SOLUTIONS TO MODEL QUESTIONS 2068 (SET I)

Time: 3 hrs | Full Marks: 100 | Pass Marks: 35

Candidates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks.

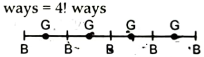
Attempt ALL questions of group A and group B or C.

Group A

1. a. It is required to seat 5 boys and 4 girls in a row so that the girls occupy the even places. How many such arrangements are possible? [2]

Solution

Since the girls occupy only the even places, they can take 2nd, 4th, 6th and 8th seats. Hence, 4 girls in 4 places can be arranged in P(4,4) ways = 4! ways



And, 5 boys in remaining 5 seats can be arranged in P(5,5) ways = 5! ways.

∴ Total number of arrangements $= 4! \times 5!$

$= 4 \times 3 \times 2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 2880$

b. Prove that: $\frac{1}{1\cdot3}+\frac{1}{2\cdot5}+\frac{1}{3\cdot7}+\frac{1}{4\cdot9}+\ldots = 2(1-\ln 2)$. [2]

Solution

$$\text{L.H.S.} = \frac{1}{1\cdot3}+\frac{1}{2\cdot5}+\frac{1}{3\cdot7}+\frac{1}{4\cdot9}+\ldots$$

$$= 2\left(\frac{1}{2\cdot3}+\frac{1}{4\cdot5}+\frac{1}{6\cdot7}+\frac{1}{8\cdot9}+\ldots\right) = 2\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\ldots\right]$$

$$= 2\left[1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\ldots\right)\right] = 2[1-\ln(1+1)] = 2(1-\ln 2)$$

c. Let a * b = 3a + 2b for a, b ∈ Z. Verify that * is a commutative binary operation on Z. [2]

Solution

Let a, b ∈ Z. Then 3a, 2b ∈ Z.

Since sum of any two integers is again an integer, a*b = 3a + 2b ∈ Z.

Hence, for all a, b ∈ Z, a * b = 3a + 2b ∈ Z uniquely. So, * is a binary operation.

Again, take a = 2, b = 3 in Z. Then,

$a * b = 2 * 3 = 3 \times 2 + 2 \times 3 = 12$

$b * a = 3 * 2 = 3 \times 3 + 2 \times 2 = 13$

∴ $a * b \neq b * a$

Hence, * is not commutative.

2. a. Find the equation of a hyperbola in standard position such that the length of transverse axis is 6 and it passes through (4, 2). [2]

Solution

Let the equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \ldots(i)$$

By question, length of transverse axis = 6

i.e. 2a = 6

∴ a = 3

Putting the value of a in (i), we get

$$\frac{x^2}{9}-\frac{y^2}{b^2}=1$$

If the hyperbola passes through the point (4, 2), then

$$\frac{16}{9}-\frac{4}{b^2}=1$$

or, $\frac{16}{9}-1=\frac{4}{b^2}$

$\therefore$ $b^2=\frac{36}{7}$

Putting the value of a and b in (i), we get

$$\frac{x^2}{9}-\frac{y^2}{36/7}=1$$

or, $\frac{x^2}{9}-\frac{7y^2}{36}=1$

$\therefore$ $4x^2-7y^2=36$

which is the required equation of hyperbola

b. Find the locus of points which are equidistant from the points (1, 2, 3) and (3, 2, –1). **[2]**

Solution

Let P(x, y, z) be any point on the locus.

By given, distance of (1, 2, 3) from P(x, y, z) = distance of (3, 2, –1) from P (x,y,z)

or, $(x-1)^2+(y-2)^2+(z-3)^2=(x-3)^2+(y-2)^2+(z+1)^2$

or, $x^2-2x+1+y^2-4y+4+z^2-6z+9=x^2-6x+9+y^2-4y+4+z^2+2z+1$

$\therefore$ $x-2z=0$

c. Find the cosines of the angle between the vectors: **[2]**

$\vec{a}=(1,-2,-2)$, $\vec{b}=(2,1,-2)$.

Solution

Given, $\vec{a}=(1,-2,-2)$, $\vec{b}=(2,1,-2)$

Then, $\vec{a}\cdot\vec{b}=(1,-2,-2)\cdot(2,1,-2)=2-2+4=4$

$a=|\vec{a}|=\sqrt{1^2+(-2)^2+(-2)^2}=3$

$b=|\vec{b}|=\sqrt{2^2+1^2+(-2)^2}=3$

If θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}=\frac{4}{(3)\,(3)}=\frac{4}{9}$$

$\therefore$ Cosine of the angle between the vectors is $\frac{4}{9}$.

3. a. Find the derivative of $(\ln x)^{\sinh x}$. **[2]**

Solution

Let $y=(\ln x)^{\sinh x}$

Taking 'ln' on both sides, we get,

$\ln y=\sinh x\ \ln(\ln x)$

Differentiating both sides with respect to 'x'

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}\{\sinh x\ \ln(\ln x)\}$$

$$\frac{d}{dy}(\ln y)\cdot\frac{dy}{dx}=\sinh x\cdot\frac{d}{dx}\{\ln(\ln x)\}+\ln(\ln x)\frac{d}{dx}(\sinh x)$$

or, $\frac{1}{y}\frac{dy}{dx}=\sinh x\cdot\frac{d\{\ln(\ln x)\}}{d(\ln x)}\cdot\frac{d(\ln x)}{dx}+\ln(\ln x)\cdot\cosh x$

or, $\frac{1}{y}\frac{dy}{dx}=\sinh x\,\frac{1}{\ln x}\cdot\frac{1}{x}+\cosh x\cdot\ln(\ln x)$

or, $\frac{dy}{dx}=y\left\{\frac{\sinh x}{x\ln x}+\cosh x.\ \ln(\ln x)\right\}$

$\therefore$ $\frac{dy}{dx}=(\ln x)^{\sinh x}\left\{\frac{\sinh x}{x\ln x}+\cosh x.\ \ln(\ln x)\right\}$

b. Find the integral $\int\frac{dx}{1+2\sin x}$ **[2]**

Solution

$$\text{Let } I=\int\frac{dx}{1+2\sin x}=\int\frac{dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cdot2\sin\frac{x}{2}\cos\frac{x}{2}}=\int\frac{dx}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\cdot2\sin\frac{x}{2}\cos\frac{x}{2}}\times\frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$=\int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1}$$

Put $\tan\frac{x}{2}=y$

$\frac{1}{2}\sec^2\frac{x}{2}\,dx=dy$

$\sec^2\frac{x}{2}\,dx=2dy$

$$\therefore\ I=\int\frac{2dy}{y^2+4y+1}=2\int\frac{dy}{(y+2)^2-3}=2\int\frac{dy}{(y+2)^2-(\sqrt{3})^2}=2\cdot\frac{1}{2\sqrt{3}}\log\left(\frac{y+2-\sqrt{3}}{y+2+\sqrt{3}}\right)+C$$

$$=\frac{1}{\sqrt{3}}\log\left(\frac{\tan\frac{x}{2}+2-\sqrt{3}}{\tan\frac{x}{2}+2+\sqrt{3}}\right)+C$$

c. Find the integral $\int\frac{dx}{(x+7)\sqrt{2-x}}$ **[2]**

Solution

Put $2-x=y^2$

or, $-dx=2ydy$

$\therefore$ $dx=-2ydy$

Now,

$$\int\frac{dx}{(x+7)\sqrt{2-x}}=\int\frac{-2y\,dy}{(9-y^2)\cdot y}=2\int\frac{dy}{y^2-3^2}=2\cdot\frac{1}{2\cdot3}\log\left(\frac{y-3}{y+3}\right)+C=\frac{1}{3}\log\left(\frac{\sqrt{2-x}-3}{\sqrt{2-x}+3}\right)+C$$

4. a. Solve the differential equation: $\frac{dy}{dx}=e^{x+y}+3x^2e^y$. **[2]**

Solution

Given equation is:

$$\frac{dy}{dx}=e^{x+y}+3x^2e^y$$

or, $\frac{dy}{dx}=e^y(e^x+3x^2)$

or, $e^{-y}dy=(e^x+3x^2)\,dx$

On integration, we get

$$-e^{-y}=e^x+\frac{3x^3}{3}+C$$

$\therefore$ $e^x+e^{-y}+x^3+C=0$

b. From the following data, calculate the expected value of Y when X = 25,

	X	Y
Average	5.6	12.5
Standard deviation	3.2	2.4

and correlation coefficient r = 0.95. **[2]**

Solution

Here, $\bar{X} = 5.6$, $\bar{Y} = 12.5$, $\sigma_X = 3.2$, $\sigma_Y = 2.4$, $r = 0.95$

$b_{YX} = r \cdot \frac{\sigma_Y}{\sigma_X} = 0.95 \times \frac{2.4}{3.2} = 0.7125$

