F \times \frac{1}{10} = 50 \times 10$

$\therefore$ $F = 5000 \text{ N}$

Example 5

A body of mass 5 kg falling from a certain height is brought to rest after hitting the ground with a speed of 10 ms^{-1}. If the resistance force of the ground is 500 N, find the duration of contact.

Solution :

Let F = resistance force = –500 N

m = mass of the body = 5 kg

u = the speed with which the body hits the ground = 10 ms^{-1}

v = the final speed = 0

t = required time interval

Using $\quad F = m\dfrac{v-u}{t}$, we have

$$-500 \text{ N} = 5 \text{ kg}\ \frac{0 - 10 \text{ ms}^{-1}}{t}$$

or, $\quad -500 \text{ kg ms}^{-2} = \dfrac{-50 \text{ kg ms}^{-1}}{t}$

or, $\quad 10 \text{ s}^{-1} = \dfrac{1}{t}$

$\therefore \quad t = 0.1$ s

Alternative Method

Change in momentum of the body = $m \times v = 5 \times 10$

Impulse of a force $= F \times t = 500.t$

$\therefore \quad 500t = 5 \times 10$

$\therefore \quad t = \dfrac{1}{10} = 0.1$ sec.

Example 6

A body of mass 0.5 kg, and initially at rest, is subjected to a force of 2 newtons for 1 sec. Find the velocity acquired during the second.

Solution :

Here, Mass of the body $= m = 0.5$ kg

Initial velocity $= u = 0$ ms^{-1}

Force applied = F = 2 N

Time of action $= t = 1$ sec

Final velocity $= v = ?$

Using $\quad F = \dfrac{m(v-u)}{t}$, we have

$$2\text{ N} = \frac{0.5\text{ kg} \times v}{1\text{ s}}$$

or, $2\text{ kg ms}^{-2}\text{ s} = 0.5\text{ kg } v$

$\therefore$ $v = \left(\frac{2}{0.5}\right)\text{ ms}^{-1}$

$= 4\text{ ms}^{-1}$

Alternative method

Change in momentum $= m.V = 0.5V$

Impulse of a force $= F \times t = 2 \times 1$

$\therefore$ $0.5\text{ V} = 2$

$\therefore$ $\text{V} = 4\text{ m/s}$

Example 7

Suppose a rocket moving upwards in the air loses its mass at the rate of 0.1 kgs^{-1} as its fuel burns. If the velocity of the rocket changes from 110 ms^{-1} to 10 ms^{-1}, find the force retarding the motion of the rocket.

Solution :

Suppose F = retarding force,

r = change in mass per sec

u = initial velocity

v = final velocity

Using $F = m\frac{v-u}{t} = \frac{m}{t}(v-u)$

$= r(v-u)$

We have

$F = 0.1\text{ kgs}^{-1}(110-10)\text{ ms}^{-1}$

$= 10\text{ kg ms}^{-2} = 10\text{ N}$

Example 8

Water flowing from a horizontal hose-pipe at a speed of 5 ms^{-1} comes to rest on hitting a wall normally. If the force exerted on the wall is 10 N, find the quantity of water flowing per second.

Solution :

Let u = initial velocity of water = 5 ms^{-1}

v = final velocity of water = 0

F = force exerted on the wall = 10 N

r = quantity of water flowing per sec.

Then using

$$F = \frac{m}{t}(v-u) = r(v-u)$$

We have

$$10\ N = r(5-0)\ ms^{-1}$$

or, $10\ kg\ ms^{-2} = 5\ r\ ms^{-1}$

∴ $r = 2\ kg\ s^{-1}$

Example 9

Rain drops falling vertically on a flat roof at the rate of 0.2 kg s^{-1} come to rest after hitting the roof. If the resistance force of the roof is 2 N, find the velocity of rain.

Solution :

Suppose r = mass of water falling per sec = 0.2 kgs^{-1}

u = velocity before hitting the roof = ?

v = velocity after hitting the roof = 0

F = force on the roof = –2 N (why ?)

Using, $F = \frac{m}{t}(v-u) = r(v-u)$

We have

$$-2\ N = 0.2\ kgs^{-1}(0-u)$$

or, $-2\ kg\ ms^{-2} = -0.2\ kg\ s^{-1}\ u$

∴ $u = 10\ ms^{-1}$

EXERCISE

1. Find the impulse of a force acting on a mass of 50 kg producing a change in velocity from 5 ms^{-1} to 15 ms^{-1}.
2. A girl on a bicycle, total mass 50 kg, has a velocity of 1 m/s and paddling faster for 5 seconds, the velocity increases to 3 ms^{-1}. Find the average force exerted.
3. A body of mass 50 kg falling from a certain height is brought to rest after striking the ground with a speed of 5 ms^{-1}. If the resistance force of the ground is 500 N, find the duration of contact.
4. A cart is pushed on a frictionless smooth plane with an average force of 20 N for 5 seconds. If the cart with mass 50 kg is at rest in the beginning, find the velocity acquired by the cart.
5. a) A rocket expels gas at the rate of 0.5 kgs^{-1}. If the velocity of the gas expelled is 200 ms^{-1}, what is the force produced by the rocket?

b) Sand allowed to fall vertically at a steady rate hits a horizontal floor with a speed 0.05 ms^{-1}. If the force exerted on the floor is 0.005 N, find the mass of sand falling per second.

c) Suppose a rocket moving upwards in the air loses its mass as its fuel burns. If the velocity of the rocket is reduced from 210 ms^{-1} to 110 ms^{-1} by a force of 100 N due to earth, find the mass of fuel burnt per second.

d) Suppose a steady mass of rain falls vertically on a flat roof at the rate of 0.5 kgs^{-1} and then comes to rest. If the force on the roof is 2.5 N, find the velocity of raindrops just before hitting the roof.

Answer

1. 500 kg ms^{-1}	2. 20 N	3. 0.5 s	4. 2 ms^{-1}
5. a) 100 N,	b) 0.1 kgs^{-1}	c) 1 kg	d) 5 ms^{-1}

Worked Out Examples

Example 10

A person of mass 50 kg who is jumping from a height of 5 metres is brought to rest in one tenth of a second after landing on the ground. Taking $g = 9.8$ ms^{-2}, find the resistance force of the ground.

Solution :

Here mass of the person = m = 50 kg

Height from which he jumped = h = 5 m

Acceleration due to gravity = g = 9.8 ms^{-2}

Suppose v = velocity of the person just before landing

time of contact = t = 1/10 s

Then, from $v^2 = 2gh$

we have, $v = \sqrt{2 \times 9.8 \times 5} = \sqrt{98} = 7\sqrt{2}$ ms^{-1}

Using Force $= \text{mass} \times \dfrac{\text{velocity change}}{t}$

We have $F = 50 \times \dfrac{7\sqrt{2} - 0}{1/10} = 3500\sqrt{2}$ N

Example 11

An aeroplane of mass 2 megagram (2 Mg) lands on the runway with a velocity of 50 ms^{-1} and decelerates to a velocity of 20 ms^{-1} in 3 seconds. Find the resistance of the ground.

Solution :

Here, mass of the aeroplane = 2 Mg = 2×10^6 kg

Landing velocity of the plane = u = 50 ms^{-1}

Velocity after 3 seconds = $v = 20\ ms^{-1}$

Using $\quad v = u + at$

where a is the retardation, we have

$$20 = 50 + a.3$$

or, $\quad a = -10\ ms^{-2}$

Hence, the resistance of the ground against the aeroplane

$= \text{mass} \times \text{acceleration}$

$= 2 \times 10^6\ kg \times 10\ ms^{-2}$

$= 2 \times 10^7\ N$

Example 12

A force equal to 4.9 N acting on a body changes its velocity from 3 ms^{-1} to 5 ms^{-1} when it covers a distance of 16 m. Find the mass of the body.

Solution :

Here, force F = 4.9 N

Initial velocity = $u = 3\ ms^{-1}$

Final velocity = $v = 5\ ms^{-1}$

Distance covered = $s = 16$ m

Mass of the body = m = ?

Using $\quad v^2 = u^2 + 2as$ where a is the acceleration of the body,

We have $\quad 5^2 = 3^2 + 2.a.16$

$\therefore \quad a = 0.5\ ms^{-2}$

Again, using $\quad F = ma$, we have

$4.9\ kg\ ms^{-2} = \text{mass} \times 0.5\ ms^{-2}$

$\therefore \quad m = 9.8$ kg

Example 13

A ball of mass 0.49 kg is thrown vertically upwards with an initial velocity of 30 ms^{-1}. It reached a height of 45 m in 3 s. Find the pull of the earth on the body.

Solution :

Here, Mass of the ball = 0.49 kg

Initial velocity = $u = 30\ ms^{-1}$

Height attained = 45 m

Time taken = t = 3s

The pull of the earth = F

Using $S = ut + \frac{1}{2}at^2$ where a is the deceleration due to earth.

We have $45 = 30 \times 3 + \frac{1}{2}a\,9$

$\therefore \quad a = -10 \text{ ms}^{-2}$

Hence, pull of the earth on the body = mass × acceleration

$= 0.49 \times 10$

$= 4.9 \text{ kg ms}^{-2} = 4.9 \text{ N}$

Example 14

Find the velocity of a 4 kg shot that will just penetrate through a wall 16 cms. thick, the resistance being 4 metric tonnes weight. ($g = 9.8 \text{ ms}^{-2}$)

Solution :

Resistance force = – 4 metric tonnes wt.

$= -4 \times 1000 \times g$ N

Let a be the retardation. Then,

$P = -ma$

or, $-4 \times 1000\, g = -4.a$

$\therefore \quad a = 9800 \text{ m/sec}^2$

Let u be the velocity of the shot which can just penetrate a wall of thickness 16 cms. i.e. 0.16 metre. Then,

$0 = u^2 - 2as$

or, $0 = u^2 - 2 \times 9800 \times 0.16$

or, $u^2 = 2 \times 9800 \times 0.16$

$\therefore \quad u = 56$ m/sec

Example 15

A force acts on a body of mass 5 gms at rest for the first 4 secs, in 8 secs from the start, the body is found to be 480 m from the starting point; find the value of the force.

Solution:

Let F be the force and a be the acceleration. Then,

$F = m.a$

$\Rightarrow \quad F = \frac{5}{1000}a$

$\therefore \quad a = 200$ F

V = velocity at the end of 4 secs

$= at = 200F \times 4 = 800$ F

which will be the initial velocity for the next (8–4) sec i.e. 4 sec.

Using $S = ut + \frac{1}{2}at^2$, we have

$$480 = 800\text{ F} \times 4 + \frac{1}{2} \times 200\text{ F} \times 16$$

$$\Rightarrow \quad 480 = 3200F + 1600F$$

$$\Rightarrow \quad 4800F = 480$$

$$\therefore \quad F = \frac{480}{4800} = \frac{1}{10}\text{ N}$$

Example 16

A man on a lift moves (i) up (ii) down with an acceleration of 2 ms^{-2}. In each case, calculate the reaction of the floor on a man of mass 50 kg standing on the lift. ($g = 9.8\ ms^{-2}$)

Solution :

Two forces act on a mass.

a) The weight mg of the man acting vertically downwards

b) The reaction R of the floor on a man acting vertically upwards

(i) Since the lift is moving upwards, so the

resultant upward force = $R - mg$

By Newton's second law,

$R - mg = ma$ where a is the acceleration of the lift.

or, $R = m(g + a)$

$= 50(9.8 + 2) = 590$ N

(ii) Since the lift is moving downwards, so the

resultant doward force = $mg - R$

By Newton's second law,

$mg - R = ma$

or, $R = m(g - a)$

$= 50(9.8 - 2) = 390$ N

Example 17

A stone of mass 1 kg falls from the top of a vertical cliff. After (i) falling for 3 seconds (ii) descending 800 cms, it reaches the foot of the cliff and penetrates 25 cms into the sand. Find the resistance offered by the sand. ($g = 9.8\ ms^{-2}$)

Solution :

(i) Let v be the velocity at the end of 3 seconds. Then,

$v = gt = 9.8 \times 3 = 29.4$ m/sec.

Now, with this initial velocity, the stone enters into the sand. Let a be the retardation of the stone when it enters into the sand. After penetrating 25 cms; i.e. 0.25 m, it comes to rest.

$$0 = v^2 - 2as$$

or, $(29.4)^2 = 2 \times a \times 0.25$

$$\therefore \quad a = \frac{29.4 \times 29.4}{2 \times 0.25} = 1728.72 \text{ m/sec}^2$$

While the stone is within the sand, two forces act on it.

(i) The weight mg of the stone acting vertically downwards.

(ii) The resistance R offered by the sand acting vertically upwards.

The resultant upward force = $R - mg$

Now, by Newton's second law,

$$R - mg = ma$$

or, $R = m(g + a)$

$= 1 \times (9.8 + 1728.72)$

$= 1738.52 \text{ N} = 177.4 \text{ kg.wt.}$

(ii) Let v be the velocity after descending 800 cms. i.e. 8 m. Then

$$v^2 = 2gh = 2 \times g \times 8 = 16g$$

$$\therefore \quad v = \sqrt{16g}$$

Now, with this initial velocity, the stone enters into the sand. Let a be the retardation. The stone comes to rest after penetrating 25 cms i.e. 0.25 m.

$$0 = v^2 - 2a \times 0.25$$

or, $16g = 0.5a$

$$\therefore \quad a = 32\,g$$

Let R be the resistance force offered by the sand on the stone, then the resultant upward force is $R - mg$

By Newton's secon law, $R - mg = ma$

or, $R = m(g + a)$

$= 1(g + 32g) = 33\text{ g N} = 33 \text{ kg wt}$

EXERCISE

1. State Newton's Laws of motion and explain how from the first law we obtain a definition of the force and from the second a measure of force.

2. How large a force is required to bring a 1000 kg car moving with a velocity of 100 ms^{-1} to rest at a) a distance of 1000 m; (b) in 20 second.

3. A constant force of 10 N acting on an object reduces its velocity from 15 ms^{-1} to 5 ms^{-1} in 2s. Find the mass of the object.

4. A 12.0 g bullet is accelerated from rest to a speed of 700 ms^{-1} as it travels 20 cm in the barrel of a gun. Assuming the acceleration to be uniform, find how large the accelerating force was ?

5. A bullet moving at 250 ms^{-1} penetrates 5 cm into a tree trunk before coming to rest. Assuming that the force exerted by the tree trunk is uniform, find its magnitude. Mass of the bullet is 10 g.

6. A force of 25 newtons acts on a mass of 0.50 kg starting from rest. Find
   a) the acceleration in ms^{-2}
   b) the final velocity after 20 s.
   c) the distance moved in 20 s.

7. A train of mass 327 tonnes moves at the rate of 108 kmh^{-1}, after the steam is shut off, it is brought to rest by the brakes in 50 m. Find the force exerted, assuming it to be uniform and assuming $g = 9.81\ ms^{-2}$.

8. On turning a corner, a motorist rushing at 36 km/hr finds a child 51 m ahead, he stops the car within one metre of the child by the application of the brakes. Calculate the retarding force and the time required to stop the car. The total mass of the car and the passenger = 2000 kg.

9. A force equal to a weight of 1 kg acts on a body continuously for 10 secs and causes it to distance 10 metres in that time, find the mass of the body.

10. Due to the application of a force of 12 kg.wt. a body of mass 4.9 kg changes its velocity from 12 m/s to 20 m/s. Find the distance through which the body describes.

11. Find the velocity of a 4 kg shot that will just penetrate through a wall 25 cms thick, the resistance being 36 tonnes wt. ($g = 9.8\ ms^{-2}$)

12. A body of mass 1 kg is falling under gravity at the rate of 28 ms^{-1}. What is the uniform force that will stop it – (i) 0.1 second, (ii) 20 cm. ($g = 10\ ms^{-2}$) Instead of falling under gravity if the body is moving at the rate of 28 ms^{-1} along a horizontal line, what will be the force required in above two cases.

13. A mass of 5 kg falls 300 cms from rest and is then brought to rest by penetrating 30 cms into some sand; find the average thrust of the sand on it.

14. A particle of mass 15 kg falls from a height of 18 metres and penetrates into some sand. If the average resistance offered by the sand is equal to a force of 150 kg.wt., find how far it penetrates into the sand.

15. A mass 'm' kg is acted on by a constant force 'P' kg wt and in 't' secs; it moves a distance of x metres from rest and acquires a velocity of v m/s. Show that $x = \dfrac{gt^2P}{2m} = \dfrac{v^2m}{2gP}$.

16. A balloon is rising with an acceleration f. Prove that the fraction of the weight of the balloon which must be emptied out of the balloon in order to double the acceleration is $\dfrac{f}{g+2f}$.

Answers

2. (a) 5000 N, (b) 5000 N 3. 2 kg 4. 14700 N
5. 6250 N 6. a) 50 ms^{-2} b) 1000 ms^{-1} c) 10000 m
7. 300 tonnes wt 8. 2000 N, 10 sec 9. 49 kg. 10. $5\frac{1}{3}$ m 11. 210 ms^{-1}
12. 29 kg wt, 197 kg wt; 28 kg wt; 196 kg wt 13. 55 kg wt 14. 2 m

Worked Out Examples

Example 18

A body of mass 50 kg falling from a certain height acquires a velocity of 10 ms^{-1} just before hitting the ground, and then comes to rest in one tenth of a second. Find the momentum of the mass before hitting the ground and also the impulse of the force on the mass and the force.

Solution :

Here, mass of the body = 50 kg

velocity v before hitting the ground = 10 ms^{-1}

velocity after $\frac{1}{10}$ s = 0

So, momentum of the body before hitting the ground

= mass × velocity

= 50 kg × 10 ms^{-1}

= 500 kg ms^{-1}

momentum after impact = 50 kg × 0 = 0

So, change in momentum = 500 kg ms^{-1} – 0

= 500 kg ms^{-1}

But impulse = Force × time

= Change in momentum = 500 kg ms^{-1}

So, Force = $\frac{500 \text{ kg ms}^{-1}}{1/10 \text{ s}}$

= 5000 kg ms^{-2} = 5000 N

Example 19

The value of a cylinder containing 18 kg of compressed gas is opened and the cylinder empties in 1 min 30 s. If the gas issues from the exit nozzle with an average velocity of 25 ms^{-1}, find the force exerted on the cylinder.

Solution :

Here, the velocity of gas changes from rest to 25 ms^{-1}

Change in momentum = mass × change in velocity

$$= 18 \text{ kg} \times 25 \text{ ms}^{-1} = 18 \times 25 \text{ kg ms}^{-1}$$

Time taken = 1 min 30 sec = 90 s

Since, the force required to push the gas out of the cylinder

$$= \text{mass} \times \text{acceleration}$$

$$= \text{mass} \times \frac{\text{change in velocity}}{\text{time}}$$

$$= \frac{\text{change in momentum}}{\text{time}}$$

Average force on gas $= \dfrac{18 \times 25}{90}$ newtons = 5 N

By Newton's third law of motion, an equal reaction force is exerted on the cylinder. Thus, average force on the cylinder = 5 N

Example 20

A bullet of mass 15 g is fired from a rifle of mass 3 kg with a velocity of 100 kmh^{-1}. Find the velocity of recoil of the rifle.

Solution :

We know, mass of bullet × muzzle velocity

= mass of the rifle × recoil velocity

so, $15 \times 100 = 3000 \times v$, where

v is the recoil velocity

$$\therefore \quad v = \frac{1500}{3000} = 0.5 \text{ km h}^{-1}$$

Example 21

A bullet of mass 8 g is fired horizontally into a block of wood of mass 5 kg and stikes in it. The block, which is free to move, gains a velocity of 50 cms^{-1}. Find the initial velocity of the bullet.

Solution :

Before impact momentum of the bullet = mass × velocity of bullet

$= 0.008 \text{ kg} \times v,$

where v is the velocity of the bullet

momentum of the block = 5 kg × 0 = 0

$\therefore$ momentum before impact = 0.008 kg v + 0

After impact, momentum of the system (block + bullet)

$= (5 + 0.008) \text{ kg} \times 0.5 \text{ ms}^{-1}$

Since momentum of the system before impact

= momentum of the system after impact

or, $0.008 \text{ kg } v = 5.008 \text{ kg} \times 0.50 \text{ ms}^{-1}$

So, $v = 313 \text{ ms}^{-1}$

Example 22

An object of mass 2 kg is moving in the +ve x–direction with a velocity of 3 ms^{-1} while an object of mass 1 kg is moving in –ve x direction with a velocity of 4 ms^{-1}. They collide head on and stick together. Find their common velocity after collison.

Solution :

By the law of conservation of linear momentum, we have

momentum before impact = momentum after impact

So, $(2 \text{ kg})(3 \text{ ms}^{-1}) + (1 \text{ kg}) \times (-4 \text{ ms}^{-1}) = (3 \text{ kg})\, v$

where v is the common velocity

$\therefore \quad 3v = 2 \text{ ms}^{-1}$

$\therefore \quad v = \frac{2}{3} \text{ ms}^{-1}$

Example 23

A body of mass 8 kg, moving with a velocity of 6 metre/sec meets a body of mass 24 kg, moving with a velocity of 2m/sec in the same direction as the first; if after the impact they coalensce into one body, show that the velocity of the compound body is 3 m/sec.

If they were moving in opposite direction, show that after impact the compound body is at rest.

Solution :

Here, $m_1 = 8$ kg, $u_1 = 6$ m/sec

$m_2 = 24$ kg, $u_2 = 2$ m/sec

Let v be the velocity of the combination. Then using the principle of conservation of linear momentum,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\, v$$

or, $8 \times 6 + 24 \times 2 = (8 + 24)\, v$

or, $96 = 32\, v$

$\therefore \quad v = 3$ m/sec

Again if the two bodies move in opposite direction,

$$m_1u_1 + m_2u_2 = (m_1 + m_2)\, v$$

or, $8 \times 6 + 24 \times (-2) = (8 + 24)\, v$

$\therefore \quad v = 0$

$\therefore$ after impact, the combination is at rest.

Example 24

A shot of 400 kg is projected from a 40 metric tons gun with a velocity of 600 metre/sec; find the velocity with which the gun would commence to recoil, if free to move in the line of projection.

Solution :

Here, m = mass of shot = 400 kg

v = velocity of the shot = 600 m/sec

M = mass of the gun = 40 metric tons = (40 × 1000) kg

V = velocity of the gun = ?

Momentum of the shot = mv = 400 × 600

Momentum of the gun = MV = 40 × 1000 × V

Momentum of the shot = Momentum of the gun (in magnitude)

or, 400 × 600 = 40 × 1000 × V

∴ V = 6 m/sec

Example 25

A cricket ball of mass 0.2 kg moving with a velocity of 20 ms^{-1} is brought to rest by a player in 0.1 second. Find the impulse to the ball and the average force applied by the player.

Solution :

Here, m = mass of the ball = 0.2 kg

u = initial velocity of the ball = 20 ms^{-1}

v = final velocity of the ball = 0

t = time duration = 0.1 sec

Impulse of a force = mass × change in velocity

= 0.2 × 20

= 4 MKS units of impulse

Let R be the average force applied by the player

Impulse of a force = R × t = R × 0.1

∴ R × 0.1 = 4

∴ R = 40 N

Example 26

A glass marble, whose mass is 1/10 kg, falls from a height of 3.6 metre and rebounds to a height of 2.5 metre, find the impulse and the average force between the marble and the floor if the time during which they are in contact be 1/10 of a second. (g = 9.8 ms^{-2})

Solution :

Let v be the velocity of the glass marble just before hitting the ground.

Then, $v^2 = 2gh = 2 \times 9.8 \times 3.6$

$\therefore \quad v = 8.4\ ms^{-1}$

Let u be the velocity of the glass marble just after hitting the ground. Then,

$$0 = u^2 - 2gh$$

or, $u^2 = 2gh = 2 \times 9.8 \times 2.5$

$\therefore \quad u = 7.0\ ms^{-1}$

Change in velocity = 8.4 – (–7) = 15.4

Impulse $= \frac{1}{10} \times 15.4 = 1.54$ MKS units of impulse.

Let R be the average force between the marble and the floor which acts upwards. The resultant upward force $= R - mg = R - \frac{1}{10}g$

$$\text{Impulse} = \left(R - \frac{1}{10}g\right) \times \frac{1}{10}$$

$$\therefore \quad \left(R - \frac{1}{10}g\right) \times \frac{1}{10} = 1.54$$

or, $R - \frac{1}{10}g = 15.4$

or, $R - 0.98 = 15.4$

$\therefore \quad R = 15.4 + 0.98 = 16.38$ N

EXERCISE

1. A force of 12 N acts for 5 s on a mass of 2 kg. What is the change in momentum of the mass ? What would be the change in momentum of a mass of 10 kg under the same condition ?

2. A body of mass 0.5 kg and initially at rest, is subjected to a force of 2 newtons for 1 sec. Calculate the change in momentum of the body.

3. A bullet of mass 0.006 kg travelling at 120 ms^{-1} penetrates deeply into a fixed target, and is then brought to rest in 0.01 s. Find
   a) change in momentum of the bullet
   b) the average retarding force exerted on the bullet
   c) the distance of penetration of the target.

4. An inflated balloon contains 2.0 g of air which is allowed to escape from a nozzle at a speed of 4.0 ms^{-1}. Assuming that the balloon deflates at a steady rate in 2.5 s, what is the force exerted on the balloon ?

5. A bullet of mass 10 g is fired from a rifle of mass 1000 g with a velocity of 50 kmh^{-1}. Find the velocity of the recoil of the rifle.

6. A ball of mass 0.1 kg moving with a velocity of 6 ms^{-1}, collides directly with a ball of mass 0.2 kg at rest. Calculate their common velocity if both balls move together in the same direction.

7. Two bodies of masses 8 and 4 kg move along the x-axis in opposite directions with velocities 11 ms^{-1} and –7 ms^{-1} respectively. They collide and stick together. Find their velocity just after collision.

8. A 10 g bullet is fired from a kilogram gun suspended to move freely. This bullet now enters a block of wood of mass 990 g. If the speed of the bullet is 500 ms^{-1}, find the speed of the gun and the common velocity of the wood.

9. A gun of mass 400 kg fires a shot of mass 3 kg, with a velocity of 200 ms^{-1}, find the constant force which acting on the gun would stop it after a recoil of 2.5 metres.

10. A shot of 2 kg is discharged by a gun of mass 400 kg with a velocity of 800 m/sec. Find the constant force which would be required to stop the recoil of the gun in (i) 2 metres, (ii) $1\frac{1}{4}$ sec.

11. A gun of mass 1 metric tonne, fires a shot of mass 14 kg and recoils up smooth inclined plane, rising to a height of 1.6 m, find the initial velocity of the projectile.

12. A cricket ball of mass 150 g is moving with a velocity of 12 ms^{-1} and is hit by a bat so that the ball is turned back with a velocity of 20 ms^{-1}. The force of the blow acts for 0.01 s. Find the impulse and the average force exerted on the ball by the bat.

13. A gun of mass 40 metric tonnes resting on an incline of 3 in 5, fires a shot of 100 kg horizontally with a velocity of 700 m/sec. Find the velocity of the recoil of the gun and the distance it moves up the incline before coming to rest. ($g = 9.8$ ms^{-2})

A. List three applications of Newton's law of motion and relate them in our daily life cases.

Answers

1. 60 kg ms^{-1}, 300 kg ms^{-1} 2. 2 Ns
3. a) 0.72 Ns, b) 72 N, c) 0.6 m 4. 0.0032 N 5. 0.5 kmh^{-1}
6. 2 ms^{-1} 7. 5 ms^{-1} 8. 5 ms^{-1}, 5 ms^{-1}. 9. 180 N 10. (i) 1600 N, (ii) 1280 N
11. 400 m/s 12. 4.8 MKS units, 480 N 13. 1.4 ms^{-1}, $\frac{1}{6}$ m.

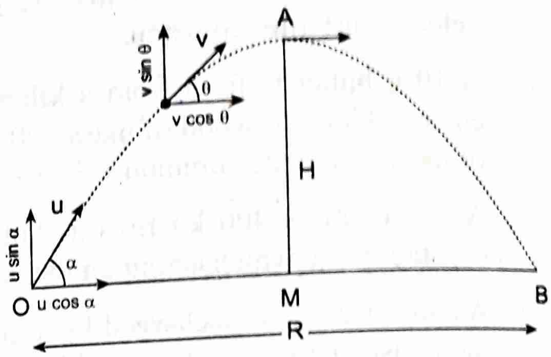
Introduction : Projectiles

An object, thrown vertically upwards with a given velocity, begins to fall vertically downwards after reaching the highest point (i.e. after the velocity has reduced to zero). If the object is supposed to be near the surface of the earth and the resistance due to air is neglected, the acceleration due to gravity is then almost constant. It is directed vertically downwards and its magnitude is $g = 9.8$ ms^{-2} (or 10 ms^{-2} approximately).

If the object is thrown obliquely (i.e. at an angle α to the horizon) with a certain velocity near the surface of the earth, it moves along a curved path. At any point on the path, the only effective acceleration is the acceleration due to gravity. As said above, it is almost constant and directed downwards in this case also.

Motion of projectiles

A *projectile* is any object having a given initial velocity and moving along a path determined by the force of gravity and frictional resistance of the air. In an idealized situation, the object is assumed so small that it may be regarded as a particle. The resistance due to air is neglected, so that the particle may be considered to be moving in vacuum. Lastly, the particle is supposed to be close to the surface of the earth so that the acceleration due to gravity remains almost constant. In our further discussion, unless otherwise stated, we consider only such an idealized model of physical situation.



A particle thrown obliquely near the surface of the earth is called a *projectile*, the path along which the projectile moves, is called its *trajectory*. The angle α which the direction of projection makes in the plane of projection with the horizontal plane through the point of projection is called the *angle of projection*; the time taken by the particle to come back to the horizontal plane again is called the *time of flight*; and the distance between the point of projection and the point where the particle strikes the horizontal plane again is called the *range* or *horizontal range*.

Let a particle be projected from a point O with an initial velocity u and making an angle α with the horizontal plane through the point of projection O. Further, suppose v is the velocity of the projectile at some point P on the trajectory OAB and inclined to the horizontal OB at an angle θ.

a) Velocity at any height

The vertical component of u is $u \sin \alpha$, the acceleration $a = -g$. If t is the time taken to reach the point P, then the vertical component of v is

$$v \sin \theta = u \sin \alpha - gt \qquad \text{......... (1)}$$

Since, the acceleration due to gravity along the horizontal is zero

$$v \cos \theta = u \cos \alpha \qquad \text{......... (2)}$$

Squaring and adding the above equations, we have

$$v^2(\sin^2\theta + \cos^2\theta) = u^2(\sin^2\alpha + \cos^2\alpha) - 2u \sin \alpha\, gt + g^2t^2$$

or, $$v^2 = u^2 - 2u \sin \alpha\, gt + g^2t^2$$

$$= u^2 - 2g\left(u \sin \alpha\, t - \frac{1}{2} gt^2\right)$$

$$= u^2 - 2gy$$ where y is the vertical height of the particle above OB.

Also, $\dfrac{v \sin \theta}{v \cos \theta} = \dfrac{u \sin \alpha - gt}{u \cos \alpha}$

or, $\tan \theta = \dfrac{u \sin \alpha - gt}{u \cos \alpha}$

Thus, we have formulae for finding both magnitude and direction of velocity of a projectile at a given point or time.

b) Time to attain the highest point

At the highest point, the vertical component of the velocity, i.e. $v \sin \theta = 0$.

So, by $v \sin \theta = u \sin \alpha - gt$

We have

$$0 = u \sin \alpha - gt$$

or, $t = \dfrac{u \sin \alpha}{g}$

This is the time required to attain the greatest height.

c) Time of flight

The *time of flight*, i.e. the time taken by the projectile to reach the horizontal plane through the point of projection again, can be obtained from

$h = u \sin \alpha t - \dfrac{1}{2} gt^2$ by taking $h = 0$. Thus

$$t(u \sin \alpha - \frac{1}{2} gt) = 0$$

$\therefore$ $t = 0$ or $\dfrac{2u \sin \alpha}{g}$

but $t = 0$ implies it is at the point of projection itself. So, the required time of flight, usually denoted by T, is

$$T = \frac{2u \sin \alpha}{g}$$

d) Horizontal range

The *horizontal range* R i.e. the distance between the point of projection and the point where the particle strikes the horizontal again, is the distance travelled in time $T = \dfrac{2u \sin \alpha}{g}$, the *time of flight*, with a uniform horizontal velocity $u \cos \alpha$. Thus

$$R = OB = u \cos \alpha \, T$$

$$= u \cos \alpha . \frac{2u \sin \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}$$

e) Maximum horizontal range

The range R is maximum when sin 2α is maximum. This will happen if $\sin 2\alpha = 1$, i.e. when $\alpha = \frac{\pi}{4}$ or $45°$.

$\therefore$ the *maximum horizontal range* $= \frac{u^2}{g}$

f) Greatest height

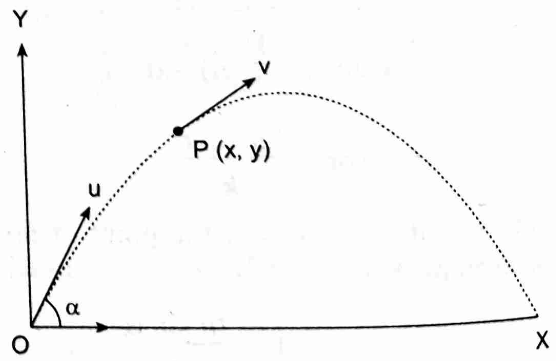
At the *greatest height* H, the vertical component of the velocity is zero.

So, $\quad 0 = u^2 \sin^2\alpha - 2gH$

or, $\quad H = \frac{u^2 \sin^2\alpha}{2g}$

Position of the projectile

Newton's second law of motion can be used to find the position of a projectile in its trajectory. For convenience, we select the origin as the point of projection, the x-axis as horizontal and the y-axis as the vertical. Then, there is no component of the force due to gravity in the x-direction. The y-component (i.e. vertical) of the force on the projectile is its weight, $-mg$ (why?). With this much preliminary idea, we now proceed to find the position of the projectile on its path.



Let P(x, y) be the position of a particle at the end of time t and projected from the point O on the xy-plane. Suppose that its initial velocity at O is u and is inclined at an angle α with OX. If F_x is the x-component of the force on the projectile at P(x, y) and F_y the y-component, then by Newton's second law of motion,

acceleration of the projectile a_x

$$= F_x/m = 0 \qquad \text{...... (i)}$$

and acceleration of the projectile a_y

$$= F_y/m = \frac{-mg}{m} = -g \qquad \text{...... (ii)}$$

where m is the mass of the particle.

Since, $\quad u_x = u \cos\alpha \quad$ and $\quad u_y = u \sin\alpha$

From (i), we notice that the horizontal velocity is always $u \cos\alpha$. So, the distance travelled along the x-axis in time t is

$$x = (u \cos\alpha)\, t \qquad \text{...... (iii)}$$

Also, the vertical height gained after time t is

$$y = (u \sin \alpha)\, t - \frac{1}{2}\, g\, t^2 \qquad \ldots\ldots \text{(iv)}$$

The four equations describe the motion and position of the projectile at any time t.

As explained in the previous article, we have

velocity v of the particle at $P(x, y) = u^2 - 2gy$

direction of v at $P(x, y) = \theta = \tan^{-1}\left(\frac{u \sin \alpha - gt}{u \cos \alpha}\right)$

The equations (iii) and (iv) give the position of the particle in terms of the parameter t.

Eliminating t between (iii) and (iv), we get

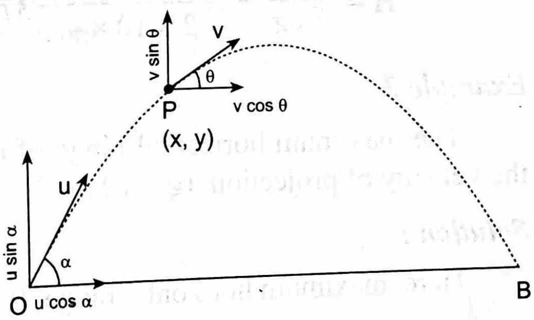
$$y = (\tan \alpha)x - \frac{g}{2u^2 \cos^2\alpha}\, x^2$$

Putting a and b for the fixed quantities $\tan \alpha$ and $\frac{g}{2u^2 \cos^2\alpha}$ respectively, we have

$$y = ax - bx^2 \qquad \ldots\ldots \text{(v)}$$

This is obviously the equation of a parabola. In other words, the *trajectory of a projectile is a parabola*.

Let a particle of mass m be projected from the origin O with an initial velocity u at an angle α with the x-axis. Suppose it is at the point $P(x, y)$ at the end of time t.



If F_x is the x-component of the force of gravity on the particle and a_x the corresponding component of the acceleration due to gravity, then, by Newton's second law of motion, $\quad a_x = \frac{F_x}{m} = 0$

Similarly, if F_y is the y-component of the force of gravity on the particle and a_y the corresponding component of the acceleration due to gravity, then

$$a_y = \frac{F_y}{m} = \frac{-mg}{m} = -g$$

Let v be the velocity of the particle at $P(x, y)$ and θ its inclination to the horizontal

Then, $\quad a_x = \frac{v \cos \theta - u \cos \alpha}{t} = 0$

or, $\quad v \cos \theta = u \cos \alpha \qquad \ldots\ldots \text{(i)}$

and $\quad a_y = \frac{v \sin \theta - u \sin \alpha}{t} = -g$

or, $\quad v \sin \theta = u \sin \alpha - gt \qquad \ldots\ldots \text{(ii)}$

Worked Out Examples

Example 1

A particle is thrown with an initial velocity of 100 ms^{-1} at an angle of 30° above the horizontal. Find a) the time to attain the highest point, b) the time of flight, c) the horizontal range, d) the greatest height reached. ($g = 10$ ms^{-2}).

Solution :

Here, initial velocity = u = 100 ms^{-1}

inclination above the horizontal = α = 30°

Suppose, t, T, R and H have their usual meanings. Then,

$$t = \frac{u \sin \alpha}{g} = \frac{100 \times \frac{1}{2}}{10} = 5 \text{ s}$$

$$T = \frac{2u \sin \alpha}{g} = 10 \text{ s}$$

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{100^2 \times \sqrt{3}}{10 \times 2} = 500\sqrt{3} \text{ m}$$

$$H = \frac{u^2 \sin^2\alpha}{2g} = \frac{100^2 \times 3}{2 \times 10 \times 4} = 375 \text{ m}$$

Example 2

The maximum horizontal range of a particle thrown with a certain velocity is 10 m. Find the velocity of projection. ($g = 10$ ms^{-2})

Solution :

Here, maximum horizontal range, R = 10 m

Velocity of projection, u = ?

We know, $R = \frac{u^2}{g}$

$$\therefore \quad 10 = \frac{u^2}{10}$$

Hence $u = 10$ ms^{-1}

Example 3

The highest point reached by a projectile is 20 m above the horizontal. If the initial velocity is $20\sqrt{2}$ ms^{-1}, find the angle of projection. ($g = 10$ ms^{-2})

Solution :

Here, highest point reached, H = 20 m

Initial velocity, $u = 20\sqrt{2}$ ms^{-1}

Angle of projection, α = ?

Since, $H = \dfrac{u^2 \sin^2\alpha}{2g}$

$$20 = \frac{(20\sqrt{2})^2 \sin^2\alpha}{2.10}$$

So, $\sin^2\alpha = \left(\dfrac{1}{\sqrt{2}}\right)^2$

$\therefore$ $\sin\alpha = \dfrac{1}{\sqrt{2}}$

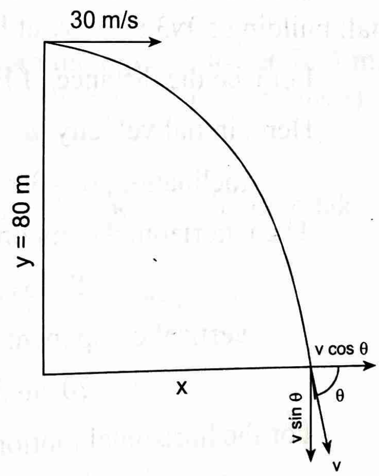
Hence $\alpha = 45°$

Example 4

A projectile is fired with a horizontal velocity of 30 ms^{-1} from the top of a cliff 80 m high (a) How long will it take to strike the level ground at the base of the cliff ? (b) How far from the foot of the cliff will it strike ? (c) With what velocity will it strike ? (g = 10 ms^{-2})

Solution :

Suppose that the projectile fired with a horizontal velocity of 30 ms^{-1} strikes the level ground at a distance x metres from the base of the cliff and y metres below the point of projection. Then,



a) Vertical motion :

Vertical velocity (component) = 0

Acceleration, $a = g$

time of fall, t = ?

Vertical distance = 80 m

So, from $y = \dfrac{1}{2} gt^2$,

We have $80 = \dfrac{1}{2} \cdot 10 \cdot t^2$

$\therefore$ $t = 4$ s

b) Horizontal motion

Horizontal velocity (component) = 30 ms^{-1} (constant)

Time of motion = 4 s

From $x = ut$,

We have $x = 30 \times 4 = 120$ m

c) If v is the velocity with which it strikes the ground at an angle θ with the horizontal, then

$$v\cos\theta = 30\ ms^{-1} \text{ (why ?)}$$

and $\quad v \sin\theta = 0 + 10 \times 4 = 40 \text{ ms}^{-1}$

Hence,

$$v^2(\cos^2\theta + \sin^2\theta) = 30^2 + 40^2$$

or, $\quad v^2 = 2500$

$\therefore \quad v = 50 \text{ ms}^{-1}$

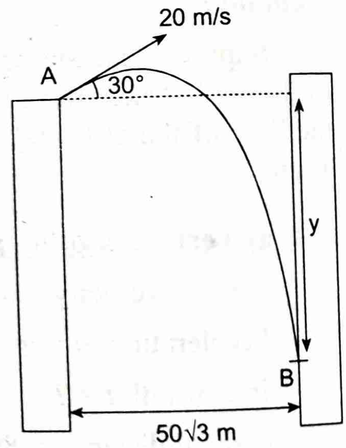
Also, $\quad \tan\theta = \frac{40}{30} = \frac{4}{3}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{4}{3}\right)$

Example 5

A ball is thrown from the top of a building towards a tall building $50\sqrt{3}$ m away. The initial velocity of the ball is 20 ms^{-1} at 30° above the horizontal. How far above or below its original level will the ball strike the opposite wall? ($g = 10 \text{ ms}^{-2}$)

Solution :



Let the ball thrown from the top A of a building strike another tall building $50\sqrt{3}$ m away at B after t seconds.