The regression equation of Y on X is

$Y - \bar{Y} = b_{YX}(X - \bar{X})$

or, $Y - 12.5 = 0.7125\,(X - 5.6)$

or, $Y - 12.5 = 0.7125X - 3.99$

or, $Y = 0.7125X + 12.5 - 3.99$

$\therefore$ $Y = 0.7125X + 8.51$

When X = 25, $Y = 0.7125 \times 25 + 8.51 = 26.3225$

Required expected value of Y is 26.3225.

c. The average percentage of failures in a certain examination is 40. What is the probability that out of 5 candidates, at least 3 will be passed in the examination? **[2]**

Solution

Here, $p = P(\text{failure}) = 40\% = 0.4$

$\therefore$ $q = P(\text{not failure}) = 1 - p = 1 - 0.4 = 0.6$

$n = 5$

Therefore, the probability that at least 3 will be passed in the examination i.e. at most 2 will be failed is given by

$P(r \le 2) = P(0) + P(1) + P(2) = {}^5C_0\, p^0 q^5 + {}^5C_1\, p^1 q^4 + {}^5C_2\, p^2 q^3$

$= 1 \times (0.6)^5 + 5 \times (0.4) \times (0.6)^4 + 10 \times (0.4)^2 \times (0.6)^3$

$= 0.07776 + 0.2592 + 0.3456 = 0.683$

5. a. Show that the number of combinations of n different objects taken r at a time is given by $C(n, r) = \frac{n!}{(n-r)!\,r!}$. Also, prove that C (n, n −r) = C (n, r). **[4]**

Solution

Consider any one of the C(n,r) combinations. This combination contains r objects. These r objects among themselves can be arranged in r! different ways. So, for each combination, there are r! permutations. Hence, for the C(n,r) combinations, there are C(n,r) · r! different permutations. Since these are all possible permutations of n objects taken r at a time. So, we have

$C(n,r) \cdot r! = P(n,r)$

or, $C(n,r) \cdot r! = \frac{n!}{(n-r)!}$

$\therefore$ $C(n,r) = \frac{n!}{(n-r)!\,r!}$

Second Part

$C(n, n-r) = \frac{n!}{(n-(n-r))!(n-r!)} = \frac{n!}{(n-r)!\,r!} = C(n,r)$

OR

State the multiplication principle of counting. Prove that the number of circular permutations of n different objects taken all at a time is (n −1)! **[4]**

Solution

Multiplication Principle of Counting

If one thing can be done independently in n_1 different ways and if a second thing can be done in n_2 different ways and if a third thing can be done in n_3 ways and so on for any finite number of things, then the total number of ways in which all the things can be done in the given order is $n_1 \cdot n_2 \cdot n_3 \cdot \ldots$

Next Part

Let $x_1, x_2, \ldots, x_n$ be the objects arranged in a circle, then the arrangements $x_1x_2x_3 \ldots x_n$, $x_2x_3x_4 \ldots x_nx_1$, $x_3x_4x_5 \ldots x_nx_1x_2$, … are not different. But if they are arranged in a row (straight line), to each arrangement in a circle, there are n arrangements in a straight line. Thus, if P is the number of arrangements in a circle, we have

$nP = n!$

or, $P = \frac{n!}{n} = \frac{n \cdot (n-1)!}{n} = (n-1)!$

b. What is a group? If a binary operation * is defined on a set S = {a, b, c} by the following Caley's table.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Show that (S, *) is a group. **[4]**

Solution

Group: Let G be a non empty set and * is an operation defined on G.

Then,

(G, *) is said to be a group if * satisfies the following axioms.

1. Closure axiom: G is closed under the operation *.
   i.e. $a*b \in G$ for all $a, b \in G$
2. Associative axiom
   $a*(b*c) = (a*b)*c$ for all $a, b, c \in G$
3. Identity axiom: For all $a \in G$ there exists an element $e \in G$ such that
   $a*e = e*a = a$
   The element e is called the identity element.
4. Inverse axiom: Each element of G possesses inverse i.e. for all $a \in G$, there exists $b \in G$ such that $a*b = b*a = e$.
   The element b is called the inverse of a. We write $b = a^{-1}$.

Second Part

Here, S = {a, b, c}

i. Closure property: Since all the entries in the Cayley's table are element of S, S is closed under *.

ii. Associative property:

$a*(b*c) = a*a = a$

$(a*b)*c = b*c = a$

$\therefore$ $a*(b*c) = (a*b)*c$

This result is true for all elements of S.

$\therefore$ Associativity is satisfied.

iii. From table,

$a*a = a$

$b*a = a*b = b$

& $c*a = a*c = c$

$\therefore$ a is the identity element.

iv. $a*a = a$

$b*c = c*b = a$

$\therefore$ a, b, c are inverse element of a, c and b respectively.

From (i) – (iv) we conclude (S, *) is a group.

OR

Let a, b, c and x be elements of a group G. Solve for x if axb = c and $x^2b = xa^{-1}c$. **[4]**

Solution

Let, $a, b, c, x \in G$.

Here, $x^2b = xa^{-1}c$

or, $x^{-1} \cdot (x \cdot x \cdot b) = x^{-1} \cdot (xa^{-1}c)$

or, $(x^{-1} \cdot x) \cdot (xb) = (x^{-1} \cdot x) \cdot a^{-1}c$ (by associativity)

or, $e \cdot xb = e \cdot a^{-1}c$

or, $xb = a^{-1}c$

or, $(xb) \cdot b^{-1} = (a^{-1}c) \cdot b^{-1}$

or, $x(b \cdot b^{-1}) = a^{-1}cb^{-1}$

or, $x \cdot e = a^{-1}cb^{-1}$

$\therefore$ $x = a^{-1}cb^{-1}$

Putting the value of x in axb = c, we get

$a(a^{-1}cb^{-1})b = c$

or, $(aa^{-1})c(b^{-1}b) = c$

or, $ece = c$

$\therefore$ $c = c$ (true)

$\therefore$ $x = a^{-1}cb^{-1}$

which is the required solution.

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6. a. Find the integral $\int \frac{x}{x^3+1}dx$. **[4]**

Solution

Here, $\int \frac{x}{x^3+1}dx = \int \frac{x}{(x+1)(x^2-x+1)}dx$

Let $\frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ ...(i)

or, $\frac{x}{(x+1)(x^2-x+1)} = \frac{A(x^2-x+1)+(Bx+C)(x+1)}{(x+1)(x^2-x+1)}$

or, $x = A(x^2-x+1) + (Bx+C)(x+1)$

Put x = –1, then $A = -\frac{1}{3}$

Again, put x = 0, then $C = \frac{1}{3}$

And, put x = 1, then $B = \frac{1}{3}$

Now, from (i)

$$\frac{x}{(x+1)(x^2-x+1)} = \frac{-1/3}{x+1} + \frac{1/3\,x+1/3}{x^2-x+1} = -\frac{1}{3(x+1)} + \frac{1}{3}\frac{(x+1)}{(x^2-x+1)}$$

$$\therefore \int \frac{x}{x^3+1}dx = \int \frac{x}{(x+1)(x^2-x+1)}dx = -\frac{1}{3}\int \frac{1}{x+1}dx + \frac{1}{3}\int \frac{(x+1)\,dx}{x^2-x+1}$$

$$= -\frac{1}{3}\int \frac{1}{x+1}dx + \frac{1}{3}\cdot\frac{1}{2}\int \frac{2x+2}{x^2-x+1}dx \quad = -\frac{1}{3}\int \frac{1}{x+1}dx + \frac{1}{6}\int \frac{2x-1+1+2}{x^2-x+1}dx$$

$$= -\frac{1}{3}\int \frac{1}{x+1}dx + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{6}\int \frac{3}{x^2-x+1}dx$$

$$= -\frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

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b. What is a linear differential equation? Solve: $(x^2+1)\frac{dy}{dx} + 2xy = 3x^2$. **[4]**

Linear Differential Equation

Solution

A differential equation of the form $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x or constants but not of y is called a linear differential equation.

Next Part:

Here, $(x^2+1)\frac{dy}{dx} + 2xy = 3x^2$

or, $\frac{dy}{dx} + \frac{2x}{x^2+1}\cdot y = \frac{3x^2}{x^2+1}$ ...(i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{2x}{x^2+1}$, $Q = \frac{3x^2}{x^2+1}$

$I.F = e^{\int Pdx} = e^{\int \frac{2x}{x^2+1}} = e^{\log(x^2+1)} = x^2+1$

Multiplying both sides of (i) by (x^2+1), we get

$(x^2+1)\frac{dy}{dx} + (x^2+1)\cdot\frac{2x}{x^2+1}\cdot y = (x^2+1)\cdot\frac{3x^2}{x^2+1}$

or, $d\{(x^2+1)\cdot y\} = 3x^2\,dx$

Integrating both sides, we get

$y(x^2+1) = \int 3x^2dx + C$

$y(x^2+1) = 3\cdot\frac{x^3}{3} + C$

$\therefore$ $y(x^2+1) = x^3 + C$

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7. a. State the first mean value theorem of differential calculus and interpret it geometrically. Using it to f(x) = sinx on [0, x], prove that sin x ≤ x for x ≥ 0. **[4]**

Solution

Statement of mean value theorem

If a function f(x) is

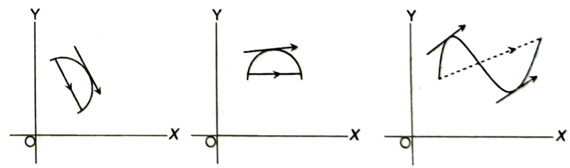
(a) continuous in the closed interval [a, b]

(b) differentiable in the open interval (a, b)

then there exists at least one c ∈ (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Second part

Geometrically, Lagrange's mean value theorem says that in a continuous curve, in which tangent can be drawn at every point, there is at least one point where the tangent is parallel to the secant joining the end points as shown in the figure.



Next Part

Let x ≥ 0. For all x ≥ 0, f(x) = sin x is defined in [0, x]. So, f(x) = sin x is continuous in [0, x]. Again, f'(x) = cos x which exists for all x ≥ 0. So, f(x) is differentiable in (0, x). Hence, all the conditions of mean value theorem are satisfied. There exists c ∈ (0, x) such that

$\frac{f(x)-f(0)}{x-0} = f'(c) = \cos c$

or, $\frac{\sin x - 0}{x-0} = \cos c$

or, $\frac{\sin x}{x} = \cos c \le 1$

or, $\frac{\sin x}{x} \le 1$

$\therefore$ $\sin x \le x$ for all $x \ge 0$.

b. An urn contains four white, eight black, six red and two green marbles. If three balls are drawn at random, find the probability of getting (i) all white marbles (ii) 2 red and 1 green marbles. [4]

Solution

Number of white marbles = 4
Number of black marbles = 8
Number of red marbles = 6
Number of green marbles = 2
Total number of marbles = 4 + 8 + 6 + 2 = 20
Total number of possible cases (n) = Number of selection of 3 marbles out of 20

$$= {}^{20}C_3 = \frac{20!}{17!\,3!} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140$$

(i) **All white marbles**
Total number of possible cases (m) = Number of selection of 3 white marbles out of 4 white marbles

$$= {}^4C_3 = \frac{4!}{1!\,3!} = 4$$

$$P(\text{all white marble}) = \frac{m}{n} = \frac{4}{1140} = \frac{1}{285}$$

(ii) **2 red and 1 green**
Total number of possible cases (m) = Number of selection of 2 red out of 6 red marbles and 1 green out of 2 green marbles $= {}^6C_2 \times {}^2C_1 = \frac{6!}{4!\,2!} \times \frac{2!}{1!\,1!} = 30$

$$\therefore\ P(\text{2 red and 1 green marbles}) = \frac{m}{n} = \frac{30}{1140} = \frac{1}{38}$$

8. a. **What is a conic section? Find the equation of the tangent to the parabola $y^2 = 8x$ which is parallel to the straight line $2x - 3y + 7 = 0$. Also find its point of contact.** [4]

Solution

Conic Section

The locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point to its distance from a fixed straight line is constant is called conic section.

Second Part

The given equation of parabola is

$y^2 = 8x$ ...(i)

Comparing (i) with $y^2 = 4ax$, we get

$4a = 8$

$a = 2$

The equation of tangent to the parabola (i) is

$$y = mx + \frac{a}{m}$$

or, $y = mx + \frac{2}{m}$ ...(ii)

If this tangent is parallel to $2x - 3y + 7 = 0$, their slopes must be equal.

$\therefore\ m = \frac{2}{3}$

Substituting the value of m in (ii)

$$y = \frac{2}{3}x + \frac{2}{2/3}$$

or, $y = \frac{2}{3}x + 3$

$\therefore\ 2x - 3y + 9 = 0$

Point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{2}{\left(\frac{2}{3}\right)^2}, \frac{2 \times 2}{\frac{2}{3}}\right) = \left(\frac{9}{2}, 6\right)$

b. **Define linearly independent vectors. Show that the following vectors are linearly dependent.** [4]

$2\vec{i} + \vec{j} - \vec{k},\ 3\vec{i} - 2\vec{j} + \vec{k},\ \vec{i} + 4\vec{j} - 3\vec{k}$

Solution

Linearly Independent Vectors

A set of vectors $\vec{a}, \vec{b}, \vec{c}, \ldots, \vec{v}$ is said to be linearly independent if in the relation

$$x\vec{a} + y\vec{b} + z\vec{c} + \ldots + t\vec{v} = 0$$

all the scalars x, y, z, … t are zero.

Next part:

Let $\vec{a} = 2\vec{i} + \vec{j} - \vec{k} = (2, 1, -1)$

$\vec{b} = 3\vec{i} - 2\vec{j} + \vec{k} = (3, -2, 1)$

$\vec{c} = \vec{i} + 4\vec{j} - 3\vec{k} = (1, 4, -3)$

Let x, y, z be the scalars such that

$x\vec{a} + y\vec{b} + z\vec{c} = 0$

or, $x(2, 1, -1) + y(3, -2, 1) + z(1, 4, -3) = 0$

or, $(2x + 3y + z, x - 2y + 4z, -x + y - 3z) = 0$

By equality of the vectors, we have

$2x + 3y + z = 0$

$x - 2y + 4z = 0$

$-x + y - 3z = 0$

$$\Rightarrow \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\Rightarrow AX = 0$

Where, $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ -1 & 1 & -3 \end{vmatrix} = 2\begin{vmatrix} -2 & 4 \\ 1 & -3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 \\ -1 & -3 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix}$$

$= 2(6 - 4) - 3(-3 + 4) + 1(1 - 2) = 4 - 3 - 1 = 0$

Thus, $X \neq 0$

i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq 0$

which shows that all the scalars are not zero. Hence, the vectors are linearly dependent.

OR

Prove that if θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} = ab\cos\theta$. [4]

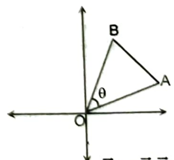
Solution

Let $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$ and $\angle AOB = \theta$. Then

$|\vec{OA}| = a,\ |\vec{OB}| = b.$

$|\vec{AB}| = |\vec{OB} - \vec{OA}|$

$= |\vec{b} - \vec{a}|$



Applying cosines law to ΔOAB,

$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta$

$|\vec{b} - \vec{a}|^2 = a^2 + b^2 - 2ab\cos\theta$

or, $(\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) = a^2 + b^2 - 2ab\cos\theta$ $\quad (\because |\vec{a}|^2 = \vec{a} \cdot \vec{a})$

or, $\vec{b} \cdot \vec{b} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{a} = a^2 + b^2 - 2ab\cos\theta$

or, $b^2 - 2\vec{a} \cdot \vec{b} + a^2 = a^2 + b^2 - 2ab\cos\theta$ $\quad (\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a})$

$\therefore\ \vec{a} \cdot \vec{b} = ab\cos\theta$

9. **For any positive integer n, prove that:**

$(a+x)^n = C(n,0)\,a^n + C(n,1)\,a^{n-1}x + C(n,2)\,a^{n-2}x^2 + \dots + C(n,n)\,x^n$.

Find the term containing x^2, if any, in the expansion of $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$. [6]

Solution

When n = 1, $(a+x)^1 = a + x = C(1,0)\,a^1 + C(1,1)\,x^1$

This shows that theorem holds for n = 1.

Assume that the theorem is true for n = k.

Then, $(a+x)^k = C(k,0)\,a^k + C(k,1)\,a^{k-1}x + \dots + C(k,k)\,x^k$ ...(i)

We wish to prove that the theorem is true for n = k + 1

Multiplying both sides of (i) by (a + x), we get

$(a+x)^{k+1} = (a+x)\{C(k,0)\,a^k + C(k,1)a^{k-1}x + \dots + C(k,k)\,x^k\}$

$= C(k,0)\,a^{k+1} + \{C(k,1) + C(k,0)\}\,a^kx + \{C(k,2) + C(k,1)\}\,a^{k-1}x^2 + \dots$

$+ \{C(k,k) + C(k,k-1)\}\,ax^k + C(k,k)\,x^{k+1}$

$= C(k+1,0)a^{k+1} + C(k+1,1)\,a^kx + C(k+1,2)\,a^{k-1}x^2 + \dots + C(k+1,k)\,ax^k + C(k+1,k+1)\,x^{k+1}$

Using $C(k,0) = 1 = C(k+1,0)$

$C(k,k) = 1 = C(k+1,k+1)$

& $C(k,r) + C(k,r-1) = C(k+1,r)$

which shows that theorem is true for n = k + 1 whenever it is true for n = k.