Let y be the distance of B from the original level.

Here, initial velocity, $u = 20 \text{ ms}^{-1}$

inclination, $\alpha = 30°$

Then, horizontal component of u

$= 20\cos 30° = 10\sqrt{3} \text{ ms}^{-1}$

vertical component of u

$= 20 \sin 30° = 10 \text{ ms}^{-1}$

For the horizontal motion,

$50\sqrt{3} = 10\sqrt{3}\, t$

$\therefore \quad t = 5\text{s}$

During this time, the vertical distance covered

$$y = 10.5 + \frac{1}{2}(-10)\ 25$$

$$= 50 - 125 = -75 \text{ m}$$

The negative sign shows that it strikes the opposite wall 75 m below the original level.

Example 6

For a given velocity of projection and a given horizontal range, prove that there are, in general, two directions of projection which are equally inclined to the direction of maximum range.

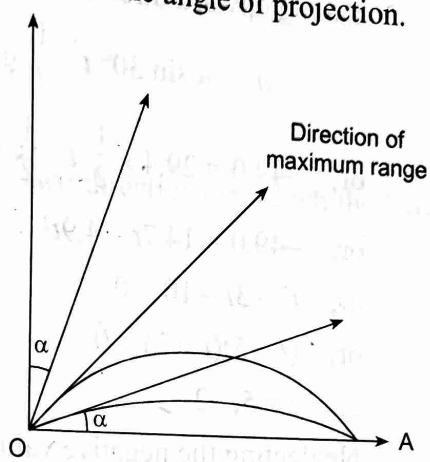
Solution :

Let u be the given velocity of projection, R the given range and α the angle of projection.

Then, $R = \frac{u^2}{g} \sin 2\alpha$

$= \frac{u^2}{g} \sin (\pi - 2\alpha)$

$= \frac{u^2}{g} \sin 2\left(\frac{\pi}{2} - \alpha\right)$



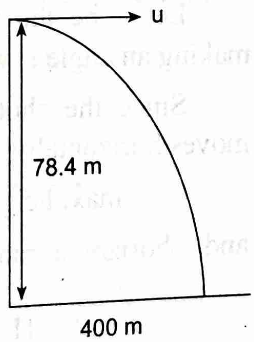
i.e. the value of the range remains the same if α is replaced by $\frac{\pi}{2} - \alpha$. Moreover, these two directions are equally inclined to the horizontal and the vertical respectively. Hence, the direction of maximum range (45°) bisects the angle between these two directions. In other words, for a given velocity of projection and for a given range, there are, in general, two directions of projections which are equally inclined to the direction of maximum range.

Example 7

With what velocity must a stone be projected horizontally from the top of a tower 78.4 m. high so as to reach a point on the ground 400 metres from the foot of the tower ? ($g = 9.8\ ms^{-2}$)

Solution :

Let u be the velocity with which a stone be projected horizontally and t be the time taken by the stone to reach the ground



$h = \frac{1}{2} gt^2$

or, $78.4 = \frac{1}{2} \times 9.8\ t^2$

or, $t^2 = 16$

$\therefore$ $t = 4$ sec.

Also, horizontal distance $= s = ut$

$400 = u \times 4$

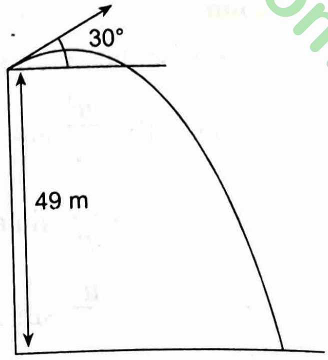
$\therefore$ $u = 100\ ms^{-1}$

Example 8

From the top of a tower, 49.0 metre high, a particle is projected with a velocity of 29.4 metres/sec making an angle of 30° with the horizon; find when and where will it hit the ground. ($g = 9.8\ ms^{-2}$)

Solution :

The horizontal and vertical components of the velocity of projection u are $u \cos 30°$ and $u \sin 30°$ respectively.



Let t be the time taken by the particle to hit the ground.

Taking upward direction as positive,

$$-h = u \sin 30^\circ\, t - \frac{1}{2} gt^2$$

or, $-49.0 = 29.4 \times \frac{1}{2}\, t - \frac{1}{2} \times 9.8\, t^2$

or, $-49.0 = 14.7t - 4.9t^2$

or, $t^2 - 3t - 10 = 0$

or, $(t-5)(t+2) = 0$

$\therefore\ t = 5, -2$

Neglecting the negative value of t, we have

$t = 5$ secs.

Let the particle hit at a distance of s from the foot of the tower.

$$s = u \cos 30^\circ\, t$$

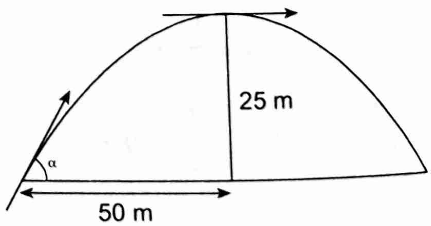
$$= 29.4 \times \frac{\sqrt{3}}{2} \times 5 = 73.5\sqrt{3} \text{ metres}$$

Example 9

Find the velocity and the direction of projection of a shot which passes in a horizontal direction just over the top of a building which is 50 metre off and 25 metre high. ($g = 9.8$ ms^{-2})

Solution :

Let u be the velocity with which a shot is projected making an angle α with the horizon.



Since the shot just passes the top of the building, it moves horizontally, so

max. height = H = 25 m

and horizontal range = R = 2 × 50 m = 100 m

$$H = \frac{u^2 \sin^2\alpha}{2g}$$

$$25 = \frac{u^2 \sin^2 \alpha}{2g} \qquad \ldots\ldots \text{(i)}$$

and $$R = \frac{u^2 \sin 2\alpha}{g}$$

$$100 = \frac{u^2 \sin 2\alpha}{g} \qquad \ldots\ldots \text{(ii)}$$

From (i) and (ii), we have

$$\frac{1}{4} = \frac{1}{4} \tan \alpha \quad \text{or, } \tan \alpha = 1 \qquad \therefore\ \alpha = 45^\circ$$

Now from (i), $25 = \dfrac{u^2 \sin^2 45°}{2 \times 9.8}$

$$u^2 = 25 \times 2 \times 9.8 \times 2$$

$$\therefore \quad u = 14\sqrt{5} \text{ m/sec}$$

Example 10

A ball is thrown with the velocity of 29.4 m/sec, find the two directions in which the ball may be thrown so as to give a range of 44.1 m. ($g = 9.8$ m/s^2)

Solution :

Let α be the angle of projection.

Then, horizontal range (R) $= \dfrac{u^2 \sin 2h}{g}$

$$\Rightarrow \quad 44.1 = \frac{29.4^2 \sin 2\alpha}{9.8}$$

$$\Rightarrow \quad \sin 2\alpha = \frac{44.1 \times 9.8}{29.4^2} = \frac{1}{2}$$

$$\Rightarrow \quad \sin 2\alpha = \sin 30° \text{ or } \sin 150°$$

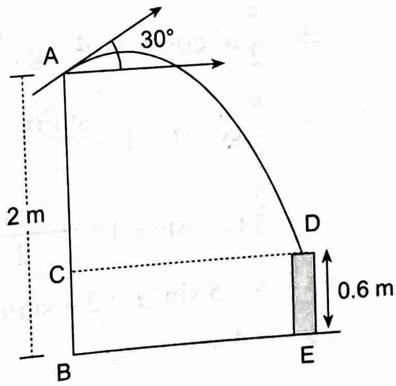
$$\therefore \quad 2\alpha = 30° \text{ or } 150°$$

i.e. $\alpha = 15°$ or $75°$

Example 11

A cricket ball thrown from a height of 2 metres at an angle of 30° to the horizon with a speed of 1820 cm per sec is caught by another fieldsman at a height 60 cm from the ground. How far apart were the men? ($g = 9.8$ m/s^2)

Solution:



Let a cricket ball projected from A be caught by a fieldsman at D such that DE = CB = 0.6 m. Then the cricket ball descends = AC = (2 – 0.6) m = 1.4 m.

Let t be the time taken by the cricket ball to descend AC.

The vertical component of $u = u \sin 30°$

Taking upward direction as positive, we have

$$-h = u \sin \alpha \, t - \frac{1}{2} g t^2$$

$$\Rightarrow \quad -1.4 = 18.20 \sin 30° \, t - \frac{1}{2} \times 9.8 \, t^2$$

$$\Rightarrow \quad 4.9 t^2 - 9.1 t - 1.4 = 0$$

$$\Rightarrow \quad 7 t^2 - 13t - 2 = 0$$

$$\Rightarrow \quad (t - 2)(7t + 1) = 0$$

$\therefore \quad t = 2, -\frac{1}{7}$

$\therefore$ the possible value of $t = 2$ secs

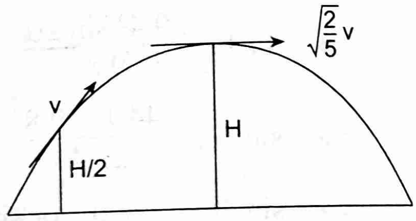
CD = BE $= u \cos 30^\circ . t = 18.20 \cos 30^\circ \times 2$

$= 18.20 \times \frac{\sqrt{3}}{2} \times 2 = 1820\sqrt{3}$ cm

Example 12

The velocity of a particle when at its greatest height is $\sqrt{\frac{2}{5}}$ of its velocity when at half its greatest height. Show that the angle of projection is 60°.

Solution:



Let u be the velocity and α, angle of projection of a particle. Let H be the greatest height. If v be the velocity at $\frac{H}{2}$, then the velocity at H is $\sqrt{\frac{2}{5}}\, v$.

$$\therefore \quad \sqrt{\frac{2}{5}}\, v = u \cos \alpha$$

$$\Rightarrow \quad v^2 = \frac{5}{2} u^2 \cos^2\alpha \qquad \ldots\ldots(i)$$

$$\text{Also, } H = \frac{u^2 \sin^2\alpha}{2g} \qquad \ldots\ldots(ii)$$

$$\text{And } v^2 = u^2 - 2g\frac{H}{2}$$

$$\Rightarrow \quad \frac{5}{2} u^2 \cos^2\alpha = u^2 - g \cdot \frac{u^2 \sin^2\alpha}{2g} \quad \text{(from (i) and (ii))}$$

$$\Rightarrow \quad \frac{5}{2} \cos^2\alpha = 1 - \frac{\sin^2\alpha}{2}$$

$$\Rightarrow \quad \frac{5}{2}(1 - \sin^2\alpha) = \frac{2 - \sin^2\alpha}{2}$$

$$\Rightarrow \quad 5 - 5\sin^2\alpha = 2 - \sin^2\alpha$$

$$\Rightarrow \quad 4\sin^2\alpha = 3$$

$$\Rightarrow \quad \sin^2\alpha = \frac{3}{4}$$

$$\Rightarrow \quad \sin\alpha = \frac{\sqrt{3}}{2}$$

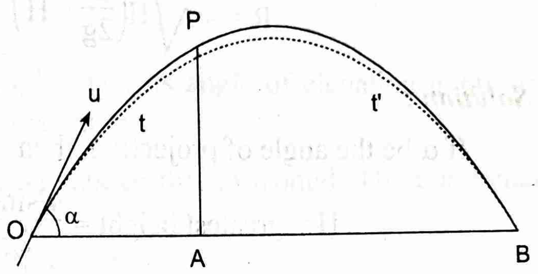
$$\therefore \quad \alpha = 60^\circ$$

Example 13

If t be the time in which a projectile reaches a point P of its path and t' be the time from P till it strikes the horizontal plane through the point of projection, show that the height of P above the plane is $\frac{1}{2} gtt'$.

Solution :

Let u and α be the velocity and the angle of projection respectively.



$t + t'$ = time from O to P + time from P to B

= time of flight

$= \dfrac{2u \sin \alpha}{g}$

$\therefore \quad u \sin \alpha = \frac{1}{2} g(t + t') \; \ldots\ldots \text{(i)}$

The vertical component of $u = u \sin \alpha$

If PA = h, then

$$h = u \sin \alpha \, t - \frac{1}{2} gt^2$$

$$= \frac{1}{2} g(t + t')\, t - \frac{1}{2} gt^2 = \frac{1}{2} gtt'$$

Example 14

If R be the horizontal range and T, the time of flight of a projectile, show that $\tan \alpha = \dfrac{gT^2}{2R}$, where α is the angle of projection.

Solution :

If u be the velocity of projection, then

$$R = \text{horizontal range} = \frac{u^2 \sin 2\alpha}{g} \quad \ldots\ldots \text{(i)}$$

$$\text{and } \; T = \text{time of flight} = \frac{2u \sin \alpha}{g} \quad \ldots\ldots \text{(ii)}$$

$$\text{Now,} \quad \frac{gT^2}{2R} = \frac{g\left(\dfrac{2u \sin \alpha}{g}\right)^2}{2 . \dfrac{u^2 \sin 2\alpha}{g}} \quad \text{(From (i) and (ii))}$$

$$= \frac{g \,.\, \dfrac{4u^2 \sin^2 \alpha}{g^2}}{2 \,.\, \dfrac{u^2 . 2 \sin \alpha \cos \alpha}{g}} = \tan \alpha$$

$$\therefore \quad \tan\alpha = \frac{gT^2}{2R}$$

Example 15

A particle is projected with a velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is

$$R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$$

Solution :

If α be the angle of projection, then

$$H = \text{greatest height} = \frac{u^2\sin^2\alpha}{2g}$$

and $$R = \text{horizontal range} = \frac{u^2 \sin 2\alpha}{g}$$

Then, $$\frac{u^2}{2g} - H = \frac{u^2}{2g} - \frac{u^2\sin^2\alpha}{2g}$$

$$= \frac{u^2(1-\sin^2\alpha)}{2g} = \frac{u^2\cos^2\alpha}{2g}$$

Now, $$4\sqrt{H\left(\frac{u^2}{2g} - H\right)} = 4\sqrt{\frac{u^2\sin^2\alpha}{2g}\cdot\frac{u^2\cos^2\alpha}{2g}}$$

$$= \frac{4\,u\sin\alpha\,u\cos\alpha}{2g}$$

$$= \frac{u^2\sin 2\alpha}{g} = R$$

$$\therefore \quad R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$$

EXERCISE

1. A particle projected upwards from the level ground at an angle of 60° with the horizon has an initial speed of $40\sqrt{3}$ ms^{-1}. ($g = 10$ ms^{-1})
   a) How long will it be before it hits the ground?
   b) How far from the starting point will it strike?
2. A ball is projected with an initial upward velocity component of 20 ms^{-1} and a horizontal velocity component of 25 ms^{-1}. ($g = 9.8$ ms^{-2})
   a) Find the position and the velocity of the ball after (i) 2 s (ii) 3 s (iii) 4 s.
   b) How much time is required to reach the highest point of the trajectory?
   c) How high is the point?

d) How much time (after launch) is required for the ball to return to its original level?
e) How far has it travelled horizontally during this time?

Illustrate your answer with neat and sufficiently large sketch.

3. A helicopter flying horizontally with a speed of 30 ms^{-1} at an altitude of 500 m has to drop a food pracket for a person standing on the ground. At what distance from the person should the packet be dropped? The man stands in the vertical plane of the helicopter's motion. ($g = 10ms^{-2}$)

4. Prove that a projectile will rise three times as high when its angle of elevation is 60° as when it is 30°, but will cover the same horizontal distance.

5. A spring gun projects a golf ball at an angle of 45° above the horizontal. The horizontal range is 10 m. ($g = 10\ ms^{-2}$)
   a) What is the maximum height to which the ball rises?
   b) For the same initial speed, what are the two angles of departure for which the range is 5 m?

6. A particle projected at an angle of 30° passes horizontally over a wall in 5 secs. (a) How fast was the ball thrown (b) Find the height of the wall. ($g = 10\ m/s^2$)

7. Find the angle of projection where the range on a horizontal plane is (a) 4 (b) $4\sqrt{3}$ times the greatest height.

8. A projectile thrown from a point in a horizontal plane comes back to the plane in 4 secs at a distance of 60 m in front of the point of projection; find the velocity of projection. ($g = 10\ ms^{-2}$)

9. Find the velocity and the direction of projection of a shot which passes in a horizontal direction just over the top of a wall which is 250 m off and 125 m high. ($g = 9.8\ ms^{-2}$)

10. The hozitonal and vertical components of the initial velocity of a projectile are U and V respectively. If R be the range and H the greatest height attained, prove that
    a) $\frac{4H}{R} = \frac{V}{U}$ b) $\left(\frac{R}{U}\right)^2 = \frac{8H}{g}$

11. A projectile shot at an angle of 60° above the horizontal strikes a building $30\sqrt{3}$ m away at a point 85 m above the point of projection. ($g = 10\ ms^{-2}$)
    a) Find the speed of projection.
    b) Find the magnitude and direction of the velocity of the projectile when it strikes the building.

12. A stone is thrown horizontally with velocity $\sqrt{2gh}$ from the top of a tower of height h. Find where it will strike the level ground through the foot of the tower. What will be its striking velocity?

13. From a point on the ground at a distance x from the foot of a vertical wall, a ball is thrown at an angle of 45° which just clears the top of the wall and afterwards strikes the ground at a distance y on the other side. Prove that the height of the wall is $\frac{xy}{x+y}$.

14. If R be the horizontal range of a projectile and h its greatest height, prove that its initial velocity is $\sqrt{2g\left(h + \frac{R^2}{16h}\right)}$.

15. A cannon ball has the same range R on a horizontal plane for two different angles of projection. If H and H′ are the greatest heights and t_1 and t_2 are the time of flights in two paths for which this is possible, prove that

a) $R^2 = 16HH'$ b) $R = \frac{1}{2}gtt'$.

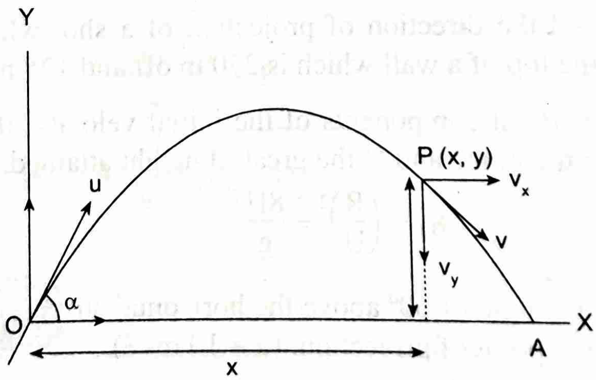
16. A ball is projected with a velocity of 49.0 m/s, find the two directions along which the ball must be projected so as to have a range of 122.5 m.

17. With what velocity must a body be projected at an angle of 45° from the top of a tower 65 m high, if it is to reach a point on the ground 180 m from the base of the tower.

18. A body is projected with a velocity of $16\sqrt{2}$ m/sec from the top of a tower 16 m high at an elevation of 45°. Find where and with what velocity will it strike the ground? ($g = 10$ m/s²)

Answers

1. a) 12 secs b) $240\sqrt{3}$ m
2. a) i) $x = 50$ m, $y = 20.4$ m, $v_x = 25$ ms^{-1}, $v_y = 0.40$ ms^{-1}
ii) $x = 75$ m, $y = 15.9$ m, $v_x = 25$ ms^{-1}, $v_y = -9.4$ ms^{-1}
iii) $x = 100$ m, $y = 1.6$ m, $v_x = 25$ ms^{-1}, $v_y = -19.2$ ms^{-1}



b) 2.04 secs c) 20.4 m d) 4.08 secs e) 102 m
3. 300 m 5. a) 2.5 m b) 15° 75° 6. a) 100 m/s b) 125 m
7. a) 45° b) 30° 8. 25 ms^{-1} 9. 70 ms^{-1}, 45°
11. a) $60\sqrt{3}$ ms^{-1} b) $10\sqrt{91}$ ms^{-1}, $\tan^{-1}\left(\frac{8}{3\sqrt{3}}\right)$ 12. 2h, $2\sqrt{gh}$ at 45° to the horizon.
16. 15°, 75° 17. 36 m/s 18. 64 m, $8\sqrt{13}$ m/s, $\tan^{-1}\left(\frac{3}{2}\right)$

Chapter 20
Mathematics for Economics and Finance

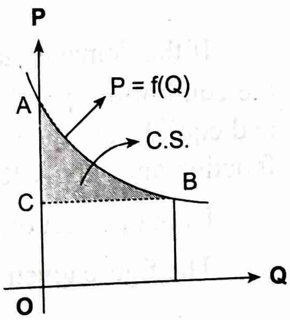
Consumer and Producer Surplus

Discussion of consumer and producer surplus can be made by considering the area of the triangle if demand function and supply function are linear. When demand and supply function are not linear, they are discussed and studied with the help of definite integral. But definite integral can be used in case of demand and supply function in linear form as well.