Hence, by principle of mathematical induction, the theorem is true for all $n \in N$.

Next Part

Let t_{r+1} be the general term in the expansion of $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$

Then, $t_{r+1} = C(6,r)\left(\frac{2x}{3}\right)^{6-r}\left(\frac{-3}{2x}\right)^r$

$= C(6,r)\left(\frac{2}{3}\right)^{6-r} \cdot x^{6-r} \cdot \left(\frac{-3}{2}\right)^r \cdot \frac{1}{x^r} = C(6,r)\left(\frac{2}{3}\right)^{6-r} \cdot \left(\frac{-3}{2}\right)^r \cdot x^{6-2r}$

For the coefficient of the term containing x^2, we have

$2 = 6 - 2r$

or, $2r = 4$

$\therefore$ $r = 2$

$\therefore$ $t_3 = 3^{rd}$ term contains x^2.

Coefficient of $x^2 = C(6,2)\left(\frac{2}{3}\right)^{6-2} \cdot \left(\frac{-3}{2}\right)^2 = \frac{6!}{4!\,2!}\left(\frac{2}{3}\right)^4 \cdot \left(\frac{-3}{2}\right)^2 = \frac{6\times5}{2\times1} \times \frac{16}{81} \times \frac{9}{4} = \frac{20}{3}$

10. **Find the direction cosines of two lines which are connected by the relations 2l + 2m − n = 0, mn + nl + lm = 0.** [6]

Solution

Given relations are:

$2l + 2m - n = 0$ ...(i)

$mn + nl + lm = 0$ ...(ii)

Eliminating n between (i) and (ii)

$m(2l + 2m) + (2l + 2m)l + lm = 0$

or, $2l^2 + 5lm + 2m^2 = 0$

or, $(l + 2m)(2l + m) = 0$

Either $l + 2m = 0$ ...(iii)

or $2l + m = 0$ ...(iv)

From (i) and (iii)

$2l + 2m - n = 0$

$l + 2m + 0\cdot n = 0$

By the rule of cross multiplication, we have

$\frac{l}{0+2} = \frac{m}{-1-0} = \frac{n}{4-2}$

or, $\frac{l}{2} = \frac{m}{-1} = \frac{n}{2} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{2^2+(-1)^2+2^2}} = \frac{1}{3}$

$\therefore$ $l = \frac{2}{3}$, $m = -\frac{1}{3}$, $n = \frac{2}{3}$ which gives the d.c's of the first line.

Again, from (i) and (iv), we have

$2l + 2m - n = 0$

$2l + m + 0\cdot n = 0$

By the rule of cross multiplication, we have

$\frac{l}{0+1} = \frac{m}{-2-0} = \frac{n}{2-4}$

or, $\frac{l}{1} = \frac{m}{-2} = \frac{n}{-2} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+(-2)^2+(-2^2)}} = \frac{1}{3}$

$\therefore$ $l = \frac{1}{3}$, $m = -\frac{2}{3}$, $n = -\frac{2}{3}$

which gives the d.c's of the second line.

OR

Prove that a plane through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

Also, find the angle between planes 2x − y + z = 6 and x + y + 2z = 3. [6]

Solution

First Part

The equation of plane through the point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ... (i)

Since (i) passes through the points (x_2, y_2, z_2) and (x_3, y_3, z_3), so

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$ ...(ii)

& $a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0$ ... (iii)

Eliminating a, b and c from (i), (ii) and (iii), we have

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$ is the required equation of plane.

Second Part:

Given equations of plane are:

$2x - y + z = 6$ and $x + y + 2z = 3$

$\therefore$ $a_1 = 2, b_1 = -1, c_1 = 1$ and $a_2 = 1, b_2 = 1, c_2 = 2$

Let θ be the angle between the planes. Then

$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{2\times1 + (-1)\times1 + 1\times2}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+1^2+2^2}} = \frac{3}{6} = \frac{1}{2} = \cos\frac{\pi}{3}$

$\therefore$ $\theta = \frac{\pi}{3}$

11. **Lives of two models of refrigerators turned in for new models in a recent survey are**

No. of years	No. of refrigerators	
	Model A	Model B
0 – 2	5	2
2 – 4	16	7
4 – 6	13	12
6 – 8	7	19
8 – 10	5	9
10 – 12	4	1

What is the average life of each model of these refrigerators? Which model has more uniformity? [6]

Solution

No. of year	Mid value (x)	Model A			Model B		
		f	fx	fx^2	f	fx	fx^2
0–2	1	5	5	5	2	2	2
2–4	3	16	48	144	7	21	63
4-6	5	13	65	325	12	60	300
6-8	7	7	49	343	19	133	931
8-10	9	5	45	405	9	81	729
10-12	11	4	44	484	1	11	121
		N = 50	Σfx =256	Σfx^2=1706	N =50	Σfx = 308	Σfx^2=2146

For Model A

$\bar{x}(A) = \frac{\Sigma fx}{N} = \frac{256}{50} = 5.12$

$\sigma(A) = \sqrt{\frac{\Sigma fx^2}{N} - (\bar{x})^2} = \sqrt{\frac{1706}{50} - (5.12)^2} = \sqrt{34.12 - 26.2144} = \sqrt{7.9056} = 2.81$

$C.V(A) = \frac{\sigma(A)}{\bar{x}(A)} \times 100 = \frac{2.81}{5.12} \times 100 = 54.88\%$

For Model B

$\bar{x}(B) = \frac{\Sigma fx}{N} = \frac{308}{50} = 6.16$

$\sigma(B) = \sqrt{\frac{\Sigma fx^2}{N} - (\bar{x})^2} = \sqrt{\frac{2146}{50} - (6.16)^2} = 2.23$

$C.V(B) = \frac{\sigma(B)}{\bar{x}(B)} \times 100 = \frac{2.23}{6.16} \times 100 = 36.20\%$

Average life of model A = 5.12 years, Average life of model B = 6.16 years

Since C.V(B) < C.V(A), model B has more uniformity.

Group B

12. a. **Three forces P, Q and R acting on a particle are in equilibrium, the angle between the P and Q is 60° and that between Q and R is 150°. Find the ratios of the forces.** [2]

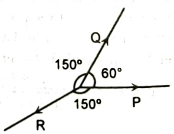
Solution

Given, angle between P and Q is 60° and angle between Q and R is 150°.

∴ Angle between R and P = 360° – 60° – 150° = 150°

Since P, Q and R are in equilibrium, by Lami's theorem, we have

$\frac{P}{\sin 150°} = \frac{Q}{\sin 150°} = \frac{R}{\sin 60°}$



or, $\frac{P}{1/2} = \frac{Q}{1/2} = \frac{R}{\sqrt{3}/2}$

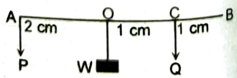
or, $\frac{P}{1} = \frac{Q}{1} = \frac{R}{\sqrt{3}}$

∴ $P : Q : R = 1 : 1 : \sqrt{3}$

b. **A uniform beam, 4m long, is supported in a horizontal position by two props which are 3m apart, so that the beam projects one meter beyond one of the props. Show that the force on one of the props is double of that on the other.** [2]

Solution

Let O be the centre of a uniform beam AB = 4 cm supported by two props at A and C. Then,



$OA = \frac{4}{2} = 2$ m, OC = OB – BC = 2 – 1 = 1 m

If W be the weight of the beam, P and Q be the forces on the props at A and C respectively.

Then,

$\frac{P}{CO} = \frac{Q}{AO} = \frac{W}{AC}$

or, P × AO = Q × CO

or, P × 2 = Q × 1

∴ Q = 2P

Therefore, the force on the prop at C is double of than that of the other.

c. **A pump having a power of 392 W pumps water at the rate of 100 litres per minute. Find the height to which the water is raised. (g = 9.8m/s², 1 litre of water = 1kg)** [2]

Solution

Mass (m) = 100

Time (t) = = 1 minute = 60 secs

Power (P) = 392 W

g = 9.8 m/s²

Height (h) = ?

We know that

$P = \frac{mgh}{t}$

or, $392 = \frac{100 \times 9.8 \times h}{60}$

or, $h = \frac{392 \times 60}{100 \times 9.8} = 24$ m

Required height (h) = 24m.

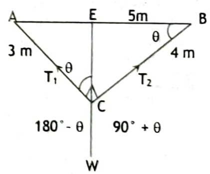
13. a. **A body of weight w is suspended by strings of length 3 m and 4 m attached to two points in the same horizontal line whose distance apart is 5m. Find the tensions along the strings.** [4]

Solution

Let CA and CB be the strings so that CA = 3m, CB = 4m and AB = 5m.

Since $3^2 + 4^2 = 5^2$, so ∠ACB = 90°.

Let the line of action of weight W be produced to meet AB at E. Let∠ACE = θ so that ∠CBE = θ. Let T_1 and T_2 be the tensions along the strings CA and CB respectively. Since the three forces T_1, T_2 and W acting at C are in equilibrium, so by Lami's theorem,



$\frac{T_1}{\sin BCD} = \frac{T_2}{\sin ACD} = \frac{W}{\sin ACB}$

or, $\frac{T_1}{\sin(90° + \theta)} = \frac{T_2}{\sin(180 - \theta)} = \frac{W}{\sin 90°}$

or, $\frac{T_1}{\cos\theta} = \frac{T_2}{\sin\theta} = W$

∴ $T_1 = W\cos\theta$ and $T_2 = W\sin\theta$

But $\cos\theta = \frac{BC}{BA} = \frac{4}{5}$ and $\sin\theta = \frac{AC}{AB} = \frac{3}{5}$

Hence, $T_1 = W \times \frac{4}{5} = \frac{4W}{5}$ N

$T_2 = W \times \frac{3}{5} = \frac{3W}{5}$ N

b. **A body of mass 49 kg is falling freely under gravity at the rate of 20 m/s. What is the uniform force that will stop it (i) in 2 sec (ii) in 50 cm? (g = 9.8 ms/²)** [4]

Solution

Here, m = 49 kg

u = 20 m/s

v = 0 m/s

i. t = 2 sec

Let a be the retardation. Then

$a = \frac{u - v}{t} = \frac{20 - 0}{2} = 10$

∴ Retardation (a) = 10 m/s²

Resistance force (F) = m(g + a) = 49(9.8 + 10) = 49 × 19.8 = 970.2 N = $\frac{970.2}{9.8}$ kg-wt = 99 kg-wt

ii. $s = 50$ cm $= 0.5$ m
We have,
$v^2 = u^2 - 2as$
or, $0^2 = 20^2 - 2 \times a \times 0.5$
or, $-400 = -a$
$\therefore$ $a = 400$

Therefore, the resistance force $F = m(a + g) = 49(400 + 9.8) = 20{,}080.2 \text{ N} = \dfrac{20{,}080.2}{9.8}$ kt wt
$= 2{,}049$ kg wt

OR

A bullet fired into a target losses half its velocity after penetrating 3cm. How much further will it penetrate? **[4]**

Solution

Let u be the velocity of the bullet just before the penetration and a be its retardation. Then,

$u = u$ cm/s, $v = \dfrac{u}{2}$ cm/s, $s = 3$ cm

We have,
$v^2 = u^2 - 2as$

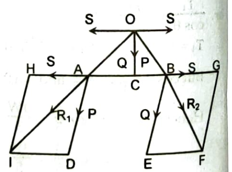
or, $\left(\dfrac{u}{2}\right)^2 = u^2 - 2 \cdot a \cdot 3$

$\therefore$ $u^2 = 8a$ ...(i)

Let $(3 + x)$ cms be the thickness penetrated just before coming to rest. Then,
$0^2 = u^2 - 2a(3 + x)$
or, $u^2 = 2a(3 + x)$
or, $8a = 2a(x + 3)$ [using (i)]
$\therefore$ $x = 1$
$\therefore$ Required further penetration = 1 cm.

14. The resultant of two like parallel forces P and Q acting on rigid body is a force of magnitude P + Q in the same direction as P and Q are. If A and B are any points on the lines of action of P and Q respectively, prove that the resultant divides line segment AB internally in the inverse ratio of the forces. **[6]**

Solution



Let P and Q be two like parallel forces acting respectively at the points A and B of a rigid body represented by the lines AD and BE. Join AB. Let us introduce two equal and opposite forces each equal to S along BA and AB respectively. These two forces being equal and opposite have no effect upon the system. Let these forces be represented by AH and BG. Complete the parallelogram ADIH and BEFG. Also let the diagonals IA and FB be produced to meet at O. Through O, draw OC parallel to the given forces P and Q meeting AB at point C.

Now, the resultant of the forces P and Q is equivalent to the resultant of R_1 and R_2. But the point of application of R_1 may be transferred from A to O and it can be resolved into two components S and P which are parallel to their original directions. Similarly, R_2 may be taken to act at O and resolved into two forces Q and S.

The two equal and opposite forces each equal to S acting at the point O balance each other and we have finally left with the force P and Q acting along OC whose resultant is P + Q.

Now, the triangles OAC and AID are similar

$\dfrac{OC}{AC} = \dfrac{AD}{DI}$

or, $\dfrac{OC}{AC} = \dfrac{P}{S}$

$\therefore$ $S \times OC = P \times AC$ ...(i)

Similarly, the triangles OBC and BEF are similar,

$\dfrac{OC}{CB} = \dfrac{BE}{EF}$

or, $\dfrac{OC}{CB} = \dfrac{Q}{S}$

$\therefore$ $S \times OC = Q \times CB$ ...(ii)

From (i) and (ii), we have
$P \times AC = Q \times CB$

or, $\dfrac{AC}{CB} = \dfrac{Q}{P}$

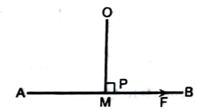
i.e. C divides AB internally in the inverse ratio of the forces.

OR

Define the moment of a force. Forces 1, 2, 4, 5 kg-wts. act along the sides of a square taken in order. Prove that their resultant is parallel to a diagonal and find where it cuts the side along which the first force acts. **[6]**

Solution

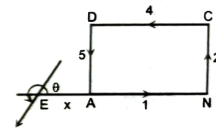
Moment of a force



The moment of a force about a given point is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force.

Thus if F be a force and p be the perpendicular distance of the point O from AB then the moment of F about O is $F \times OM = F \times p$

Second Part



Let the forces 1, 2, 4, 5 kg wts act along the sides AB, BC, CD and DA respectively of a square ABCD.

Since 1 and 4 kg wts are horizontal parallel forces, so
$X = 1 - 4 = -3$ kg wts
Again, since 2 and 5 kg wts are vertical parallel forces, so
$Y = 2 - 5 = -3$ kg wts
Let R be the resultant. Then
$R = \sqrt{X^2 + Y^2} = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$ kg wt
If R makes an angle of θ with AB, then

$\tan\theta = \dfrac{Y}{X} = \dfrac{-3}{-3} = 1 = \tan 225°$ $(\because X < 0, Y < 0)$

$\therefore$ $\theta = 225°$

Hence, the resultant is parallel to the diagonal CA.

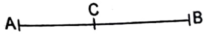
Suppose that the resultant cuts BA produced at E and AE = x. Then the moment of the resultant about E is zero. Hence
$2(EA + AB) + 4 \times AD - 5 \times EA = 0$
or, $2(x + a) + 4a - 5x = 0$, where a is the length of the square ABCD.
or, $2x + 2a + 4a - 5x = 0$
or, $-5x = -8a$
$\therefore$ $x = \dfrac{8}{5}a$

15. A man travels from A to B in 45 minutes. At C, somewhere between A and B, it attains its maximum velocity of 45 m per hr. If he travels with uniform acceleration from A to C and uniform retardation from C to B, find the distance between A and B, it being supposed that the man starts from rest at A and comes to rest at B. **[6]**

Solution

Let a, s and t denote the acceleration, distance and time taken from A to C respectively. Also, let f, x and T be the retardation, the distance and time taken from C to B respectively,
Then, $t + T = 45$ min $= 45 \times 60 = 2700$s ...(i)

From A to C



$u = 0, v = 45 \text{ m/hr} = \frac{45}{60 \times 60} \text{m/s} = \frac{1}{80} \text{m/s}$

We have,

$v = u + at$

or, $\frac{1}{80} = 0 + at$

$\therefore$ $at = \frac{1}{80}$

Again, we have

$s = ut + \frac{1}{2}at^2$

or, $s = 0 + \frac{1}{2} \times \frac{1}{80} \times t$ $\left[\because at = \frac{1}{80}\right]$

$\therefore$ $s = \frac{1}{160}t$ ...(ii)

From B to C

$u = \frac{1}{80}\text{m/s}, v = 0 \text{ m/s}$

We have,

$v = u - fT$

or, $0 = \frac{1}{80} - fT$

$\therefore$ $fT = \frac{1}{80}$

Again, we have

$x = uT - \frac{1}{2}fT^2$

or, $x = \frac{1}{80}T - \frac{1}{2} \times \frac{1}{80} \times T$

or, $x = \frac{1}{160}T$ ...(iii)

From (ii) and (iii)

$s + x = \frac{1}{160}t + \frac{1}{160}T$

$= \frac{1}{160}(t + T)$

$= \frac{1}{160} \times 2700$ [using (i)]

$= \frac{135}{8} = 16\frac{7}{8}$

Therefore, the total distance from A to B is $16\frac{7}{8}$ m.