Consumer's Surplus

The difference between the expenditure which a consumer is ready to pay for the use of goods from $Q = 0$ to $Q = Q_0$ and the actual amount paid for Q_0 units of goods at the present market price P_0 per unit.

Let the demand function $P = f(Q)$ be represented by the curve AB. Let $P_0 = OC = DB$ be the market price whose corresponding demand is Q_0.



Total cost the consumer is ready to pay

$= \text{Area AODB}$

$= \int_0^{Q_0} P.dQ = \int_0^{Q_0} f(Q)\, dQ$

Total cost actually paid by the consumer

$= \text{total revenue} = \text{Area of the rectangle CODB}$

$= P_0.Q_0$

Consumer Surplus = C.S. = Area of the figure ACBA

$= \text{Area of fig. AODB} - \text{Area of the rectangle CODB}$

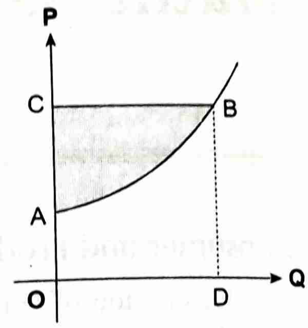
$= \int_0^{Q_0} P.dQ - P_0Q_0$

$= \int_0^{Q_0} (\text{demand function})\, dQ - P_0Q_0$

Producer Surplus

The difference between the total revenue actually received and the total revenue that would have been willing to receive is known as the producer surplus. That is, it is the difference between the total revenue received by the producer for Q_0 units of goods when the market price is P_0 and the total revenue that the producer was willing to accept for the goods from $Q = 0$ to $Q = Q_0$ at the market price P_0.

Let the supply function $P = f(Q)$ be represented by the curve AB. let $P_0 = OC = DB$ be the market price whose corresponding supply is Q_0 units of goods.



Total revenue actually received for Q_0 units

$= \text{Area of the rectangle CODB} = P_0Q_0$

Total revenue that the producer is willing to accept

$= \text{Area of AODB}$

$= \int_0^Q P\,dQ = \int_0^Q (\text{supply function})\,dQ$

Now, producer surplus (P.S.) = Area ABC

$= \text{Area of the rectangle CODB} - \text{Area AODB}$

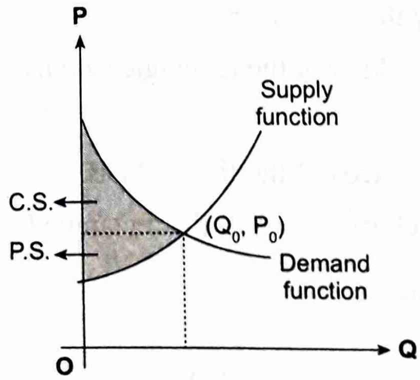
$= P_0Q_0 - \int_0^{Q_0} P\,dQ$

$= P_0Q_0 - \int_0^{Q_0} f(Q)\,dQ$

If the demand curve and the supply curve meet at the point (Q_0, P_0), the point is said to be the equilibrium point. The price and the quantity at the point is known as the equilibrium price and equilibrium quantity respectively. This point is determined by making equations of demand function and supply function equal. This condition is known as market equilibrium condition.

Under perfect or pure competition, market equilibrium condition is used.

The figure given below is the intersection of the demand curve and the supply curve.



Worked Out Example

Example 1.

The demand equation of a certain commodity is $p = 60 - 2x - x^2$ where p is the price and x, the quantity. Find the consumer surplus when i) demand is 6 ii) the price is 45.

Solution:

Here, $p = 60 - 2x - x^2$

i) When $x = 6$, $\quad p = 60 - 2 \times 6 - (6)^2 = 12$

Here, $P_0 = 12, Q_0 = 6$

The required consumer surplus

$$= \int_0^{Q_0} P\, dQ - P_0Q_0$$

$$= \int_0^{6} (60 - 2x - x^2)dx - 12 \times 6$$

$$= \left[60x - x^2 - \frac{x^3}{3}\right]_0^6 - 72$$

$$= 60 \times 6 - (6)^2 - \frac{(6)^3}{3} - 72$$

$$= 180$$

ii) When $p = 45$

$p = 60 - 2x - x^2$

$\Rightarrow 45 = 60 - 2x - x^2$

$\Rightarrow x^2 + 2x - 15 = 0$

$(x + 5)(x - 3) = 0 \quad \therefore x = -5, 3$

$x \neq -5, \quad \therefore x = 3$

Here, $P_0 = 45, Q_0 = x = 3$

Now, the required consumer surplus

$$= \int_0^{Q_0} P\, dQ - P_0Q_0$$

$$= \int_0^{3} (60 - 2x - x^2)dx - 45 \times 3$$

$$= \left[60x - x^2 - \frac{x^3}{3}\right]_0^3 - 135$$

$$= 60 \times 3 - (3)^2 - \frac{(3)^3}{3} - 135 = 27$$

Example 2.

If the supply function of a commodity is $P = 36 + 24(Q - 1)^2$. What is the producer surplus at $Q = 3$.

Solution:

$P = 36 + 24(Q - !)^2$

When $Q = 3$, $P = 36 + 24(3 - 1)^2 = 132$

Here $P_0 = 152$, $Q_0 = 3$

$$\text{Producer Surplus} = P_0Q_0 - \int_0^{Q_0} [36 + 24(Q - 1)^2]\, dQ$$

$$= 132 \times 3 - \int_0^3 [36 + 24(Q - 1)^2]\, dQ$$

$$= 396 - [36Q + 8(Q - 1)^3]_0^3$$

$$= 396 - [36 \times 3 + 8(3 - 1)^3]$$

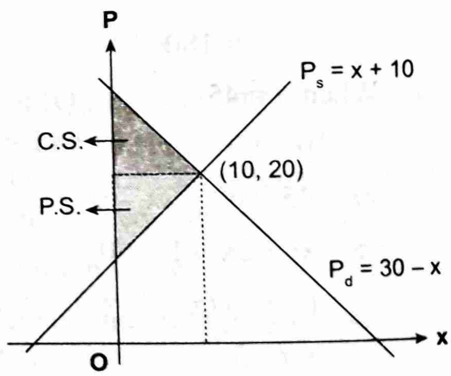
$$= 396 - [108 + 64] = 224$$

Example 3.

Given the demand function $P_d = 30 - x$ and the supply function $P_s = x + 10$ where x is the quantity in units. Find the consumer as well as producer surplures at market equilibrium point.

Solution:

$P_d = 30 - x$ and $P_s = x + 10$



At the equilibrium point,

$P_d = P_s$

$\Rightarrow$ $30 - x = x + 10$

$\Rightarrow$ $2x = 20$

$\therefore$ $x = 10$

When $x = 10$, $P_d = 30 - 10 = 20 = P_s$

For Consumer Surplus : $x_0 = 10$, $P_0 = 20$

The required consumer surplus

$$= \int_0^{x_0} P_d\, dx - P_0x_0$$

$$= \int_0^{10} (30 - x)dx - 20 \times 10$$

$$= \left[30x - \frac{x^2}{2}\right]_0^{10} - 200$$

$$= 30 \times 10 - \frac{(10)^2}{2} - 200$$

$$= 300 - 50 - 200 = 50$$

For Producer Surplus

Here $x_0 = Q_0 = 10$, $P_0 = 20$

The required producer surplus

$$= P_0Q_0 - \int_0^{Q_0} P_s \, dQ$$

$$= 20 \times 10 - \int_0^{10} (x + 10)\, dx$$

$$= 200 - \left[\frac{x^2}{2} + 10x\right]_0^{10}$$

$$= 200 - \left[\frac{(10)^2}{2} + 10 \times 10\right]$$

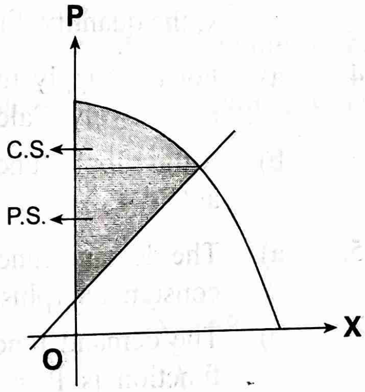
$$= 200 - 50 - 100 = 50$$

Example 4.

Find the consumer surplus and the producer surplus under pure comption if the demand and the supply functions are $D(x) = 16 - x^2$ and $S(x) = 4 + x$ respectively.

Solution:

For pure competition,



$D(x) = S(x)$

$\Rightarrow \quad 16 - x^2 = 4 + x$

$\Rightarrow \quad x^2 + x - 12 = 0$

$\Rightarrow \quad (x + 4)(x - 3) = 0$

$\therefore \quad x = -4, 3$

$x \neq -4 \qquad \therefore x = 3$

When $x = 3$, $p = 16 - x^2 = 16 - (3)^2 = 7$

Here $p_0 = 7$, $x_0 = 3$

Consumer surplus (C.S.) $= \int_0^{x_0} D(x)\, dx - p_0x_0$

$$= \int_0^{3} (16 - x^2)dx - 7 \times 3$$

$$= \left[16x - \frac{x^3}{3}\right]_0^3 - 21 \quad = 16 \times 3 - \frac{(3)^3}{3} - 21 = 18$$

Again, producer surplus (P.S.) $= p_0x_0 - \int_0^{x_0} S(x)\, dx$

$$= 7 \times 3 - \int_0^{3} (4 + x)\, dx$$

$$= 21 - \left[4x + \frac{x^2}{2}\right]_0^3$$

$$= 21 - \left[4 \times 3 + \frac{(3)^2}{2}\right]$$

$$= 21 - 12 - \frac{9}{2} = 4.5$$

EXERCISE

1. Calculate the consumer surplus for the demand functions given below
   a) $P = 45 - 2Q$ at $Q = 10$
   b) $2P = 24 - 3Q$ at $P = 6$
   c) $P = 50 - Q^2$ at $Q = 3$
   d) $P = 100 - Q^2$ at $P = 36$
   e) $P = \frac{100}{(Q+1)^2}$ at $Q = 9$
2. Calculate the producer surplus for the following supply function:
   a) $P = 3 + 2Q$ at $P = 33$
   b) $Q = -13 + 2P$ at $Q = 11$
   c) $P = Q^2 + 10$ at $Q = 6$
   d) $P = Q^2 + Q$ at $Q = 4$
3. a) If the demand law is $p = 85 - 4x - x^2$ where p is the price and x the quantity demanded, find the consumer surplus if
      i) demand is 5
      ii) market price is 64
   b) The demand function for a new product is $p = 50 - 2x - 3x^2$ where p is the price and x, the quantity. Find the consumer surplus if demand is 3.
4. a) For the supply function $P = Q^2 + 2Q$, where P and Q stand for price and quantity respectively. Calculate producer surplus at $Q = 3$.
   b) If the supply function for a product is $P = 40 + 30(Q + 1)^2$, what is producer surplus at $Q = 4$.
5. a) The demand function is $p = 20 - 3x$ and the supply function is $p = 4 + x$. Find the consumer surplus and producer surplus at the market equilibrium price.
   b) The demand function for a commodity is given by $P = 120 - 4Q$ and the supply function is $P = 2Q$ where P and Q represent price and the quantity demanded respectively. Calculate consumer surplus and producer surplus under pure competition.
   c) The demand and the supply functions under perfect competition are $P_d = 16 - x^2$ and $P_s = 2x^2 + 4$ respectively. Find the market price, consumer surplus and producer surplus. Also find the total surplus.

Answers

1. a) 100 b) 12 c) 18 d) $341\frac{1}{3}$ e) 81
2. a) 225 b) $30\frac{1}{4}$ c) 144 d) $50\frac{2}{3}$
3. a) i) $133\frac{1}{3}$ ii) 36 b) 63 4. a) 27 b) 1760 5. a) 24, 8 b) 800, 400 c) $\frac{16}{3}, \frac{32}{3}$

Quadratic Functions in Economics

A polynomial function of degree two is known as quadratic function. The general form of a quadratic function is

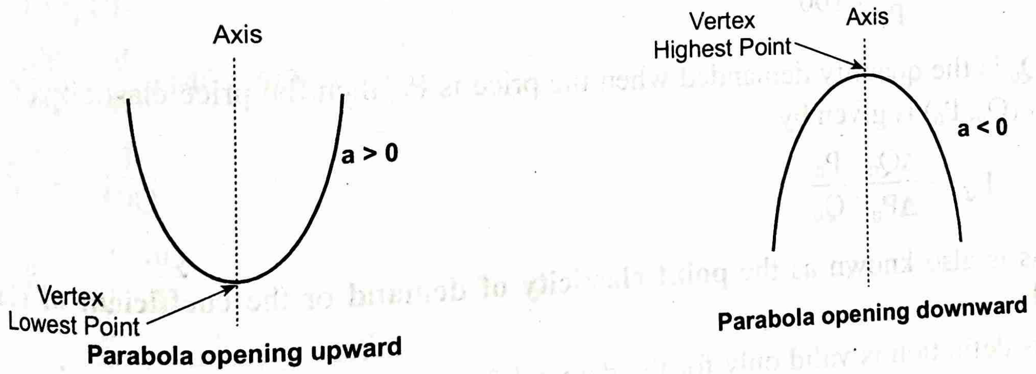
$y = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$.

Let us see below the graph of the quadratic function $y = x^2$

x	0	±1	±2	±3	±4
y	0	1	4	9	16

The graph is a parabola.

The graph of the quadratic function $y = ax^2 + bx + c$ is a parabola. It may open up or down according as $a > 0$ or $a < 0$. If $a > 0$, the parabola opens up and termed as **concave upward** and if $a < 0$, it opens down and termed as **concave downward**. If the parabola opens upward, it has the lowest point and if it opens downward, it has the highest point. The highest point or the lowest point of the parabola is known as its **vertex**. The coordinates of the vertex are $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. The line through the vertex and parallel to y-axis is known as its **axis**. This is also known as **axis of symmetry**. Its equation is $x = -\frac{b}{2a}$. At the vertex, the maximum or the minimum value of the quadratic function occurs. The y-coordinate of the vertex will give the maximum or minimum value of the function.



Application of Quadratic Functions

As an application, we have already discussed and used the linear functions in Economics and management in the form of demand, supply, cost, revenue, profit etc. But now in this section, we use the quadratic function in terms of economic variable mentioned above. We also deal with the break-even point, market equilibrium condition and the maximum or minimum values of the economic variables defined in quadratic forms.

If the economic variable is in the form of

$$y = ax^2 + bx + c$$

it represents a parabola and the maximum or minimum value occurs at the vertex = (x, y)

$$= \left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$$

i) If $a > 0$, the minimum value occurs at the vertex.

ii) If $a < 0$, the maximum value occurs at the vertex.

The maximum or minimum value will be $y = \frac{4ac - c^2}{4a}$ at $x = -\frac{b}{2a}$

We can use the quadratic functions in elasticity of demand also.

Elasticity of Demand

Demand and price have a relation that when one increases, the other decreases. The measure of the rate of change of demand with respect to the rate of change of price is the elasticity of demand.

Thus, the percentage change in the quantity demanded per unit percentage change in price is known as the price eleasticity of demand or coefficient of price elasticity. It is denoted by E_d.

If ΔQ be the change in quantity demanded Q_d when the corresponding ΔP is the change in price P, the price elasticity of demand is

$$E_d = \frac{\text{percentage change in quantity demanded}}{\text{percentage change in price}} = \frac{\Delta Q_d\%}{\Delta P\%}$$

$$= \frac{\frac{\Delta Q}{Q} \times 100}{\frac{\Delta P}{P} \times 100} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

If Q_0 is the quantity demanded when the price is P_0, then the price elasticity of demand at the point (Q_0, P_0) is given by

$$E_d = \frac{\Delta Q_0}{\Delta P_0} \cdot \frac{P_0}{Q_0}$$

This is also known as the **point elasticity of demand** or the **coefficient of elasticity of demand**.

This definition is valid only for the demand function in the linear form. But if the demand function is in the non-linear form, the above formula may not be true. Thus in case of a non-linear demand function $P = f(Q)$, ΔP is made to approach for zero and then the price elasticity of demand E_d is given by

$$E_d = \frac{P}{Q} \cdot \frac{dQ}{dP}$$

where $\frac{dQ}{dP}$ is the derivative of Q with respect to P.

Also the formula can be used even for the demand function which is in the linear form.

The magnitude of E_d denoted by $|E_d|$ gives the measurement of the rate of change of quantity demanded with respect to the rate of the change of price. The negative sign shows the deviation of the movement i.e. increase in price $\Rightarrow$ decrease in demand and decrease in price $\Rightarrow$ increase in demand.

We have the following conclusion from the value of E_d

i) If $E_d = -1$ (i.e. $|E_d| = 1$) then the demand is **unit elastic** i.e. percentage change in demand = percentage change in price.

ii) If $E_d < -1$ (i.e. $|E_d| > 1$), then the demand is **elastic** i.e. percentage change in demand is greater than the percentage change in price.

iii) If $-1 < E_d < 0$ (i.e $|E_d| < 1$) then the demand is less than percentage change in price.

Besides the elasticity of demand, there is elasticity of supply as well. Definition of elasticity of supply is same as the definition of elasticity of demand. The only difference is in place of demand, supply is to be used.

For example: Given the demand function $P = 140 - 5Q^2$, calculate the coefficient of point elasticity of demand for $P = 90$.

Solution:

Here, $P = 140 - 5Q^2$

When $P = 90$,

$$90 = 140 - 5Q^2$$

$$\Rightarrow 5Q^2 = 50$$

$$\Rightarrow Q^2 = 10$$

$$\therefore Q = \sqrt{10}$$

Again, $\frac{dP}{dQ} = \frac{d}{dQ}(140 - 5Q^2) = -10Q$

$$\Rightarrow \frac{dQ}{dP} = -\frac{1}{10Q}$$

Now, $E_d = \frac{P}{Q} \cdot \frac{dQ}{dP}$

$$= \frac{P}{Q} \cdot \left(-\frac{1}{10Q}\right) = -\frac{P}{10Q^2}$$

When $P = 90$, $Q = \sqrt{10}$

$$E_d = -\frac{90}{10 \times 10} = -0.9$$

This result shows that 1% increase in price $\Rightarrow$ 0.9% decrease in quantity demanded.

Equilibrium Condition

Customers will demand the goods for the sake of purchasing goods from the market. Producers will supply the goods in the market to sale their production. There is a relation between the price and demand and also between the price and the supply. When the price increases, the demand will decrease but when the price increases the supply will increase.

The demand of the customers may or may not be equal to the supply of the producers. But it happens that the **quantity demanded = quantity supplied**, this situation or condition is known as **level of market equilibrium** or **market equilibrium condition**. In this situation, price of the demand = price of the supply.

Let Q_d and Q_s be the quantity demanded and the quantity supplied respectively whose corresponding prices are P_d and P_s. Then under the **equilibrium condition**

$$P_d = P_s \quad \text{and} \quad Q_d = Q_s$$

Break-even Point

In the business system, the producers or the firms produce different types of goods. For the production of the goods, there needs costs for the inputs such as raw materials, labour charge etc. The total sum invested by the producer or the firm in course of production of goods is known as the total cost and is denoted by TC or simply C.

When the goods produced by the firms are sold in the market, the amount of money that the firm or producer obtain is known as the total revenue and is denoted by TR or simply R.

In the business activity if the total revenue obtained by the firm is equal to the total cost invested i.e. if there is not profit or not loss situation happens **break-even point** is said to occur.

Thus in the **break-even point** situation

$$TR = TC$$

On the other hand, if TR > TC then the firm is said to be in **profit** while if TR < TC, the firm is said to be in **loss**.

Worked Out Examples

Example 1

The total cost (TC) of producing x units of a product is defined by $C(x) = x^2 - 150x + 10000$ when C(x) is measured in rupees.

a) What is the concavity of the graph of the cost function?

b) At what quantity will the total cost be minimum and what will be the minimum cost?

Solution:

$$C(x) = x^2 - 150x + 10000$$

Comparing it with $y = ax^2 + bx + c$, we have

$$y = C(x), \quad x = x, \quad a = 1, \quad b = -150, \quad c = 10000$$

a) Since $a = 1 > 0$, so it opens upwards.