Group C

16. a. Determine graphically the solution set of the following system of inequalities:
$x + y \geq 2, 3x + 2y \leq 4, x \geq 0, y \geq 0$ **[2]**

Solution

The corresponding equations of boundary lines are:

$2x + y = 2$ ...(i)

$3x + 2y = 4$ ...(ii)

$x = 0$ ...(iii)

$y = 0$ ...(iv)

From (i) $2x + y = 2$

x	0	1
y	2	0

Taking testing point (0, 0) in $x + 2y \geq 2$, we get

$0 + 2 \times 0 \geq 2$ (false)

From (ii) $3x + 2y = 4$

x	0	2
y	2	-1

Taking testing point (0, 0) in $3x + 2y \leq 4$, we get

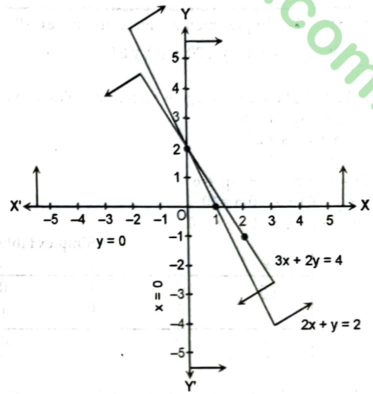
$3 \times 0 + 2 \times 0 \leq 4$ (true)

From (iii) $x = 0$ which is y-axis

From (iv) $y = 0$ which is x-axis

$x \geq 0$ represents the closed right half plane

and $y \geq 0$ represents the closed upper half plane.



The shaded portion in the figure represents the required solution.

b. Write a short note on accuracy of a numerical method. **[2]**

Solution

Everywhere we have to do measurements. Sometimes it becomes very hard or impossible to get the exact value. In later case, we find some kind of result, that may not be exact but as close as possible to the exact value. Such a result is called approximation.

Such type of approximation can be done by using numerical method. The absolute error due to approximation is given by |exact value – approximate value|. The error may arise due to truncation or rounding off the numbers.

Every method of numerical computation produces results with some errors. The results we obtain must have the degree of accuracy as required. In order to minimize the error, we should apply the suitable method according to the problem.

c. Apply the Simpons's rule to approximate the value of $\int_1^4 e^x \ln x \, dx$ with n = 3. **[2]**

Solution

Here, $f(x) = e^x \ln x, n = 3$

$a = 1, n = 4$

We have, $h = \frac{b - a}{n} = \frac{4 - 1}{3} = 1$

The four points to be considered are $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4$. The values of the function at these points are as follows

x	1	2	3	4
$y = f(x) = e^x \ln x$	0	5.1217	22.0662	75.6891

Applying Simpson's $\frac{3}{8}$ rule, we have

$$\int_1^3 e^x \ln x\, dx \approx \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3] = \frac{3\times 1}{8}[0 + 3(5.1217 + 22.0662) + 75.6891] = 58.9698$$

17. a. sing the simplex method, maximize $p = 6x - 9y$ subject to $2x - 3y \le 6,\ x + y \le 20,\ x \ge 0,\ y \ge 0.$ [4]

Solution

Let r and s be the non-negative slack variables. Then given LPP can be written as:

$2x - 3y + r = 6$
$x + y + s = 20$
$p = 6x - 9y$

$\Rightarrow$ $2x - 3y + r + 0\cdot s + 0\cdot p = 6$
$x + y + 0\cdot r + s + 0\cdot p = 20$
$-6x + 9y + 0\cdot r + 0\cdot s + p = 0$

Simplex tableau

x	y	r	s	p	R.H.S
[2]	-3	1	0	0	6
1	1	0	1	0	20
-6	9	0	0	1	0
↑					

Here, −6 is the most negative entry in the last row. So, first column is the pivot column. Since $\frac{6}{2} = 3$, $\frac{20}{1} = 20$ and $3 < 20$, so 2 is the pivot element.

Applying $R_1 \to \frac{1}{2} R_1$

x	y	r	s	p	R.H.S
1	-3/2	1/2	0	0	3
1	1	0	1	0	20
-6	9	0	0	1	0

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 6R_1$

x	y	r	s	p	R.H.S
1	-3/2	1/2	0	0	3
0	5/2	-1/2	1	0	17
0	0	3	0	1	18

In the last row, there is no negative numbers, so this table gives the optimal solution. Hence, maximum value of p = 18 at (3, 0).

b. Use Bisection method to find solution accurate to within 10^{-2} for $^3 - 7x^2 + 14x - 6 = 0$ on the interval [1, 3.2] [4]

Solution

Let $f(x) = x^3 - 7x^2 + 14x - 6$
$f(1) = 1^3 - 7 \times 1^2 + 14 \times 1 - 6 = 2$
$f(3.2) = (3.2)^3 - 7(3.2)^2 + 14 \times 3.2 - 6 = -0.112$

Since f(1) and f(3.2) have opposite signs, so there is a root between 1 and 3.2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	3.2	2.1	2	-0.112	1.791
2.1	3.2	2.65	1.791	-0.112	0.5521
2.65	3.2	2.925	0.5521	-0.112	0.0858
2.925	3.2	3.0625	0.0858	-0.112	-0.0544
2.925	3.0625	2.9937	0.0858	-0.0544	0.0063

Here, $|f(m)| = |0.0063| = 0.0063 < 10^{-2}$
So, the required solution is 2.9937

OR

Write three methods for measuring error. Approximate $\sqrt{11}$ by Newton-Raphson's method with accuracy 0.00001. [4]

Solution

Let us select an error tolerance $\varepsilon > 0$, we generate $x_1, x_2, x_3, \ldots$, until one of the following 3 conditions is met

i. $|x_n - x_{n-1}| < \varepsilon$

ii. $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon$

iii. $|f(x_n)| < \varepsilon$

Next part

Let $x = \sqrt{11}$
Then, $x^2 = 11$
or, $x^2 - 11 = 0$
$\therefore$ $a = 11$

By Newton's-Raphson method, we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$$\therefore\ x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) = \frac{1}{2}\left(3 + \frac{11}{3}\right) = 3.333$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = \frac{1}{2}\left(3.333 + \frac{11}{3.333}\right) = 3.31667$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{a}{x_2}\right) = \frac{1}{2}\left(3.31667 + \frac{11}{3.1667}\right) = 3.31662$$

$$x_4 = \frac{1}{2}\left(x_3 + \frac{a}{x_3}\right) = \frac{1}{2}\left(3.31662 + \frac{11}{3.1662}\right) = 3.31662$$

$$\text{Error} = \left|\frac{x_4 - x_3}{x_4}\right| < 0.00001$$

So, 3.31662 is approximate value of $\sqrt{11}$.

18. Find the approximate solution of the following system of equation by matrix inversion method: $x - y + z = -2,\ x + y - 2z = -9,\ x + 2y + z = 9.$ [6]

Solution

Writing the given equations in matrix form

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -9 \\ 9 \end{bmatrix}$$

which is in the form of AX=B

$\Rightarrow$ $X = A^{-1}B$

Now, we agument A with identity matrix I

$$[A|I] = \left[\begin{array}{ccc:ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & -1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -3 & 5 & 1 & -2 & 0 \\ 0 & 1 & 3 & 0 & -1 & 1 \end{array}\right] \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -5/3 & -1/3 & 2/3 & 0 \\ 0 & 1 & 3 & 0 & -1 & 1 \end{array}\right] R_2 \to \left(-\frac{1}{3}\right) R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & -5/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 14/3 & 1/3 & -5/3 & 1 \end{array}\right] \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim\begin{bmatrix} 1 & 0 & -1/3 & : & 1/3 & 1/3 & 0 \\ 0 & 1 & -5/3 & : & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & : & 1/14 & -5/14 & 3/14 \end{bmatrix} R_1 \to \frac{3}{14}R_3$$

$$\sim\begin{bmatrix} 1 & 0 & 0 & : & 15/42 & 9/42 & 1/14 \\ 0 & 1 & 0 & : & -9/42 & 3/42 & 5/14 \\ 0 & 0 & 1 & : & 1/14 & -5/14 & 3/14 \end{bmatrix} \begin{matrix} R_1 \to R_1 + \frac{1}{3}R_3 \\ R_2 \to R_2 + \frac{5}{3}R_3 \end{matrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 15/42 & 9/42 & 1/14 \\ -9/42 & 3/42 & 5/14 \\ 1/14 & -5/14 & 3/14 \end{bmatrix}$$

From (i)

$$X = A^{-1}B = \begin{bmatrix} 15/42 & 9/42 & 1/14 \\ -9/42 & 3/42 & 5/14 \\ 1/14 & -5/14 & 3/14 \end{bmatrix}\begin{bmatrix} -2 \\ -9 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -30/42 - 81/42 + 9/14 \\ 18/42 - 27/42 + 45/14 \\ -2/14 + 45/14 + 27/14 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$$

$\therefore$ x = –2, y = 3 and z = 5

19. Derive the trapezoidal rule. The capacity of a battery is a measure of $\int i\,dt$, where i is the current. Estimate, using the Trapezium rule, the capacity of a battery whose current was measured over an eight hour period with the results shown below.

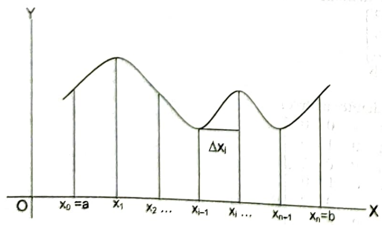
Time/hours	0	1	2	3	4	5	6	7	8
Current/Amps	25.2	29.0	31.8	36.5	33.7	31.2	29.6	27.3	28.6

Solution

Statement: If a function f is continuous on the closed interval [a, b],

then $\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$ where [a, b] has been partitioned into n equal subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, each of length $h = \frac{b-a}{n}$.

Proof: Let f be defined on [a, b]. Consider the area bounded by the graph of f, the ordinates x = a, x = b and the x-axis. This area is the geometrical representation of $\int_a^b f(x)\,dx$.



Let us divide [a, b] into n sub-intervals $[x_0, x_1], [x_1, x_2], \ldots [x_{i-1}, x_i] \ldots [x_{n-1}, x_n]$; each of length $h = \frac{b-a}{n}$. The ordinates corresponding to the points of subdivisions $x_0, x_1, x_2, \ldots x_{i-1}, x_i, \ldots, x_{n-1}, x_n$ are $f(x_0) = a$, $f(x_1), f(x_2) \ldots f(x_{i-1}), f(x_i), \ldots, f(x_{n-1}), f(x_n) = b$. If we join the consecutive points on the graph by the line segments, we get n trapezoids. The sum of areas of these n trapezoids is an approximation to the area under the curve. We know that,

the area of a trapezium = $\frac{1}{2}$ (sum of parallel sides) (distance between them).

So, A $= \frac{1}{2}[f(x_0) + f(x_1)]\,\Delta x + \frac{1}{2}[f(x_1) + f(x_2)]\,\Delta x + \ldots + \frac{1}{2}[f(x_{n-1}) + f(x_n)]\Delta x$

$$= \frac{1}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)]\,\frac{b-a}{n}$$

Hence, $\int_a^b f(x)\,dx = \frac{b-a}{2n}[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)]$

Next part:

a = 0, b = 8, n = 8

$h = \frac{b-a}{n} = \frac{8-0}{8} = 1$

Using Tapezoidal rule, we have

$$\int_0^8 i\,dt \approx \frac{h}{2}[i_0 + 2(i_1 + i_2 + i_3 + i_4 + i_5 + i_6 + i_7) + i_8]$$

$$= \frac{1}{2}[25.2 + 2(29 + 31.8 + 36.5 + 33.7 + 31.2 + 29.6 + 27.3) + 28.6] = 246$$

OR

Compute an approximate value of $\int_0^1 (1 + x^2)^{-1}\,dx$ by using the composite trapezoid rule with three points. Then compare with the actual value of the integral. Next, determine the error formula and numerically verify an upper bound on it. **[6]**

Solution

$\int_0^1 (1 + x^2)^{-1}\,dx$

Here, a = 0, b = 1

For three points, n = 2

Then, $h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

Now, 3 points to be considered are $x_0 = 0_1, x_1 = 0.5, x_2 = 1$. The value of the function at these points are

x	0	0.5	1
$y = f(x) = \frac{1}{1+x^2}$	1	0.8	0.5

Using Trapezoidal rule, we have

$$\int_0^1 (1 + x^2)^{-1}\,dx \approx \frac{h}{2}(y_0 + 2y_1 + y_2) = \frac{0.5}{2}[1 + 2 \times 0.8 + 0.5] = 0.25 \times 3.1 = 0.775$$

$$\text{Actual value} = \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1}x]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} - 0 = 0.785$$

Error in Trapezoidal rule = 0.785 – 0.775 = 0.01

Here, $f(x) = \frac{1}{1+x^2}$

$f'(x) = \frac{-2x}{(1+x^2)^2}$

$f''(x) = \frac{2(3x^2 - 1)}{(1+x^2)^3}$

The maximum value of $|f''(x)|$ occurs at x = 1. It is 2.

$\therefore$ M = 2

Error bounds $= \frac{(b-a)^3 \cdot M}{12 n^2} = \frac{(1-0)^3 \cdot 2}{12 \times 2^2} = \frac{1}{24} = 0.0417$ (upper bound)

SOLUTIONS TO MODEL QUESTIONS 2068 (SET II)

Full Marks: 100
Pass Marks: 35
Time: 3 hrs

Candidates are required to give their answer in their own words as far as practicable. The figures in the margin indicate full marks.

Attempt ALL questions of group A and group B or C.

Group A

1. a. In an examination paper containing 10 questions, a candidate has to answer 7 questions. If two questions are made compulsory, in how many ways can he choose 7 questions in all? [2]

Solution

If 2 questions are compulsory then the candidate has to select 7 – 2 = 5 questions from 10 – 2 = 8 questions.

∴ $n = 8, r = 5$

Total number of selections = $C(8, 5) = \frac{8!}{(8-5)!\,5!} = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2 \times 1} = 56$

b. Find the middle term in the expansion of $\left(2x + \frac{1}{3x^2}\right)^9$ [2]

Solution

The number of terms in the expansion of $\left(2x + \frac{1}{3x^2}\right)^9$ is 9 + 1 = 10, which is even. So, there are two middle terms. They are $t_{\frac{9+1}{2}}$ and $t_{\frac{9+1}{2}+1}$ i.e. t_5 and t_6.

We have, $t_{r+1} = C(n,r)\, a^{n-r} x^r$

∴ $t_5 = t_{4+1} = C(9,4)\,(2x)^{9-4} \cdot \left(\frac{1}{3x^2}\right)^4 = \frac{9!}{5!\,4!} \cdot 2^5 \cdot x^5 \cdot \frac{1}{3^4} \cdot \frac{1}{x^8} = \frac{448}{9x^8}$

Again,

$t_6 = t_{5+1} = C(9,5)\,(2x)^{9-5} \cdot \left(\frac{1}{3x^2}\right)^5 = \frac{9!}{4!\,5!} \cdot 2^4 \cdot x^4 \cdot \frac{1}{3^5} \cdot \frac{1}{x^{10}} = \frac{224}{27x^6}$

c. Let S = {– 1, 1} and * denote the usual operation of multiplication. Represent it by Cayley's table. Show that * is a binary operation on S. [2]

Solution

Cayley's table

*	–1	1
–1	1	–1
1	–1	1

From above Cayley's table we see that product of any two elements of S is unique and belongs to S. So, * is a binary operation on S.

2. a. Find the eccentricity and the foci of the ellipse:

$x^2 + 4y^2 - 4x + 24y + 24 = 0$. [2]

Solution

Given equation is

$x^2 + 4y^2 - 4x + 24y + 24 = 0$

or, $x^2 - 4x + 4y^2 + 24y + 24 = 0$

or, $x^2 - 4x + 4 - 4 + 4(y^2 + 6y + 9 - 9) + 24 = 0$

or, $(x-2)^2 + 4(y+3)^2 = 16$

or, $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$ ...(i)

Comparing equation (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

We get

$h = 2$, $k = -3$

$a^2 = 16$, $b^2 = 4$

⇒ $a = 4$, $b = 2$

Since a > b, so major axis is along x-axis.

Centre = (h, k) = (2, –3)

We have,

Eccentricity (e) = $\sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{16}} = \sqrt{\frac{12}{16}} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

Foci = $(h \pm ae, k) = \left(2 \pm 4 \cdot \frac{\sqrt{3}}{2}, -3\right) = (2 \pm 2\sqrt{3}, -3)$

b. Find the point where the line through the points (1, 2, 3) and (4, – 4, 9) meets the zx-plane. [2]

Solution

Let zx-plane divide the line joining the points (1, 2, 3) and (4, –4, 9) in the ratio $m_1 : m_2$. At zx-plane, we have y = 0.