So, the graph of the cost function has upward concavity.

b) $\because a = 1 > 0$, C(x) has minimum value at the vertex.

vertex = (x, C(x)) $= \left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

$= \left(-\frac{-150}{2 \times 1}, \frac{4 \times 1 \times 10000 - (-150)^2}{4 \times 1}\right)$

$= (75, 4375)$

∴ the required quantity = x = 75 units and minimum cost = C(x) = Rs. 4375.

Example 2

The demand function for a product is $Q = f(P) = 22500 - 75P$ where Q is measured in units and P in rupees. Determine the quadratic total revenue function $R = f(P)$.

a) What is the concavity of the graph of the revenue function?

b) What is the total revenue at a price of Rs. 40?

c) How many units will be demanded at this price?

d) At what price will the total revenue be maximum and what will be the maximum revenue?

e) Sketch the graph of the revenue function.

Solution:

$$Q = f(P) = 22500 - 75P$$

R = price × quantity = P.Q

$= P(22500 - 75P) = 22500P - 75P^2$

$\therefore R = f(P) = 22500P - 75P^2$

Comparing revenue function with $y = ax^2 + bx + c$

$y = R, \quad x = P, \quad a = -75, \quad b = 22500, \quad c = 0$

a) $\because a = -75 < 0$, so the curve represented by revenue function has **downward concavity.**

b) When P = 40,

R $= 22500 \times 40 - 75 \times (40)^2 = 780000$

∴ total revenue = Rs. 780000

c) When P = 40,

Q $= 22500 - 75 \times 40 = 19500$ units

d) $\because a = -75 < 0$, so the maximum value of revenue occurs at the vertex $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

i.e. at $\left(-\frac{22500}{2 \times -75}, \frac{4 \times -75 \times 0 - 22500^2}{4 \times -75}\right) = (150, 1687500)$

∴ the required price = P = Rs. 150 and the maximum revenue = R = Rs. 1687500.

e) $R = 22500P - 75P^2$

When $R = 0$, $22500P - 75P^2 = 0$

$\Rightarrow \quad P(22500 - 75P) = 0$

$\therefore \quad P = 0, \quad P = 300$

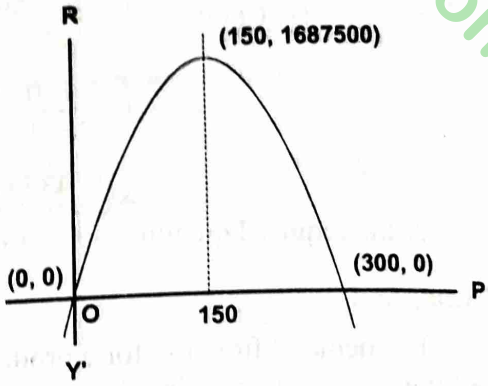
i.e. the revenue function representing parabola cuts P-axis (x-axis) at $P = 0$ and $P = 300$

$a = -75 < 0$, it opens donwards.

Vertex = (150, 1687500)

Axis is $P = x = 150$

The graph is given aside.



Example 3

The demand function for a good is $P = 60 - 2Q$. Fixed cost for a good is Rs. 192 and the variable cost for each additional unit of good is Rs. 20.

a) Write down the equation for total revenue and total cost in terms of Q.

b) Find the profit function in terms of Q.

c) Determine the maximum profit.

d) Present the graph of profit function.

Solution:

Demand function is $P = 60 - 2Q$, Q being the quantity.

Fixed cost = Rs. 192 and variable cost per unit = Rs. 20

a) TR = Total Revenue = $P.Q = (60 - 2Q).Q = 60Q - 2Q^2$

Total cost (TC) = Fixed cost + Quantity × Cost per unit good

$TC = 192 + 20Q$

b) Profit function (π) = TR – TC

$= 60Q - 2Q^2 - (192 + 20Q)$

$= -2Q^2 + 40Q - 192$

c) Comparing profit function with $y = ax^2 + bx + c$, we have

$y = \pi, \quad x = Q, \quad a = -2, \quad b = 40, \quad c = -192$

$\because a = -2 < 0$, so the profit is maximum

Profit function represents a parabola with downward concavity and vertex is at

$$\left(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\right) = \left(-\frac{40}{2 \times -2}, \frac{4 \times -2 \times -192 - 40^2}{4 \times -2}\right) = (10, 8)$$

$\therefore$ maximum profit = Rs. 8 when $Q = 10$ units.

d) Profit function (π) = $-2Q^2 + 40Q - 192$ which represent a parabola with downward concavity. $a = -2 < 0$

Here, $y = \pi$, $x = Q$, $a = -2$, $b = 40$, $c = -192$

When $\pi = 0$, $-2Q^2 + 40Q - 192 = 0$

$\Rightarrow \quad Q^2 - 20Q + 96 = 0$

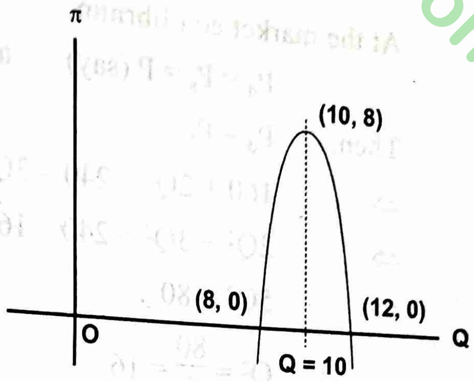
$\Rightarrow \quad (Q - 8)(Q - 12) = 0$

$\therefore \quad Q = 8, 12$

Profit function curve cuts the Q-axis at Q = 8 and Q = 12.

Max. value of $\pi = 8$ at $Q = 10$.

The graph of the profit function is given aside.



Example 4

Find the elasticity of demand for the demand function $Q = 150 - P^2$ at $P = 5$. If price increases by 10%, calculate the percentage change in quantity demanded.

Solution:

Here, $Q = 150 - P^2$

When $P = 5$, $\quad Q = 150 - 25 = 125$

Also, $\quad \frac{dQ}{dP} = \frac{d}{dP}(150 - P^2) = -2P$

Now, $\quad E_d = \frac{P}{Q} \cdot \frac{dQ}{dP} = \frac{P}{Q}(-2P)$

When $P = 5$, $Q = 125$;

$$E_d = \frac{P}{Q}(-2P) = \frac{5}{125} \times (-2 \times 5) = -0.4$$

Since $E_d = -0.4$, so 1% increase in price $\Rightarrow$ 0.4% decreases in quantity demanded.

Again, $E_d = \frac{\text{Percentage change in quantity demanded}}{\text{Percentage change in price}}$

$$-0.4 = \frac{\text{Percentage change in quantity demanded}}{10\%}$$

$\therefore$ Percentage change in quantity demanded = –4%

$\therefore$ 10% increase in price causes 4% decrease in quantity demanded.

Example 5

The supply function is $P_s = 160 + 2Q^2$ and the demand function is $P_d = 240 - 3Q^2$. Estimate the market equilibrium price and quantity.

Solution:

$P_s = 160 + 2Q^2 \quad$ and $\quad P_d = 240 - 3Q^2$

At the market equilibrium,

$P_d = P_s = P$ (say) and $Q_d = Q_s = Q$ (say)

Then, $P_d = P_s$

$\Rightarrow$ $160 + 2Q^2 = 240 - 3Q^2$

$\Rightarrow$ $2Q^2 + 3Q^2 = 240 - 160$

$\Rightarrow$ $5Q^2 = 80$

$\therefore$ $Q^2 = \frac{80}{5} = 16$

$\therefore$ $Q = 4$ ($Q = -4$ is not possible)

When $Q = 4$, $P_d = P_s = 160 + 2 \times (4)^2 = 192$

Example 6

The demand function for a good is $P = 60 - 2Q$. Fixed cost for a good is Rs. 192 and the variable cost for each additional unit of good is Rs. 20.

a) Write down the equations for total revenue and total cost in terms of Q.

b) Find break-even point.

c) Find the TR and TC at the break-even points.

Solution:

a) $TR = PQ = (60 - 2Q).Q = 60Q - 2Q^2$

Fixed cost = 192, Variable cost = 20 per unit good

Total cost (TC) = Fixed cost + Quantity × cost per unit good

$TC = 192 + 20Q$

b) For break-even point

$TC = TR$

$\Rightarrow$ $192 + 20Q = 60Q - 2Q^2$

$\Rightarrow$ $2Q^2 - 40Q + 192 = 0$

$\Rightarrow$ $Q^2 - 20Q + 96 = 0$

$\Rightarrow$ $(Q - 8)(Q - 12) = 0$

$\therefore$ the break-even point are at $Q = 8$ and $Q = 12$.

c) When $Q = 8$,

$TR = 60Q - 2Q^2 = 60 \times 8 - 2 \times 8^2 = 352$

$TC = 192 + 20Q = 192 + 20 \times 8 = 352$

$\therefore$ TR = TC = Rs. 352

d) When $Q = 10$

$TR = 60Q - 2Q^2 = 60 \times 10 - 2 \times 10^2 = 400$

$TC = 192 + 20Q = 192 + 20 \times 10 = 392$

When $Q = 10$, $TR > TC$, so it will be **profit.**

When $Q = 15$,

$$TR = 60Q - 2Q^2 = 60 \times 15 - 2 \times 15^2 = 450$$

$$TC = 192 + 20Q = 192 + 20 \times 15 = 492$$

When $Q = 15$, $TR < TC$, so it will be in **loss.**

EXERCISE

1. The total cost of a product is given below:

   a) $C(Q) = Q^2 - 120Q + 8670$ b) $C(x) = 15x^2 - 3600x + 486000$

   Find the quantity Q or x to be produced so that the cost is minimum. Also find the minimum cost.

2. The total revenue function R(Q) and demand function P(Q) are given below

   a) $R(Q) = 1800Q - 60Q^2$ b) $R(Q) = 36Q - \frac{1}{50}Q^2$ c) $P = 180 - 3Q$

   Find the number of items to be produced to make the revenue maximum. Also find the maximum revenue.

3. a) The demand function for a particular product is $q = f(P) = 6000 - 250p$ where q is stated in units and p in rupees. Determine the total revenue function $R = f(p)$.

   i) What is the concavity of the revenue function?

   ii) What will the total revenue be at price Rs. 20?

   iii) How many units will be demanded at this price?

   iv) At what price will the revenue be maximum? Find the maximum revenue.

   b) The demand function is modelled by $Q = f(P) = 1800 - 15P$ where Q is measured in units and P in rupees. Express the revenue function R in f(P).

   i) What will be the total revenue when P = Rs. 90?

   ii) Find the quantity demanded at this price?

   iii) What is the concavity of the graph of revenue function?

   iv) At what price will the total revenue be maximum? Find the maximum revenue?

4. a) If the revenue function and the total cost function are $R(x) = 5x - 3x^2$ and $C(x) = x^2 - 3x$, find the value of the maximum profit.

   b) A company has the total cost TC and total revenue TR which are given by $TC = \frac{1}{4}Q^2 + 10$ and $TR = 4Q$ respectively. How many units of quantities are to be produced so as to make the profit maximum? What is the maximum profit.

   c) A firm has a demand function $P = 108 - 5Q$ and the cost function $C = -12Q + Q^2$. Find the price at which the profit is maximum. Also find the maximum profit.

d) The demand function for a good is $P = 24 - 2Q$. Fixed cost of the goods is Rs. 30 and the variable cost for each additional unit of goods is Rs. 8.

i) Express TC and TR in terms of Q.

ii) Find the maximum revenue.

iii) Express profit function in terms of Q. Determine the maximum profit.

iv) Sketch the graph of revenue function and profit function.

e) The demand function for a good is given by $P = 18 - 2Q$. Fixed cost for a good is Rs. 24 and the variable cost for each unit produced costs an additional Rs. 2.

i) Write down the equation for total renevue and total cost in terms of Q.

ii) Obtain an expression for the profit function in terms of Q and sketch its graph.

iii) Determine the maximum profit.

5. a) For the demand law (i) $p = 20 - 3x^2$ (ii) $p = 5 + 6x - 2x^2$ (iii) $p = 3x - x^2$ find the elasticity of demand when the level of output $(x) = 2$.

b) Find the elasticity of demand for the demand function $Q = 5P - P^2$ when $P = 3$. If price increases by 20%, calculate the percentage change in quantity demanded.

c) For the demand function $P = -\frac{1}{4}Q^2 + 2Q + 5$, find the elasticity of demand when (i) the level of output $(Q) = 6$ (ii) the price $(P) = 5$.

6. For what value of Q will make the elasticity of demand equal to -0.8 if the demand function is $Q = 1400 - P^2$?

7. a) The supply function and demand function are given below:

i) Supply function : $P = Q^2 + 40$; Demand function : $P = 240 - Q^2$

ii) Supply function : $P = 2Q + 5$; Demand function : $P = Q^2 - 10Q + 25$

Calculate the market equilibrium price and quantity.

b) The demand and supply functions for a commodity are given by $P_d = 950 - 20Q$ and $P_s = Q^2 + 150$ respectively. Find the equilibrium price and quantity algebraically and graphically.

c) The supply and demand function for a particular item are given by the equations $P_s = Q^2 + 4Q + 4$ and $P_d = Q^2 - 12Q + 36$. Find the equilibrium price and the quantity.

8. a) The revenue and cost functions of a product are given below

i) $R = 80 - Q^2$; $C = 10Q + 5$

ii) $TR = 200Q - 3Q^2$; $TC = 1440 + 8Q + 3Q^2$

Find the quantity to be produced to make break-even point.

b) Given below are the cost function C(x) (or C(Q)) and revenue function R(x) (or R(Q))

i) $C(x) = x^2 + 8x$, $R(x) = 40x - 3x^2$

ii) $TC = Q^2 + 6Q - 48$, $TR = 22Q - 3Q^2$

iii) $TC = \frac{x^2}{4} + 12$, $TR = 4x$

Find the quantity to be produced to make break-even point.

9. a) A firm's total revenue and the total cost function are given by $TR = 90Q - 3Q^2$ and $TC = 30Q + 273$ respectively.
   i) How many quantities are to be produced in order to meet break-even point?
   ii) What will happen if the quantities produced be of 6, 10 and 14 units?

   b) A shop which sell T-shirts has a demand function and a total cost function given by the equations $P = 120 - 10Q$ and $TC = 120 + 8Q$.
   i) Write down the equations for TR and profit.
   ii) Find the number of T-shirts to be sold so that there will neither be loss nor be profit.
   iii) Obtain the total revenue and the total cost.
   iv) If the shop sells 8 units, 12 units, what will be the condition of the shop?

10. a) The demand function for a good is $P = 24 - 2Q$. Fixed cost of the goods is Rs. 30 and the variable cost for each additional unit of goods is Rs. 8.
   i) Express TC and TR in terms of Q.
   ii) Find the break even point.
   iii) Present TC, TR and break-even point in the same graph.

   b) The total variable cost (TVC) $= \frac{1}{5}Q^2 + 2Q$ and the total fixed cost (TFC) = 20 where Q is the quantity of goods purchased. If the goods are sold at Rs. 7 per good
   i) Express the total cost and total revenue in terms of Q.
   ii) Find the number of goods to be sold so that the break-even points will occur.
   iii) Find the TR and TC at the break-even points.

A. A work of finding the profit of the function is given to the students. The question is as follows:

A quadratic cost function $C = px^2 + qx + r$ and the linear revenue function $R = a + bx$ are given, where x is the quantity. Use different values of p, q, r, a and b. Answer the following questions

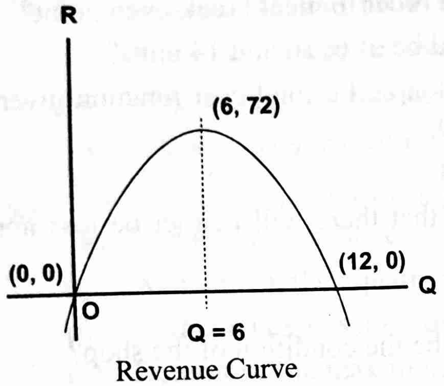
a) Find the profit function.
b) Draw the graph of the profit function.
c) What does the graph of the profit function represent?
d) What is the concavity of the curve?
e) At what level, the profit will be maximum?
f) Find the maximum profit.

Answers

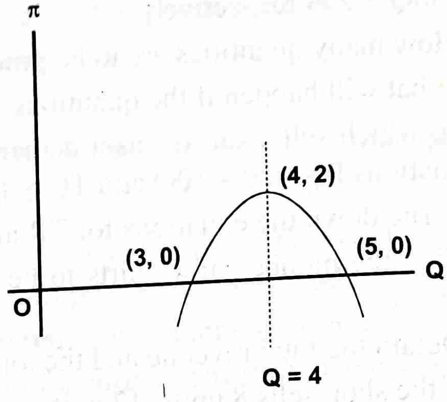
1. a) $Q = 60$, min. cost = 5070 b) $x = 120$, min. cost = 270000 c) $Q = 30$, $R = 2700$
2. a) $Q = 15$, $R = 13500$ b) $Q = 900$, $R = 16200$
3. a) i) Downward concavity ii) Rs. 20000 iii) 1000 units iv) Rs. 12, Rs. 36000
   b) i) Rs. 40500 ii) 450 units iii) downward concavity iv) Rs. 60, Rs. 54000
4. a) 4 b) $Q = 16$, profit = 22 c) 58; max. profit = 600 d) i) $TR = 24Q - 2Q^2$, $TC = 8Q + 30$

ii) Max. revenue = Rs. 72 iii) $\pi(Q) = -2Q^2 + 16Q - 30, 2$

iv)



Revenue Curve



Profit Curve

e) i) $18Q - 2Q^2, 24 + 2Q$ ii) $-2Q^2 + 16Q - 24$ iii) 8

5. a) i) –0.33, demand is inelastic ii) –2.25, demand is elastic iii) –1, demand is unique elastic

b) –0.5, –10% c) i) –1.33 ii) $-\frac{5}{16}$ 6. 1000

7. a) i) P = 140, Q = 10 ii) P = 9, Q = 2; P = 25, Q = 10 b) Q = 20, P = 550 c) Q = 2, P = 16

8. a) i) Q = 5 ii) Q = 12, 20 b) i) x = 8 ii) Q = 6 iii) x = 4, 12

9. a) i) Q = 7, 13 ii) Loss, Gain, Loss b) i) $TR = 120Q - 10Q^2, -10Q^2 + 112Q - 120$

ii) 10, 1.2 units iii) TR = TC = 200, TR = TC = 129.6 iv) Profit, Loss

10. a) i) $TC = 30 + 8Q, TR = 24Q - 2Q^2$ ii) Q = 3, 5 b) i) $TC = \frac{1}{5}Q^2 + 2Q + 20, TR = 7Q$

ii) Q = 5, 20 iii) TR = TC = 35; TR = TC = 140

Input-Output Analysis

For every industry, there needs inputs such as material, labour etc. for the production of goods. With the help of the inputs, the industry produces outputs i.e. outcomes such as different types of goods. Thus inputs and outputs are the terms used in the industry for the production of the goods. The input–output analysis presents the interrelationship and the interdependency between the industries. That is, the output of one industry will be the input of the other industry and vice-versa (conversely).

The objective of the input–output analysis is to deal with the problems such as "What quantity of the output that one industry should produce so that the demand of the consumers satisfy exactly. That is, the volume of production that should be made in order to balance the supply and the demand. The main assumption of the input–output analysis is "the quantity that has produced, should be consumed." The output of the industry may be consumed by (i) the industry itself and the industries (ii) other than the industries.

If the output of the industry is used only by the industries, then the demand is known as the **inter-industry demand.** But besides this, the outputs are used by the industries as well as others i.e. consumers; then the demand is known as **non-industry demand.** The model of the input–output analysis is known as the input–output model.

Types of input–output model

There are two types of models

a) Open input–output model   b) Closed input–output model

a) Open input–output model

In this type of model, some outputs of the industry will be used by the industries and rest part will be used by other i.e. consumers. In this model also, the assumption "what is produced is consumed" satisfies i.e. input and output balances. In this model, we find the volume of production to fulfill the future demand when the present demand is given i.e. known. We present below the following example:

Input / Output	A	B	Consumer's demand	Total output
A	60	90	50	200
B	75	45	30	150

Total output of A is 200 units in which 60 and 90 units of them are used as the inputs by A and B respectively and 50 units by other consumers. Again the total output of B is 150 units in which 75 and 45 units of them are used by A and B respectively and 30 units by other consumers.

From the table, we see that for the fulfilment of the 50 units of consumer's demand, 200 units of output from A are to be produced. In the same way, 150 units of output from B are to be produced for the fulfilment of consumer's demand of 30 units.

Thus for the production of 1 unit of A, $\frac{60}{200}$ unit of A's product and $\frac{75}{150}$ of B's product are required. Similarly, for the production of 1 unit of B, $\frac{90}{150}$ unit of A's product and $\frac{45}{150}$ unit of B's product are required. Now, a matrix is formed with the fraction of A's input and B's input for a unit production of A and B as the elements. The matrix thus formed is known as the **input-output matrix** and the fraction of the inputs of A and B are known as the **technical input coefficient**. So, the input–output matrix is also known as **matrix of technical coefficient** or technology matrix.

Now we have the following input–output matrix

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 60/200 & 90/150 \\ 75/200 & 45/150 \end{pmatrix} \end{array} \quad \text{i.e.} \quad \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} 0.3 & 0.6 \\ 0.375 & 0.3 \end{pmatrix} \end{array}$$

The sum of the elements (i.e. entries) in each column must be less than 1 otherwise, the production will not be feasible.

Let the total output of A and B be x_1 and x_2 units respectively. then to produce x_1 units of A, $0.3x_1$ units of A and $0.6x_2$ units of B are taken as the inputs and rest 50 units are sold to the consumers i.e. other thant the industries.

$\therefore \quad x_1 = 0.3x_1 + 0.6x_2 + 50$

Similarly, $x_2 = 0.375x_1 + 0.3x_2 + 30$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.6 \\ 0.375 & 0.3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 50 \\ 30 \end{pmatrix}$$

$$\Rightarrow \quad X = AX + D$$

$$\Rightarrow \quad (I - A)X = D$$

where I is the unit matrix, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = vector of output necessary for the fulfillment of the future demand, $A = \begin{pmatrix} 0.3 & 0.6 \\ 0.375 & 0.3 \end{pmatrix}$ is the matrix of technical coefficients or input coefficient matrix and $D = \begin{pmatrix} 50 \\ 30 \end{pmatrix}$ is the demand vector.

If $I - A$ is non-singular, then $(I - A)^{-1}$ exists.

$\therefore \quad X = (I - A)^{-1}D$

General two sector Input-Output Model

We consider here the interrelationship and the interdependency of the economy of two industries. The interrelationship is presented below

Industries	A	B	Final demand	Total output
A	x_{11}	x_{12}	d_1	x_1
B	x_{21}	x_{22}	d_2	x_2

Here, x_1 = total output of industry A $\qquad$ x_2 = total output of industry B

x_{11} = output of A is taken as input of A $\qquad$ x_{12} = output of A is taken as input of B

x_{21} = output of B is taken as input of A $\qquad$ x_{22} = output of B is taken as input of B

d_1 = final demand of A $\qquad$ d_2 = final demand of B

Since output of A is used as input of A and B and the rest by the external bodies, so

$$\left.\begin{aligned} x_1 &= x_{11} + x_{12} + d_1 \\ \text{Similarly } x_2 &= x_{21} + x_{22} + d_2 \end{aligned}\right\}$$

Since $\quad a_{11}$ = input coefficient $= \dfrac{x_{11}}{x_1}$, $\quad a_{12} = \dfrac{x_{12}}{x_2}$

$a_{21} = \dfrac{x_{21}}{x_1}$ $\quad$ and $\quad a_{22} = \dfrac{x_{22}}{x_2}$

so, $x_{11} = a_{11}x_1$, $\qquad x_{12} = a_{12}x_2$

$x_{21} = a_{21}x_1$ $\qquad x_{22} = a_{22}x_2$

Using these results in (i), we have

$x_1 = a_{11}x_1 + a_{12}x_2 + d_1$

and $x_2 = a_{21}x_1 + a_{22}x_2 + d_2$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$\Rightarrow X = AX + D$$

where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ = vector containing the outputs for the fulfillment of future demands

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ = input coefficient matrix

and $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ = demand vector

$\Rightarrow (I - A)X = D$ where I = unit matrix

If $I - A = T$ be non-singular, then

$(I - A)^{-1} = T^{-1}$ exists.

$\therefore X = (I - A)^{-1}D = T^{-1}D$

I - A is a matrix known as Leontief's technology matrix or Leontief's matrix only.

Note: In each column, the sum of the elements must be less than 1, otherwise, the production is not feasible.

Note 2: If the output of one industry is not used by the other industry as the input, the entry in the leading diagonal will be zero.

b) Closed Input–Output Model

In this type of model, it is assumed that whatever is produced by one industry will be used only by the industries as the inputs and no part of the production will be used by the external bodies. So, in this case there will be no consumer's demand. The model of the input–output model here will be of the following form.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow X = AX$

$\Rightarrow (I - A)X = 0$

$\Rightarrow TX = 0$

Note: In this type of model, sum of the entry's in each column will be unity i.e. 1, as no final demand occurs.

Hawkin's–Simon Condition

This is the condition where the viability of the input–output model is examined. The following are the two conditions of viability.

i) If A = input coefficient matrix and I = unit matrix of same order as that of A, then each diagonal elements of the matrix I – A is positive.

ii) The determinant of I – A i.e. | I – A | is positive.

If the above two conditions are satisfied, then the input–output model i.e. system is viable, otherwise it is not.

The above two conditions are known as **Hawkin's–Simon condition.**

Worked Out Examples

Example 1.

Given below is the input coefficient matrix

$$A = \begin{pmatrix} 0.7 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}$$

a) What type of model does the given input coefficient matrix represent?

b) Find Leontief's technology matrix.

c) Test whether the given system is viable as per Hawkin's-Simon condition.

Solution:

a) Since sum of the elements in each column of the given input coefficient matrix A is not equal to 1, so the given input matrix is the open type of model.

b) $A = \begin{pmatrix} 0.7 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}$

Leontief's technology matrix

$$= T = I - A$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.7 & 0.5 \\ 0.2 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-0.7 & 0-0.5 \\ 0-0.2 & 1-0.3 \end{pmatrix} = \begin{pmatrix} 0.3 & -0.5 \\ -0.2 & 0.7 \end{pmatrix}$$

c) Each element in the leading diagonal of the matrix T is positive.

$$\text{Also } |T| = \begin{vmatrix} 0.3 & -0.5 \\ -0.2 & 0.7 \end{vmatrix}$$

$$= 0.3 \times 0.7 - (-0.5)(-0.2)$$

$$= 0.21 - 0.10 = 0.11 > 0$$

∴ the given input–output system is viable.

Example 2.

From the following input–output table

Industry	X	Y	Final demand
X	120	80	200
Y	140	160	100

a) Determine input–output coefficient matrix (A)

b) Find I – A.

Solution:

a) Total output of X = x_1 = 120 + 80 + 200 = 400

Total output of Y = x_2 = 140 + 160 + 100 = 400

Here, $x_{11} = 120$, $\quad x_{12} = 80$

$x_{21} = 140$, $\quad x_{22} = 160$

$$a_{11} = \frac{x_{11}}{x_1} = \frac{120}{400} = 0.3, \qquad a_{12} = \frac{x_{12}}{x_2} = \frac{80}{400} = 0.2$$

$$a_{21} = \frac{x_{21}}{x_1} = \frac{140}{400} = 0.35, \qquad a_{22} = \frac{x_{22}}{x_2} = \frac{160}{400} = 0.4$$

Now, the input–output coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.2 \\ 0.35 & 0.4 \end{pmatrix}$$

b) $$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.2 \\ 0.35 & 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-0.3 & 0-0.2 \\ 0-0.35 & 1-0.4 \end{pmatrix} = \begin{pmatrix} 0.7 & -0.2 \\ -0.35 & 0.6 \end{pmatrix}$$

Example 3.

If the input–output coefficient matrix $A = \begin{pmatrix} 0.2 & 0.4 \\ 0.5 & 0.3 \end{pmatrix}$, find the demand vector D which is consistent with the output vector $\begin{pmatrix} 100 \\ 80 \end{pmatrix}$.

Solution:

Here, $A = \begin{pmatrix} 0.2 & 0.4 \\ 0.5 & 0.3 \end{pmatrix}$

$$I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.4 \\ 0.5 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 1-0.2 & 0-0.4 \\ 0-0.5 & 1-0.3 \end{pmatrix} = \begin{pmatrix} 0.8 & -0.4 \\ -0.5 & 0.7 \end{pmatrix}$$

By given, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 80 \end{pmatrix}$

Now, $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ be the demand vector.

Then, $(I - A)X = D$

$$\Rightarrow \begin{pmatrix} 0.8 & -0.4 \\ -0.5 & 0.7 \end{pmatrix}\begin{pmatrix} 100 \\ 80 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 80-32 \\ -50+56 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 48 \\ 6 \end{pmatrix}$$

$\therefore$ demand vector (D) $= \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 48 \\ 6 \end{pmatrix}$

Example 4.

A and D, the input-output coefficient matrix and the demand vector respectively are given below:

$$A = \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 560 \\ 320 \end{pmatrix}$$

Find the total output.

Solution:

Here, $A = \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.2 \end{pmatrix}$

$$T = I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.4 \\ 0.2 & 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-0.1 & 0-0.4 \\ 0-0.2 & 1-0.2 \end{pmatrix} = \begin{pmatrix} 0.9 & -0.4 \\ -0.2 & 0.8 \end{pmatrix}$$

$$|T| = \begin{vmatrix} 0.9 & -0.4 \\ -0.2 & 0.8 \end{vmatrix} = 0.72 - 0.08 = 0.64 \neq 0$$

$\therefore T^{-1}$ exists.

T_{11} = cofactor of 0.9 = 0.8, $\quad T_{12} = -(-0.2) = 0.2, \quad T_{21} = -(-0.4) = 0.4, \quad T_{22} = 0.9$

$$\therefore \text{matrix of cofactors} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.9 \end{pmatrix}$$

$$\text{Adjoint of } T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}$$

$$\text{i.e. Adj. } T = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}$$

$$T^{-1} = \frac{\text{Adj. } T}{|T|} = \frac{\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}}{0.64} = \frac{1}{0.64}\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}$$

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where x_1 and x_2 be the two outputs.

Then, $X = T^{-1}D$

$$= \frac{1}{0.64}\begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.9 \end{pmatrix}\begin{pmatrix} 560 \\ 320 \end{pmatrix}$$

$$= \frac{1}{0.64}\begin{pmatrix} 448 + 128 \\ 112 + 288 \end{pmatrix} = \frac{1}{0.64}\begin{pmatrix} 576 \\ 400 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 900 \\ 625 \end{pmatrix}$$

$\therefore x_1 = 900$ and $x_2 = 625$

$\therefore$ the required total outputs to meet the future demands of the consumers are 900 units and 625 units.

Example 5.

Given the following transaction matrix

Producing sector	Purchasing sector		Final demand
	Agriculture	Industry	
Agriculture	300	600	100
Industry	400	1200	400

a) Write the input coefficient matrix.

b) Find the total output to meet the final demand of 160 units of agriculture and 400 units of industry.

Solution:

Here, $x_{11} = 300, x_{12} = 600;$ $x_{21} = 400, x_{22} = 1200$

$x_1 = 300 + 600 + 100 = 1000$

$x_2 = 400 + 1200 + 400 = 2000$

$a_{11} = \frac{x_{11}}{x_1} = \frac{300}{1000} = 0.3,$ $a_{12} = \frac{x_{12}}{x_2} = \frac{600}{2000} = 0.3$

$a_{21} = \frac{x_{21}}{x_1} = \frac{400}{1000} = 0.4,$ $a_{22} = \frac{x_{22}}{x_2} = \frac{1200}{2000} = 0.6$

Now, the input coefficient matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$

b) $I - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.3 & 0.3 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 1-0.3 & 0-0.3 \\ 0-0.4 & 1-0.6 \end{pmatrix}$

$T = \begin{pmatrix} 0.7 & -0.3 \\ -0.4 & 0.4 \end{pmatrix}$

$|T| = \begin{vmatrix} 0.7 & -0.3 \\ -0.4 & 0.4 \end{vmatrix} = 0.28 - 0.12 = 0.16 \neq 0$

$\therefore T^{-1}$ exists.

$T_{11} = 0.4,$ $T_{12} = -(-0.4) = 0.4,$ $T_{21} = -(-0.3) = 0.3,$ $T_{22} = 0.7$

Matrix of cofactors $= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.4 \\ 0.3 & 0.7 \end{pmatrix}$

Adj. $T = \begin{pmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$

$T^{-1} = \frac{\text{Adj. } T}{|T|} = \frac{1}{0.16}\begin{pmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$

Let x_1 and x_2 be the total outputs of the agriculture and industry respectively. Then,

$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and final demand matrix $= D = \begin{pmatrix} 160 \\ 400 \end{pmatrix}$

Now using $X = (I - A)^{-1}D = T^{-1}D$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{0.16}\begin{pmatrix} 0.4 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}\begin{pmatrix} 160 \\ 400 \end{pmatrix}$$

$$= \frac{1}{0.16}\begin{pmatrix} 64 + 120 \\ 64 + 280 \end{pmatrix} = \frac{1}{0.16}\begin{pmatrix} 184 \\ 344 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1150 \\ 2150 \end{pmatrix}$$

$\therefore x_1 = 1150, x_2 = 2150$

$\therefore$ for the fulfillment of 160 units of agriculture and 400 units of industry, the total output of 1150 units of agriculture and 2150 units of industry are required.

EXERCISE

1. Given below are the input coefficients matrix

a) $A = \begin{pmatrix} 0.7 & 0.1 \\ 0.2 & 0.5 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.3 & 0.2 \\ 0.4 & 0.7 \end{pmatrix}$

c) $A = \begin{pmatrix} 3/5 & 7/10 \\ 1/2 & 2/5 \end{pmatrix}$

Test Hawkin's-Simon condition for the viability of the system.

2. a) From the following input-output table

Sectors	X	Y	Final demand
X	50	30	20
Y	40	72	8

i) Calculate the input coefficient matrix A.

ii) Determine I – A, I being 2 × 2 unit matrix.

b) Given the following inter-industrial transaction taken for two industries

Input / Output	A	B	Final demand
A	90	120	40
B	125	105	70

i) Obtain the input coefficient matrix A.

ii) Find I – A.

3. The input-output coefficient matrix A and the total output vector X are given below

a) $A = \begin{pmatrix} 0.2 & 0.4 \\ 0.6 & 0.3 \end{pmatrix}, X = \begin{pmatrix} 120 \\ 180 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.5 \end{pmatrix}, X = \begin{pmatrix} 85 \\ 90 \end{pmatrix}$

c) $A = \begin{pmatrix} 0.25 & 0.5 \\ 0.45 & 0.4 \end{pmatrix}, X = \begin{pmatrix} 640 \\ 780 \end{pmatrix}$

Find the demand vector D for each of the above cases.

4. Given the input-output coefficient matrix A and the demand vector D, find the total output for each of the following cases:

a) $A = \begin{pmatrix} 0.5 & 0.4 \\ 0.3 & 0.4 \end{pmatrix}, D = \begin{pmatrix} 45 \\ 81 \end{pmatrix}$ b) $A = \begin{pmatrix} 0.3 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}, D = \begin{pmatrix} 40 \\ 50 \end{pmatrix}$

c) $A = \begin{pmatrix} 0.15 & 0.45 \\ 0.6 & 0.4 \end{pmatrix}, D = \begin{pmatrix} 30 \\ 48 \end{pmatrix}$

5. a) The following table shows the inter-relationship between the product of two industries A and B in a year

Industry	A	B	Consumer's demand	Total output
A	45	50	55	150
B	30	40	30	100

i) Find the input coefficient matrix.

ii) Find the gross output of the two industries A and B to satisfy the demands of 72 and 48 units.

b) The inter-industry transaction table presented below was formed for an economy of the two industries P and Q for a certain year

Producers	Users		Final	Total
	P	Q		
P	250	160	90	500
Q	200	120	80	400

i) Find the input coefficient matrix.

ii) Find the total output to be produced by the industries P and Q when the final demands are 152 of P and 114 units of Q.

A. There are three producing sectors A, B and C. Let A use 30% of what it produces, 40% of B's production and 10% of C's production. Again B uses 20% of A's production, 25% of B's production and 45% of C's production, If C uses 10%, 35% and 25% of A, B and C's production respectively, form the input-output coefficient matrix.

In an economy consisting of industries X and Y, the input coefficients are $a_{11} = 0.4$, $a_{12} = 0.1$, $a_{21} = 0.2$ and $a_{22} = 0.3$. If the final demand of the product X is 20 units and that of the product Y is 15 units, construct the input-output table for the given economy. Also construct the input-output matrix equation for the model.

Answers

1. a) viable b) viable c) No

2. a) i) $\begin{pmatrix} 0.5 & 0.25 \\ 0.4 & 0.6 \end{pmatrix}$, $\begin{pmatrix} 0.5 & -0.25 \\ -0.4 & 0.4 \end{pmatrix}$ b) i) $\begin{pmatrix} 0.36 & 0.4 \\ 0.5 & 0.35 \end{pmatrix}$ ii) $\begin{pmatrix} 0.64 & -0.4 \\ -0.5 & 0.65 \end{pmatrix}$

3. a) $\begin{pmatrix} 24 \\ 54 \end{pmatrix}$ b) $\begin{pmatrix} 7 \\ 11 \end{pmatrix}$ c) $\begin{pmatrix} 90 \\ 180 \end{pmatrix}$

4. a) $x_1 = 330$, $x_2 = 300$ units b) $x_1 = 180$, $x_2 = 250$ units c) $x_1 = 165$, $x_2 = 245$ units

5. a) i) $\begin{pmatrix} 0.3 & 0.5 \\ 0.2 & 0.4 \end{pmatrix}$ ii) $x_1 = 210$, $x_2 = 150$ units

   b) i) $\begin{pmatrix} 0.5 & 0.4 \\ 0.4 & 0.3 \end{pmatrix}$ ii) $x_1 = 800$, $x_2 = 620$ units

Dynamics of Market Price

In this section, we deal with the model involving differential equation with t as the independent variable.

Dynamics of Market Model

The following are the notations used in the model:

d = quantity demanded = d(t)

s = quantity supplied = s(t)

p = price in time t = p(t)

P_0 = price at t = 0

The one-commodity market model has the following equations

$$Q_d = a - bP \; (a, b > 0) \qquad \ldots\ldots(i)$$

$$Q_s = -c + dP \; (c, d > 0) \qquad \ldots\ldots(ii)$$

Here we assume that the equilibrium condition is easily obtained. Hence, we have

$$Q_d = Q_s$$

$$\Rightarrow a - bP = -c + dP$$

$$\Rightarrow a + c = (b + d)P$$

$$\therefore P = \frac{a + c}{b + d}$$

But in practice, the equilibrium condition may not easily be obtained. When price changes, it effects Q_d as well as Q_s and hence $Q_d - Q_s$ too.

$\therefore$ rate of change of price (P) is proportional to $Q_d - Q_s$.

i.e. $\frac{dP}{dt} = k(Q_d - Q_s)$

where k is the positive constant and is known as the adjustment coefficient.

$\frac{dP}{dt} = k\{a - bP - (-c + dP)\}$ (From (i) and (ii))

$= k(a + c) - k(b + d)P$

$\Rightarrow \frac{dP}{dt} + k(b + d)P = k(a + c)$ ......(iii)

Comparing it with $\frac{dy}{dt} + Ay = B$, we have

$A = k(b + d)$ and $B = k(a + c)$, $y = P$ and $t = t$

Now, the solution of the differential equation (iii) is

$y = C.e^{-At} + \frac{B}{A}$

$\Rightarrow P = C.e^{-k(b+d)t} + \frac{a + c}{b + d}$ ......(iv)

If $P = \overline{P}$ be the particular integral (PI), then $\frac{dP}{dt} = 0$. Use of this reault reduces the equation (iii) to

$k(b + d)\overline{P} = k(a + c)$

$\Rightarrow \overline{P} = \frac{a + c}{b + d}$

From (iv), $P = C.e^{-k(b+d)t} + \overline{P}$

When $t = 0$, $P(0) = C.1 + \overline{P}$

$\therefore C = P(0) - \overline{P}$

Now the complete solution is

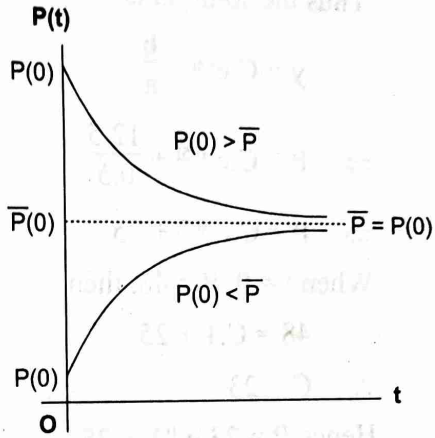
$$P = \{P(0) - \overline{P}\}\, e^{-k(b+d)k} + \overline{P}$$

where $\overline{P}$ is the equilibrium and equal to $\frac{a + c}{b + d}$.

As P(0) and $\overline{P}$ are constants, so when $t \to \infty$, $e^{-k(b+d)t} \to 0$ for $k > 0$.

In this situation, $P(t)$ converges to $\overline{P}$ and $P(t)$ is called the inter-temporal equilibrium price.

Now, we have the following three cases:



i) If $P(0) = \overline{P}$, then $P(t) = \overline{P}$ and hence the time path of $P(t)$ is horizontal i.e. the price is constant.

ii) If $P(0) > \overline{P}$, then $P(t) > \overline{P}$, but as t increases, the first term decreases i.e. $P(t)$ approaches $\overline{P}$ i.e. the time of $P(t)$ is decreasing and approaches $\overline{P}$ from the above.

iii) If $P(0) < \overline{P}$, then $P(t) < \overline{P}$ but as t increases, the first term decreases i.e. $P(t)$ approaches $\overline{P}$ so the time path of $P(t)$ is increasing and approaches $\overline{P}$ from the below.

Whenever $k(b + d) > 0$, $e^{-(b+d)t} \to 0$ as $t \to \infty$, so $P(t)$ converges to $\overline{P}$ and hence the price will be stable in the long-run. But if $k(b + d) < 0$, $e^{-k(b+d)t} \to \infty$ as $t \to \infty$ so $P(t)$ diverges and hence the price will not be stable.

Worked Out Examples

Example 1.

Demand and supply function for tea (kg/week) are given by

$$Q_d = 60 - P + 2\frac{dP}{dt} \quad \text{and} \quad Q_s = 2P - 15 + 8\frac{dP}{dt}$$

where P is the price at time t. If the initial price (i.e. when t = 0) is Rs. 48 per kg., find the time path of P for dynamic equilibrium. What will be its price in 2 weeks? What will be the time path of the prices in the long run?

Solution:

For equilibrium price,

$$Q_d = Q_s$$

$$\Rightarrow 60 - P + 2\frac{dP}{dt} = 2P - 15 + 8\frac{dP}{dt}$$

$$\Rightarrow 6\frac{dP}{dt} + 3P = 75$$

$$\Rightarrow \frac{dP}{dt} + 0.5P = 12.5$$

Comparing with $\frac{dy}{dt} + ay = b$, we have

$a = 0.5, \quad b = 12.5, \quad y = P, \quad t = t$

Thus the solution is

$$y = C.e^{-at} + \frac{b}{a}$$

$$\Rightarrow \quad P = C.e^{-0.5t} + \frac{12.5}{0.5}$$

$$\Rightarrow \quad P = C.e^{-0.5t} + 25$$

When t = 0, P = 48, then

$$48 = C.1 + 25$$

$$\therefore \quad C = 23$$

Hence $P = 23\ e^{-0.5t} + 25$

When t = 2,

$$P = 23 \times e^{-0.5 \times 2} + 25$$
$$= 23.e^{-1} + 25$$
$$= 23 \times 0.3679 + 25$$
$$= 33.46$$

$\therefore$ the required price in 2 weeks will be Rs. 33.46.

Again, $P = 23.e^{-0.5t} + 25$

When $t \to \infty$, $\quad e^{-0.5t} \to 0$, $\quad$ so $P \to 25$.

Hence in the long run the price level of the tea converges to Rs. 25.

Example 2.

The demand and the supply function of a new product in a competitive market are $Q_d = 120 - 2P$ and $Q_s = 3P - 40$ respectively.

At the time of unquilibrium condition, the rate of price adjustment is

$$\frac{dP}{dt} = 0.25(Q_d - Q_s)$$

Derive and solve the differential equation given that P(0) = 20

i) Find the price when t = 4.

ii) Is P(4) close to the equilibrium condition?

iii) Examine the state of stability in the long-run.

Solution:

$Q_d = 120 - 2P, \qquad Q_s = 3P - 40$

$Q_d - Q_s = 120 - 2P - 3P + 40 = 160 - 5P$

and $\frac{dP}{dt} = 0.25(Q_d - Q_s) = 0.25(160 - 5P)$

$\Rightarrow \frac{dP}{dt} = 40 - 1.25P$

$\Rightarrow \frac{dP}{dt} + 1.25P = 40$

Here $a = 1.25$, $b = 40$, $P = y$, $t = 5$

$\therefore$ the complete solution is $y = C.e^{-at} + \frac{b}{a}$

$\Rightarrow P = C.e^{-1.25t} + \frac{40}{1.25}$

$\Rightarrow P = C.e^{-1.25t} + 32$

when $t = 0$, $P = 20$

$20 = C.1 + 32$ $\quad\quad\quad \therefore C = -12$

$\therefore P = -12e^{-1.25t} + 32$

i) When $t = 4$

$P = -12e^{-1.25 \times 4} + 32 = 31.92$

ii) When $t \to \infty$, $e^{-1.25t} \to 0$, so $P(t) \to 32$

$\therefore$ equilibrium price = 32 and $P(4) = 31.92$

Difference of $P(4)$ and the equilibrium price

$= 32 - 31.92 = 0.08$ which is small.

$\therefore$ $P(4)$ is close to the equilibrium price.

iii) As $t \to \infty$, $e^{-1.25t} \to 0$ and hence the first term tends to zero, so the price in the long-run approaches to 32 and hence stable.

Example 3.

In a competitive market, the demand and the supply functions are given by the equations $Q_d = 240 - 3P$ and $Q_s = 5P - 150$. Also, the rate of change of price adjustment proportional to the excess of demand is given by $\frac{dP}{dt} = 0.05(Q_d - Q_s)$

Solve the differential equation for the time path of $P(t)$, the initial price level P_0 being Rs. 50.

i) Predict the price level for the time period 4.

ii) In how many time periods would its price level dropped by Rs. 6 than the initial price.

Solution:

$Q_d = 240 - 3P$ and $Q_s = 6P - 150$

$Q_d - Q_s = 240 - 3P - 6P + 150 = 390 - 9P$

Also, $\frac{dP}{dt} = 0.05(Q_d - Q_s)$

$\Rightarrow \frac{dP}{dt} = 0.05(390 - 9P)$

$= 19.5 - 0.45P$

$\Rightarrow \frac{dP}{dt} + 0.45P = 19.5$

Here a = 0.45, b = 19.5, y = P, t = t

Now, the complete solution is

$y = C.e^{-at} + \frac{b}{a}$

$\Rightarrow P = C.e^{-0.45t} + \frac{19.5}{0.45}$

$\Rightarrow P = C.e^{-0.45t} + \frac{130}{3}$

when t = 0, P = 50.

$50 = C.1 + \frac{130}{3}$ $\therefore C = \frac{20}{3}$

$\therefore$ the solution is $P = \frac{20}{3}e^{-0.45t} + \frac{130}{3}$

i) When t = 4,

$P = \frac{20}{3}e^{-0.45 \times 4} + \frac{130}{3}$

$= \frac{20}{3} \times e^{-1.8} + \frac{130}{3}$

$= \frac{20}{3} \times 0.1653 + \frac{130}{3} = 44.43$

ii) When P = 50 – 6 = 44

$44 = \frac{20}{3}e^{-0.45t} + \frac{130}{3}$

$\Rightarrow \frac{2}{3} = \frac{20}{3}e^{-0.45t}$

$\Rightarrow 0.1 = e^{-0.45t}$

$\Rightarrow -0.45t = \log 0.1 = -2.3026$

$\therefore t = \frac{-2.304}{-0.45} = 5$ periods

EXERCISE

1. a) If the demand and the supply functions are

$$Q_d = 60 - 2P + 4\frac{dP}{dt} \qquad \text{and} \qquad Q_s = 3P - 20 + 5\frac{dP}{st},$$

find the time path of the price P when the initial price condition is 25. Find the value of P when t = 3. Is P(3) close to equilibrium price?

b) Demand and supply functions are

$$x_d = 250 - 3P + 4\frac{dP}{dt} \qquad \text{and} \qquad x_s = -150 + 2P + 14\frac{dP}{dt}.$$

Find the time path of P for the dynamic equilibrium if initial price is Rs. 50. Compute the price in time t = 4? What will be the condition of the time path of the price in the long run? Will it be stable?

2. a) In a competitive market,

$$x_d = 450 - 1.8P \qquad \text{and} \qquad x_s = -180 + 1.2P \quad \text{respectively.}$$

If the rate of price adjustment proportional to the excess of demand is $\frac{dP}{dt} = 0.32(Q_d - Q_s)$.

i) Solve the differential equation when the currently quoted price is Rs. 200.

ii) Forecast the price for time (t) = 2.

b) The demand function and the supply function in the competitive market are

$$Q_d = 90 - 0.6P \qquad \text{and} \qquad Q_s = -30 + 0.4P$$

and the rate of price adjustment at the time when the condition is not in the market is

$$\frac{dP}{dt} = 0.5(Q_d - Q_s)$$

Solve the differential equation given that initial price is Rs. 140.

i) What will be the value of P(t) when t = 6?

ii) State the condition of the price level stability in the market.

c) In a competitive market, the demand and the supply function of fruits are

$$Q_d = 240 - 1.4P \qquad \text{and} \qquad Q_s = -60 + 2.6P$$

and the rate of price adjustment proportional to the excess of demand is given by $\frac{dP}{dt} = 0.2(Q_d - Q_s)$ with the current price level Rs. 120.

i) Find P(3).

ii) In what time period would its price level be Rs. 84?

iii) If there needs a large quantity of fruits in 4 months time, but someone offers a contract to supply the quantity of fruits at a price of Rs. 72 per kg, would it be accepted?

Answers

1. a) $P = 9.e^{-5t} + 16$, 16, yes
   b) $P = -30.e^{-0.5t} + 80$, Rs. 75.94, the time path converges to the equilibrium level of the price; yes
2. a) i) $P = -10.e^{-0.96t} + 210$ ii) Rs. 208.53
   b) $P = 20.e^{-0.5t} + 120$, 120 i) Rs. 121; ii) Converges to Rs. 120, Stable
   c) $P(t) = 45.e^{-0.8t} + 75$, i) 2 periods, ii) yes

Difference Equation

The values of some economic variables change due to the change in time. The time considered may be **discrete** or **continuous**. In most of the economic studies, focus is given to those economic variables in which we consider the discrete time. For example; today's price of an item depends upon the demand of the same item in the previous day's time. The expenditure of a man for a particular month depends upon the previous month's income. The present population of a certain country is estimated according to the population of the same country in a decade ago. These are the examples of dynamic situation but not the static.

The solution of the problem of continuous time will be solved by the help of differential equation. But the solution of the problem for discrete time will be solved by the method of **finite difference** where **finite difference** is the difference of variate values between two successive intervals of time.

Finite difference

Let t and y represent the independent and dependent variables respectively. Let y(t) and y(t + 1) be the values of the dependent variable at time t and t + 1 respectively. Then the difference y(t + 1) – y(t) is known as the finite difference and is denoted by Δy(t).

$$\therefore \quad \Delta y(t) = y(t + 1) - y(t)$$

where Δ is known as the difference operator.

For convenient, we write y_t for y(t) and y_{t+1} for y(t + 1), then

$$\Delta y_t = y_{t+1} - y_t$$

Note: y_{t-1} is the value of y at time t – 1 i.e. the value of y, one period before the time t.

Difference equation

A difference equation is an equation which involves independent variable, dependent variable and their successive differences.

If t be the independent variable and y the dependent variable then the equation relating y_t and y_{t+1} is known as the difference equation. For example:

$$y_{t+1} = 2y_t \text{ and } y_{t+1} + 2y_t = 3$$

The **order** of the difference equation is the highest difference involved in the equation. For example:

$\Delta y_t = 4$ is the **first order** difference equation.

Here $\Delta y_t = 4$

$\Rightarrow$ $y_{t+1} - y_t = 4$

The order is one, because the highest difference

$= t + 1 - t = 1$

Also, $y_t - y_{t-1} = 0$ is also of order one because highest difference $= t - (t - 1) = 1$.

A difference equation is said to be **homogeneous** if there is no separate constant term. For example,

$y_{t+1} = ay_t$ is first order homogenous difference equation.

A difference equation with separate constant term is known as **non-homogeneous** difference equation. For example:

$y_{t+1} - ay_t = b$

Solution of a Difference Equation

The solution of a difference equation is a function which satisfies the given equation.

A solution containing arbitrary constants equal in number to the order of the difference equation is called the **general solution**. But the solution obtained by assigning the particular values to the arbitrary constants is known as the **particular solution**.

A first order difference equation can be solved by the following two methods:

i) Iteration method ii) General method

i) Iteration method

In this method, we use the process of iteration i.e. repeated application of finite difference to get the solution of the difference equation.

Example: Solve the difference equation by iteration method

$\Delta y_t = -0.5\, y_t$

$\Rightarrow$ $y_{t+1} - y_t = -0.5y_t$

$\Rightarrow$ $y_{t+1} = 0.5y_t$

$\Rightarrow$ $y_t = 0.5\, y_{t-1}$ (moving 1 period back)

Put t = 1, 2, 3, 4,

$y_1 = 0.05\, y_0$

$y_2 = 0.5\, y_1 = 0.5\,(0.5\, y_0) = (0.5)^2\, y_0$

$y_3 = 0.5\, y_2 = (0.5)\,((0.5)^2\, y_0) = (0.5)^3\, y_0$

: : : : : :

In the similar manner,

$y_t = (0.5)^t\, y_0$

$\Rightarrow$ $y_t = A\,(0.5)^t$ where $y_0 = A$

ii) General Method

First order homogeneous difference equation

Let the first order homogeneous difference equation with constant coefficient be

$$y_{t+1} - a\, y_t = 0 \quad \ldots\ldots(i)$$

Let $y_t = A\beta^t$ be the trial solution of (i)

Then, $y_{t+1} = A\,\beta^{t+1}$

Substituting these values in (i), we have

$$A\,\beta^{t+1} - a.\ A\beta^t = 0$$

$$\Rightarrow \quad A\beta^t(\beta - a) = 0$$

$$A\beta^t \neq 0$$

$$\beta - a = 0 \quad \therefore \beta = a$$

$\therefore$ the general solution of (i) is

$$y_t = Aa^t$$

General method:

The general form of a non-homogeneous first order difference equation is

$$y_5 = ay_{t-1} + b \quad \ldots\ldots(i)$$

this equation is very similar to the non-homogeneous first order linear differential equaton. So, its solution is also equal to the sum of the complementary function (CF) and the particular integral (PI).

Its complementary function is

$$CF = Aa^t$$

For PI: If $y_t = k$ be the trial solution of (i), then

$$k = ak + b(y_{t-1} = k)$$

$$\Rightarrow \quad k - ka = b$$

$$\Rightarrow \quad k(1 - a) = b$$

$$\therefore \quad k = \frac{b}{1-a} \quad (a \neq 1)$$

$\therefore$ the general solution is $y_t = CF + PI$

$$\Rightarrow \quad y_t = Aa^t + \frac{b}{1-a}$$

If $a = 1$, let $y_t = kt$ be the trial solution.

Then $\quad y_{t-1} = k(t-1)$

Now for equation (i)

$$kt = k(t-1) + b \qquad (\because a = 1)$$

$k = b$

$\therefore$ PI is $= kt = bt$

$\therefore$ the general solution is

$y_t = CF + PI$

$= A + bt \qquad (\because a = 1)$

$\therefore$ the general solution of the non-homogeneous linear difference equation $y_t = ay_{t-1} + b$ is

$$y_t = \begin{cases} Aa^t + \dfrac{b}{1-a} & \text{for } a \neq 1 \\ A + bt & \text{for } a = 1 \end{cases}$$

where A is a constant.

If $Aa^t = 0$, $y_t = \dfrac{b}{1-a}$ which gives the line parallel to t-axis, known as the equilibrium level of stability. The time path of y_t depends upon the value of a^t. We have the following cases:

i) If $|a| > 1$, then $a^t \to \infty$ as $t \to \infty$, so y_t diverges. Hence, y_t is unstable. Here if a is negative i.e. $a < -1$ then the time path is oscillatory about t-axis.

ii) If a is a negative proper fraction then $a^t \to 0$ as $t \to \infty$ so y_t converges. Hence y_t is stable. Her also, the time parth of y_t is oscillatory about t-axis.

iii) If $a > 1$, then $a^t \to \infty$ as $t \to \infty$, so y_t divergence and hence unstable. The time path of y_t moves away from t-axis or equilibium level.

iv) If a is positive proper fraction, $a^t \to 0$ as $t \to \infty$ so y_t converges and hence stable. The time path of y_t moves towards t-axis or equilibrium level.

Worked Out Examples

Example 1

Find the first four terms of the sequence of solutions of the following difference equation

$y_t - 1.2y_{t-1} = 0 \quad , \quad y_0 = 18$

Also, find the general term.

Solution:

$y_t - 12.y_{t-1} = 0 \quad \Rightarrow \quad y_t = 1.2\, y_{t-1}$

When $t = 1$, $\quad y_1 = 1.2\, y_0 = 1.2 \times 18$

$t = 2$, $\quad y_2 = 1.2y_1 = 1.2\,(1.2 \times 18) = 1.2^2 \times 18$

$t = 3$, $\quad y_3 = 1.2y_2 = 1.2\,(1.2^2 \times 18) = 1.2^3 \times 18$

$t = 4$, $\quad y_4 = 1.2y_3 = 1.2(1.2^3 \times 18) = 1.2^4 \times 18$

which are the first four terms of the sequence of solution.

Proceeding in the same way $\qquad y = (1.2)^5 \times 18$

Example 2

A difference equation $y_{t+1} - 1.5y_t = 0$ is given

i) Solve the equation by iteration method for the year 2, 3 and 4 given that the income in year 1 is Rs. 25000

ii) Find the general expression for y_t in terms of t.

iii) Evaluate y_{10} when y_1 = Rs. 25000

Solution:

$y_{t+1} - 1.5y_t = 0$

$\Rightarrow y_{t+1} = 1.5\, y_t$

$\Rightarrow y_t = 1.5\, y_{t-1}$ (moving one year back)

i) when t = 2, $y_2 = 1.5y_1 = 1.5 \times 25000$ = Rs. 7500

t = 3, $y_3 = 1.5y_2 = 1.5\,(1.5\, y_1) = 1.5^2 y_1 = 1.5^2 \times 25000$ = Rs. 56250

t = 4, $y_4 = 1.5y_3 = 1.5\,(1.5^2\, y_1) = 1.5^3 y_1 = 1.5^3 \times 25000$ = Rs. 84375

ii) $y_2 = 1.5y_1$

$y_3 = 1.5^2 y_1$

$y_4 = 1.5^3 y_1$

Procceding in th same way, $y_t = 1.5^{t-1}\, y_1$

iii) When t = 10, $y_{10} = 1.5^{10-1} \times 25000$ = Rs. 961084 (approx.)

Example 3

Solve the following diffeential equation by iteration as well as general method.

$$\Delta y_t + 1.6y_t = 0$$

Solution:

Iteration method:

$\Delta y_t + 1.6y_t = 0$

$\Rightarrow y_{t+1} - y_5 + 1.6y_t = 0$

$\Rightarrow y_{t+1} = -0.6y_t$

$\Rightarrow y_t = -0.6\, y_{t-1}$ (Moving one year back)

$= -0.6\,(-0.6\, y_{t-2})$

$= (-0.6)^2\, y_{t-2}$

$= (-0.6)^2\,(-0.6\, y_{t-3})$

$y_t = (-0.6)^3\, y_{t-3}$

Proceeding in the same say,

$y_t = (-0.6)^t\, y_{t-t} = (-0.6)^5\, y_0$

$y_t = A.(-0.6)^t$ where $y_0 = A$

General method:

$$\Delta y_t + 1.6y_t = 0$$

$$\Rightarrow y_{t+1} - y_t + 1.6y_t = 0$$

$$\Rightarrow y_{t+1} = -0.6y_t$$

$$\Rightarrow y_t = -0.6\, y_{t-1} \qquad \text{(moving one year back)}$$

Comparing this equation with $y_t = ay_{t-1} + b$

we have $a = -0.6$, $b = 0$

Now the complete solution is

$$y_t = A.a^t + \frac{b}{1-a} = A.(-0.6)^t + \frac{0}{1+0.6}$$

$$\therefore\ y_t = A(-0.6)^t$$

Alternative method:

$$y_t = -0.6y_{6-1}$$

Let $y_t = A\beta^t$ be the trial solution. Then

$$A\beta^t + 0.6A\beta^{t-1} = 0$$

$$\Rightarrow A\beta^{t-1}(\beta + 0.6) = 0$$

$$A\beta^{t-1} \neq 0$$

$$\therefore\ \beta + 0.6 = 0$$

$$\Rightarrow \beta = -0.6$$

$\therefore$ the solution is $y_t = a\beta^t = A(-0.6)^t$

Example 3

Find the general as well as the particular solution of the following differential equation.

$$y_t - \frac{2}{3}y_{t-1} = 16 \quad , \qquad y_0 = 60$$

Find the behaviour of the time path.

Solution:

$$y_t = \frac{2}{3}y_{t-1} = 16$$

Comparing with $y_t = a\, y_{t-1} + b$, we have

$$a = \frac{2}{3}, \qquad b = 16$$

Now, the complete solution is

$$y_t = A.(a)^t + \frac{b}{1-a}$$

$$\Rightarrow \quad y_t = A.\left(\frac{2}{3}\right)^t + \frac{16}{1 - 2/3}$$

$$\Rightarrow \quad y_t = A.\left(\frac{2}{3}\right)^t + \frac{16 \times 3}{1}$$

$$\therefore \quad y_t = A\left(\frac{2}{3}\right)^t + 48$$

which is the required general solution.

When $t = 0$, $y_0 = 60$

$$y_0 = A.1 + 48$$

$$\Rightarrow \quad 60 = A + 48$$

$$\therefore \quad A = 12$$

Substituting the value of A, we have

$$y_t = 12\left(\frac{2}{3}\right)^t + 48$$

Since $\left(\frac{2}{3}\right)^t \to 0$ as $t \to \infty$, so y_t converges to 48 in the long-run.

Alternative method

$$y_t = \frac{2}{3} y_{t-1} + 16 \qquad \ldots\ldots(i)$$

The corresponding homogeneous linear difference equation is

$$y_t = \frac{2}{3} y_{t-1}$$

whose CF is

$$CF = A.\left(\frac{2}{3}\right)^t$$

For PI: If $y_t = k$ be the trial solution, then $y_{6-1} = k$.

$\therefore$ from (i)

$$k = \frac{2}{3}k + 16$$

$$\Rightarrow \quad k - \frac{2}{3}k = 16$$

$$\Rightarrow \quad \frac{1}{3}k = 16$$

$$\therefore \quad k = 48$$

Now, the complete solution is

$$y_t = CF + PI$$

$\Rightarrow \quad y_t = A.\left(\frac{2}{3}\right)^t + 48$

when $t = 0, y_0 = 60$

$y_0 = A.1 + 48$

$\Rightarrow \quad 60 = A + 48$

$\therefore \quad A = 12$

$\therefore$ the particular colution is $y_t = 12\left(\frac{2}{3}\right)^t + 48$.

Example 4

A finance company provides an interest at the rate of 9% p.a. compounded annually. If a man deposits Rs. 40000 initially, how much will he get at the end of 3 years?

Solution:

Since the rate is 9% p.a., so amount in t years is

$y_t = (1.09)\, y_{t-1}$

Comparing this equation with $y_t = ay_{t-1} + b$, we have

$a = 1.09 \quad , \quad b = 0$

$\therefore$ the solution is

$$y_t = A.(a)^t + \frac{b}{1-a}$$

$$y_t = A.(1.09)^t + 0 = A.(1.09)^t$$

Initially i.e. when $t = 0$, $y_0 = 40000$

$40000 = A.1 + 0$

$\therefore \quad A = 40000$

$\therefore \quad y_t = 40000\,(1.09)^t$

when $t = 3$, $y_3 = 40000\,(1.09)^3 = 51801.16$

$\therefore$ the man will get Rs. 51801.16 in 3 years.

EXERCISE

1. Find the solution of the following difference equations using iteration method for the periods $t = 1, 2, 3, 4$.
   a) $y_t = 0.4\, y_{t-1}$ given that $y_0 = 24$
   b) $y_t - 0.75\, y_{t-1} = 0$ given that $y_0 = 16$
   c) $\Delta y_t + 1.2y_t = 0$ with $y_1 = 8$

2. Find the first four terms of the sequence of solution of the following difference equation

a) $y_t - \frac{2}{5} y_{t-1} = 0$, $y_0 = 25$

b) $y_t + 0.6\, y_{t-1} = 0$, $y_0 = 30$

c) $y_t - 5y_{t-1} = 0$, with $y_0 = 12$

3. A difference equation $y_{t+1} = 1.8y_t$ is given

a) Solve the equation for the periods 2, 3, 4 given that the income in the year 1 is Rs. 12500.

b) Obtain the sequence of solutions of the difference equation for t = 2, 3, 4 in terms of y_1.

c) Find the general solution for y_t in terms of t.

d) Evaluate y_6 when y_1 = Rs. 12500.

4. Solve the following differnce equations

a) $y_t - 0.7\, y_{t-1} = 0$, $y_0 = 5$

b) $y_t = 6y_{t-1}$, $y_0 = 8$

c) $y_t = -\frac{1}{2} y_{t-1}$, $y_0 = 11$

d) $y_{t+1} + 5y_5 = 0$, $y_0 = 12$

5. Find the general and particular solutions of the following difference equations

a) $y_t = 0.9y_{t-1} + 8,$ $y_0 = 110$

b) $y_t + \frac{3}{4} y_{t-1} = 14,$ $y_0 = 28$

c) $y_t = 5y_{t-1} + 4,$ $y_0 = 14$

d) $y_{t+1} + \frac{2}{3} y_t = 25,$ $y_0 = 20$

Comment on the behaviour of the time path and then on the stability.

6. a) Find the general solution of $y_{t+1} - 0.2y_t = 0$.

b) If $y_2 = 48$, find the particular solution.

c) Evaluate y_1, y_4, y_6.

7. a) A finance company provides an interest of 8% p.a. compounded annually. If a man invests Rs. 25000 initially in the company, how much will the man receive at the end of 2 years?

b) The population of a certain city increases at the rate of 2% p.a. If the initial population of the city in the year 2010 be 150000, what will be the population of the city in 2020?

Answers

1. a) $24(0.4), 24(0.4)^2, 24(0.4)^3, 24(0.4)^4$ b) $16(0.75), 16(0.75)^2, 16(0.75)^3, 16(0.75)^4$
c) $8(-0.2), 8(-0.2)^2, 8(-0.2)^3, 8(-0.2)^4$

2. a) $25\left(\frac{2}{5}\right), 25\left(\frac{2}{5}\right)^2, 25\left(\frac{2}{5}\right)^3, 25\left(\frac{2}{5}\right)^4$ b) $30(-0.6), 30(-0.6)^2, 30(-0.6)^3, 30(-0.6)^4$

   c) $12(5), 12(5)^2, 12(5)^3, 12(5)^4$

3. a) Rs. 22500, Rs. 40500, Rs. 72900 b) $1.8y_1, (1.8)^2y_1, (1.8)^3y_1$

   c) $y_t = (1.8)^{t-1}y_1$ d) y_6 = Rs. 1236196

4. a) $y_t = 5(0.7)^5$ b) $y_t = 8(6)^t$ c) $y_t = 11\left(-\frac{1}{2}\right)^t$ d) $y_t = 12(-5)^t$

5. a) $y_t = A(0.9)^t + 80$, $y_t = 30(0.9)^t + 80$, converges to 80, stable

   b) $y_t = A\left(-\frac{3}{4}\right)^t + 8_n$, $y_t = 20\left(-\frac{3}{4}\right)^t + 8$, converges to 8, oscillatory, stable

   c) $y_t = A(-5)^5 - 1$, $y_t = 15(-5)^5 - 1$, diverges, unstable

   d) $y_t = A\left(-\frac{2}{3}\right)^t + 15$, $y_t = 5\left(-\frac{2}{3}\right)^t + 15$, converges to 25, oscillatory, stable

6. a) $y_t = A(0.2)^t$ b) $y_t = 1200(0.2)^t$ c) 240, 1.92, 0.07681

7. a) Rs. 29160 b) 182849

The Cobweb Model

As an application of first order linear difference equation in economic analysis, an economic model known as 'Cobweb Model' is used in analyzing the economic behavior of supply and demand towards the equilibrium level.

The basic assumption of Cobweb model lies on the fact that

i) today's demand depends upon today's price P_t

ii) today's supply depends upon yesterday's decision about the output. Hence the present output is affected by yesterday's price P_{t-1}.

The model is based on the following three equations

$$Q_{dt} = a - b\,P_t \quad \ldots\ldots \text{(i)} \quad (a, b > 0)$$

$$Q_{st} = -c + d\,P_{t-1} \quad \ldots\ldots \text{(ii)} \quad (c, d > 0)$$

and $$Q_{dt} = Q_{st} \quad \ldots\ldots \text{(iii)}$$

Using (i) and (ii) in (iii), we have

$$a - b\,P_t = -c + d\,P_{t-1}$$

$$\Rightarrow \quad b\,P_t + d\,P_{t-1} = a + c$$

$$\Rightarrow \quad P_t + \frac{d}{b}P_{t-1} = \frac{a+c}{b} \quad \ldots\ldots \text{(I)}$$

which is in the form $y_t + ay_{t-1} = b$.

Hence it is a first order linear difference equation. Its solution is the sum of C.F. and P.I.

For C.F.

The corresponding homogeneous difference equation is

$$P_t + \frac{d}{b}.P_{t-1} = 0 \quad \ldots\ldots \text{(II)}$$

Its C.F. = $A\left(-\frac{d}{b}\right)^t$

For P.I.

Let $P_t = \overline{P}$ be the trial solution. Then

$$P_{t-1} = \overline{P}$$

Now from equation (I)

$$\overline{P} + \frac{d}{b}\overline{P} = \frac{a+c}{b}$$

$$\Rightarrow \quad \overline{P}\,\frac{(b+d)}{b} = \frac{a+c}{b}$$

$$\therefore \quad \overline{P} = \frac{a+c}{b+d}$$

This price is known as the equilibrium price.

∴ the general solution is

$$P_t = CF + PI$$

$$\Rightarrow \quad P_t = A\left(-\frac{d}{b}\right)^t - \frac{a+c}{b+d}$$

$$\Rightarrow \quad P_t = A\left(-\frac{d}{b}\right)^t - \overline{P}$$

When t = 0, $P_t = P(0)$, then

$$P(0) = A.1 + \overline{P}$$

$$\therefore \quad A = P(0) - \overline{P}$$

Now, the particular solution is $P_t = (P(0) - \overline{P})\left(-\frac{d}{b}\right)^t + \overline{P}$.

The convergent and the divergent of the price towards $\overline{P}$, the quilibrium upon $\left(-\frac{d}{b}\right)^t$.

i) If $\left|\frac{d}{b}\right| < 1$, as $t \to \infty$, $\left(-\frac{d}{b}\right)^t \to 0$, so the price p_t converges to $\overline{P}$ and hence stable.

ii) If $\left|\frac{d}{b}\right| > 1$, as $t \to \infty$, $\left(-\frac{d}{b}\right)^t \to \infty$, so the price p_t diverges and hence unstable.

iii) If $\left|\frac{d}{b}\right| = 1$, then $-\frac{d}{b}$ is negative, so the time path is oscillatory.

Worked Out Examples

Example 1

The demand function and supply function for a certain product are $Q_{d,t} = 62 - 10P_t$ and $Q_{s,t} = -18 + 6P_{t-1}$ respectively

a) Using the market equilibrium condition, form a differential equation in P_t.

b) Find the solutions for P_t and Q_t given that $P_0 = 7$.

c) Find the equilibrium price.

d) Show that the price P_t converges to the equilibrium price. Is the price stable?

Solution:

a) Under equilibrium condition,

$$Q_{d,t} = Q_{s,t}$$

$$\Rightarrow 62 - 10P_t = -18 + 6P_{t-1}$$

$$\Rightarrow 10P_t + 6P_{t-1} = 80$$

$$\Rightarrow P_t + \frac{3}{5}P_{t-1} = 8 \qquad \ldots\ldots \text{(i)}$$

b) The corresponding homogeneous linear difference equation is

$$P_t + \frac{3}{5}P_{t-1} = 0$$

Its CF is $CF = A\left(-\frac{3}{5}\right)^t$

For PI: If $P_t = \overline{P}$ be the trial solution of (i), then

$$\overline{P} + \frac{3}{5}\overline{P} = 8 \qquad (\because P_t = P_{t-1} = \overline{P})$$

$$\Rightarrow \frac{8}{5}\overline{P} = 8$$

$$\therefore \overline{P} = \frac{8 \times 5}{8} = 5$$

Now, the complete solution is

$$P_t = CF + PI$$

$$\Rightarrow P_t = A\left(-\frac{3}{5}\right)^t + 5$$

When $t = 0$, $P_t = P_0 = 7$

$$7 = A.1 + 5 \qquad \therefore A = 2$$

$$\therefore P_t = 2\left(-\frac{3}{5}\right)^t + 5$$

Now for Q_t,

$$Q_t = 62 - 10P_t$$

$$= 62 - 10\left\{2\left(-\frac{3}{5}\right)^t + 5\right\}$$

$$= 62 - 20\left(-\frac{3}{5}\right)^t - 50$$

$$= 12 - 20\left(-\frac{3}{5}\right)^t$$

c) The equilibrium price = $\overline{P}$ = 5(from part (b))

d) $P_t = 2\left(-\frac{3}{5}\right)^t + 5$

Since $\left(-\frac{3}{5}\right)^t \to 0$ as $t \to \infty$, so P_t converges to 5 in the long run. But $\overline{P}$ = equilibrium price = 5, so P_t converges to the equilibrium price. Hence the price is stable in the long run.

EXERCISE

1. Find the general and the particular solutions of the following Cobweb model.

   a) $Q_t^d = 11 - 3P_t$, $Q_t^s = 6P_{t-1} - 7$; $P_0 = 5$

   b) $Q_t^d = 23 - 4P_t$, $Q_t^s = 3P_{t-1} - 5$; $P_0 = 10$

   c) $Q_{s,t} = -8 + 4P_{t-1}$, $Q_{d,t} = 32 - 6P_t$; $P_0 = 12$

   d) $Q_t^s = -8 + 3P_{t-1}$, $Q_t^d = 100 - 9P_t$; $P_0 = 15$

   Examine whether the prices will be stable?

2. a) The demand the supply functions for a good at time t are $Q_t^d = 80 - 8P_t$ and $Q_t^s = -24 + 5P_{t-1}$, respectively.

      i) Using the condition of market equilibrium, deduce a difference equation in P_t.

      ii) Solve the difference equation to find the equilibrium price given that $P_0 = 12$

      iii) Also find the equilibrium quantity.

   b) Given that demand function and supply function are $Q_{s,t} = -100 + 4P_{t-1}$ and $Q_{d,t} = 170 - 5P_t$ respectively. Using equilibrium conditon, frind expressions for P_t and Q_t when $P_0 = 36$. Also find the equilibrium price and quantity. Is the price level stable?

c) Consider the supply function $Q_t^s = 5P_{t-1} - 6$ and the demand function $Q_t^d = 54 - 10P_t$. Using the condition of equilibrium, obtain the solutions of P_t and Q_t given that $P_0 = 6$. Is the system stable?

Answers

1. a) $P_t = A(-2)^t + 2,\ P_t = 3(-2)^t + 2$; unstable b) $P_t = A\left(-\frac{3}{4}\right)^t + 4,\ P_t = 6\left(-\frac{3}{4}\right)^t + 4$; stable

c) $P_t = A\left(-\frac{2}{3}\right)^t + 4,\ P_t = 8\left(-\frac{2}{3}\right)^t + 4$, stable d) $P_t = A\left(-\frac{1}{3}\right)^t + 9,\ P_t = 6\left(-\frac{1}{3}\right)^t + 9$, stable

2. a) i) $P_t + \frac{5}{8}P_{t-1} = 13$ ii) Equilibrium price = 8 iii) Quantity = 16

b) $P_t = 6\left(-\frac{4}{5}\right)^t + 30,\ Q_t = 20 - 30\left(-\frac{4}{5}\right)^t$; 30, 20; yes

c) $P_t = 2\left(-\frac{1}{2}\right)^t + 4;\ Q_t = 14 - 20\left(-\frac{1}{2}\right)^t$; yes

Lagged Keynesian Macroeconomic Model

The income determination model also known as Lagged Keynesian macro-economical model is assumed to be based on the fact that the notional income y_t depends upon two factors: consumption c_t and the investment I_t. That is, the national income is equal to the sum of the consumption and investment. Also it is assumed that consumption is a lagged function of income i.e. a function of income of previous period y_{t-1} or it is simply assumed that c_t is proportional to y_{t-1}. Further it is assumed that the investment I_t is constant and is denoted by I_0.

The model has the following three equations

$$y_t = c_t + I_t \qquad \ldots\ldots\text{(i)}$$

$$c_t = a + by_{t-1} \qquad \ldots\ldots\text{(ii)}$$

and $$I_t = I_0 \qquad \ldots\ldots\text{(iii)}$$

where b is the marginal propensity to consume and $0 < b < 1$. Using (ii) and (iii) in (i), we have

$$y_t = a + by_{t-1} + I_0$$

$$\Rightarrow \quad y_t - by_{t-1} = a + I_0 \qquad \ldots\ldots\text{(iv)}$$

This is a non-homogeneous first order linear difference equation. The corresponding homogeneous linear difference equation of (iv) is

$$y_t - by_{t-1} = 0$$

Its complementary function is

$$CF = A(b)^t$$

For PI: Let $y_t = k$ be the trial solution.

Then $y_{t-1} = k$

Now from (iv) $k - bk = a + I_0$

$\Rightarrow \quad k(1-t) = a + I_0$

$\therefore \quad k = \dfrac{a + I_0}{1-b} \qquad (b \neq 1)$

$\therefore$ the general solution of Keynesian economic model is

$$y_t = CF + PI$$

$$y_t = A(b)^t + \frac{a + I_0}{1-b}$$

The constant value A can be determined by putting t = 0 when y_0 is given.

Note: If a = 0, then the solution is $y_t = Ab^t + \dfrac{I_0}{1-b}$

Since b < 1, so $b^t \to 0$ as $t \to \infty$. Hence this model always provides us y_t to tend $\dfrac{a_0 + I_0}{1-b}$, the equilibrium level. This is the main features of this model which is different from Cobweb model.

Worked Out Examples

Example 1

A simple national income model is given below:

$y_t = c_t + I_t$, $c_t = 0.36y_{t-1} + 48$, $I_t = 240$

a) Express the given national income model as the difference equation in y_t.

b) Find the general as well as particular solution of the difference equation when y_0 = 585.

c) Find the nature of the time path. Is the system stable?

Solution:

a) $y_t = c_t + I_t$

$= 0.36y_{t-1} + 48 + 240$

$\Rightarrow \quad y_t = 0.36y_{t-1} + 288 \qquad \ldots\ldots(i)$

b) The corresponding homogeneous linear differnce equation is

$$y_t - 0.36y_{t-1} = 0$$

Its complementary function is

$$CF = A.(0.36)^t$$

For the particular integral (PI)

Let $y_t = k$ be the trial solution. Then from (i)

$$k = 0.36k + 288$$

$$\Rightarrow \quad k - 0.36k = 288$$

$$\Rightarrow \quad 0.64k = 288$$

$$\Rightarrow \quad k = \frac{288}{0.64} = 450$$

$\therefore$ the general solution is

$$y_t = CF + PI$$

$$\Rightarrow \quad y_t = A(0.36)^t + 450$$

When $t = 0$, $y_t = y_0 = 585$

$$585 = A.1 + 450$$

$$\therefore \quad A = 235$$

$\therefore$ the particular solution is $y_t = 235(0.36)^5 + 450$

c) Since $0.36 < 1$, so as $t \to \infty$, $(0.36)^t \to 0$ and hence y converges to 450. That is, the time path of y_t approaches to the equilibrium level of 450.

$\therefore$ the system is stable.

EXERCISE

1. The following are the simple national income model
   a) $y_t = c_t + I_t$, $\quad c_t = 0.50y_{t-1}$, $\quad I_t = 40$ and $y_0 = 96$
   b) $y_t = c_t + I_t$, $\quad c_t = 0.40y_{t-1}$, $\quad I_t = 360$ and $y_0 = 650$
   c) $y_t = c_t + I_t$, $\quad c_t = 0.25y_{t-1} + 32$, $I_t = 118$ and $y_0 = 240$
   d) $y_t = c_t + I_t$, $\quad c_t = \frac{3}{5}y_{t-1} + 40$, $\quad I_t = 160$ and $y_0 = 620$
      i) Express each of the above national income equation as a differnce equation in y_t.
      ii) Solve each of the difference equatons.
2. a) Consider a two sector income model
      $y_t = c_t + I_t$, $\quad c_t = 156 + 0.2y_{t-1}$, $\quad I_t = 356$
      i) Find an expression for y_t
      ii) Solve the difference equation when $y_0 = 900$
      iii) Discuss the nature of the time path. Is the system stable?
   b) Assume a simple national income model
      $y_t = c_t + I_t$, $\quad c_t = 0.84y_{t-1} + 70$, $\quad I_t = 130$
      i) Express the income model as a difference equation in y_t.
      ii) Find the general solution as well as particular solution when $y_0 = 2200$.
      iii) Discuss the nature of the time path. Will the system stabilize?

Answers

1. a) i) $y_t = 0.50y_{t-1} + 40$ ii) $y_t = 16(0.50)^t + 80$
   b) i) $y_t = 0.40y_{t-1} + 360$ ii) $y_t = 50(0.40)^t + 600$
   c) i) $y_t = 0.25y_{t-1} + 150$ ii) $y_t = 40(0.25)^t + 200$
   d) i) $y_t = \frac{3}{5}y_{t-1} + 200$ ii) $y_t = 120(0.6)^t + 500$
2. a) i) $y_t = 0.2y_{t-1} + 512$ ii) $y_t = 260(0.2)^t + 640$ iii) y converges to 640, stable
   b) i) $y_t = 0.84y_{t-1} + 200$ ii) $y_t = A(0.84)^t + 1250$, $y_t = 950(0.84)^t + 1250$
   iii) y_t converges to 1250, stable

कोरोना भाइरस रोगबाट बच्ने उपायहरु

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