Now, by section formula, we have

$y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2}$

or, $0 = \frac{m_1 \cdot (-4) + m_2 \cdot 2}{m_1 + m_2}$

or, $-4m_1 + 2m_2 = 0$

or, $4m_1 = 2m_2$

or, $\frac{m_1}{m_2} = \frac{1}{2}$

Again, by section formula, we have

$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$, $z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$

or, $x = \frac{1.4 + 2.1}{1 + 2}$ $z = \frac{1.9 + 2.3}{1 + 2}$

or, x = 2, z = 5

∴ Required point is (2, 0, 5)

c. Are the three points with position vectors $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$ collinear? Justify your answer. [2]

Solution

Let O be the origin. Let A, B and C be the three points with position vectors $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$. Then

$\vec{OA} = \vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{OB} = 2\vec{i} + 5\vec{j} - \vec{k}$

$\vec{OC} = 3\vec{i} + 8\vec{j} - 6\vec{k}$

Then, $\vec{AB} = \vec{OB} - \vec{OA} = 2\vec{i} + 5\vec{j} - \vec{k} - (\vec{i} + 2\vec{j} + 4\vec{k}) = \vec{i} + 3\vec{j} - 5\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = 3\vec{i} + 8\vec{j} - 6\vec{k} - (\vec{i} + 2\vec{j} + 4\vec{k}) = 2\vec{i} + 6\vec{j} - 10\vec{k}$

$= 2(\vec{i} + 3\vec{j} - 5\vec{k}) = 2\vec{AB}$

This shows that AB and AC are parallel. But they start from the same point A. So, A, B and C are collinear.

3. a. Using L'Hospital's rule, evaluate $\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$. [2]

Solution

$$\lim_{x\to 0} \frac{e^x + e^{-x} - 2\cos x}{\sin^2 x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{2\sin x \cos x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x - e^{-x} + 2\sin x}{2\sin 2x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^x + e^{-x} + 2\cos x}{2\cos 2x} = \frac{1+1+2}{2\times 1} = 2$$

b. Evaluate: $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $(\beta > \alpha)$. [2]

Solution

Let $I = \int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}, (\beta > \alpha)$

Put $x - \alpha = y^2$

$dx = 2y\,dy$

$$\therefore I = \int \frac{2y\,dy}{\sqrt{y^2(y^2+\alpha-\beta)}} = 2\int \frac{dy}{\sqrt{y^2+\alpha-\beta}} = 2\int \frac{dy}{\sqrt{y^2-(\sqrt{\beta-\alpha})^2}} = 2\int \frac{dy}{\sqrt{y^2-(\beta-\alpha)}}$$

$$= 2\log(y + \sqrt{y^2-(\beta-\alpha)^2}) + C = 2\log(\sqrt{x-\alpha} + \sqrt{x-\alpha-\beta+\alpha}) + C$$

$$= 2\log(\sqrt{x-\alpha} + \sqrt{x-\beta}) + C.$$

c. If $\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$ and $\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$ show that $\vec{a}\times\vec{b}$ is perpendicular to $\vec{a}$. [2]

Solution

Here, $\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$

$\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -5 \\ 1 & -4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -5 \\ -4 & 2 \end{vmatrix}\vec{i} - \begin{vmatrix} 6 & -5 \\ 1 & 2 \end{vmatrix}\vec{j} + \begin{vmatrix} 6 & 3 \\ 1 & -4 \end{vmatrix}\vec{k}$$

$$= (6-20)\vec{i} - (12+5)\vec{j} + (-24-3)\vec{k} = -14\vec{i} - 17\vec{j} - 27\vec{k}$$

Now,

$(\vec{a}\times\vec{b})\cdot\vec{a} = (-14\vec{i} - 17\vec{j} - 27\vec{k})\cdot(6\vec{i} + 3\vec{j} - 5\vec{k}) = -84 - 51 + 135 = 0$

This shows that $(\vec{a}\times\vec{b})$ is perpendicular to $\vec{a}$.

4. a. Solve: $x\frac{dy}{dx} + y - 1 = 0$. [2]

Solution

Here, $x\frac{dy}{dx} + y - 1 = 0$

or, $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}$ ...(i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{1}{x}, Q = \frac{1}{x}$

$I.F = e^{\int P dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Multiplying both sides of (i) by x, we get

$x\frac{dy}{dx} + y = 1$

or, $d(y\cdot x) = dx$

Integrating, $xy = x + c$

b. If $n = 10$, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$, find the correlation coefficient between the two variables. [2]

Solution

Given, $n = 10$ $\quad \Sigma X = 60$

$\Sigma Y = 60$ $\quad \Sigma X^2 = 400$

$\Sigma Y^2 = 580$ $\quad \Sigma XY = 415$

Correlation coefficient (r) = ?

We have, correlation coefficient (r) $= \frac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} = \frac{10\times 415 - 60\times 60}{\sqrt{10\times 400 - 60^2}\sqrt{10\times 580 - 60^2}}$

$$= \frac{4150 - 3600}{\sqrt{400}\sqrt{2200}} = \frac{500}{20\sqrt{2200}} = 0.59$$

c. Two dice are rolled once. What is the probability of getting a total of 9 or 6? [2]

Solutions

If two dice are rolled once, then the possible cases of turning up are

{(1, 1), (1, 2), ..., (6, 6)}

There are 36 possible cases.

There are 4 cases having a total of 9, whose set is

{(3, 6), (4, 5), (5, 4), (6, 3)}

And, there are 5 cases having a total of 6. They are

{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

$\therefore$ P(a total of 9 or 6) = P(a total of 9) + P(a total of 6) $= \frac{4}{36} + \frac{5}{36} = \frac{9}{36} = \frac{1}{4}$

5. a. In how ways can the letters of the word "COMPUTER" be arranged so that

i. all the vowels are always together?

ii. the vowels may occupy only odd positions? [4]

Solution

There are 8 letters in the word 'COMPUTER'

(i) There are 3 vowels in the word 'COMPUTER'. Consider 3 vowels as one letter. Then 6 letters namely C, M, P, T, R, (O, U, E) can be arranged in P(6, 6) ways

= 6! ways

Also, 3 vowels among themselves can be arranged in P(3, 3) ways

= 3! ways

$\therefore$ Total number of arrangements = 6! × 3!

= 6 × 5 × 4 × 3 × 2 × 1 × 3 × 2 × 1 = 4320 ways

(ii) If the three vowels O, U, E occupy only in odd positions namely 1st, 3rd, 5th and 7th i.e. 4 positions, then the total number of arrangements of 3 vowels in 4 places = P(4, 3)

Again, the remaining 5 letters in remaining 5 places can be arranged in P(5,5) ways = 5! ways

$\therefore$ Total number of arrangements = P(4, 3) × 5! $= \frac{4!}{1!} \times 5! = 4\times 3\times 2\times 1\times 5\times 4\times 3\times 2\times 1 = 2880$ ways

b. Given the algebraic structure (G, *) with G = {1, ω, ω²} where ω represents an imaginary cube root of unity and * stands for the binary operation of multiplication, show that (G, *) is a group. [4]

Solution

Here, $G = \{1, \omega, \omega^2\}$

i) Since the product of any two elements of G is also an element of G. So G is closed under *.

ii) $1 * (\omega * \omega^2) = 1 * \omega^3 = 1 * 1 = 1$

and $(1 * \omega) * \omega^2 = \omega * \omega^2 = \omega^3 = 1$

$\therefore 1 * (\omega * \omega^2) = (1 * \omega) * \omega^2$

This result is true for all elements of G.

$\therefore$ Associativity is satisfied.

iii) Here, $1 * 1 = 1$

$1 * \omega = \omega * 1 = \omega$

$1 * \omega^2 = \omega^2 * 1 = \omega^2$

So, 1 is the multiplicative identity.

iv) Here, $1 * 1 = 1$

$1 * \omega^2 = \omega^3 = 1$

$\omega^2 * \omega = \omega^3 = 1$

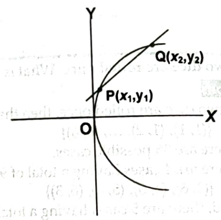
So, 1, ω and ω^2 are multiplicative inverse of 1, ω^2 and ω respectively.

Hence, from (i) - (iv), (G, *) is a group.

6. a. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1). Express it in the slope form. [4]

Solution

Let $P(x_1, y_1)$ be the point on the parabola $y^2 = 4ax$. Let us take another point $Q(x_2, y_2)$ on the parabola which is very close to P. Then,



$y_1^2 = 4ax_1$ ...(i)

& $y_2^2 = 4ax_2$ ...(ii)

Subtracting, we have

$y_2^2 - y_1^2 = 4ax_2 - 4ax_2$

or, $(y_2 - y_1)(y_2 + y_1) = 4a(x_2 - x_1)$

$\therefore \frac{y_2 - y_1}{x_2 - x_1} = \frac{4a}{y_2 + y_1}$ ...(iii)

The equation of PQ is

$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

or, $y - y_1 = \frac{4a}{y_2 + y_1}(x - x_1)$ [using (iii)]

By definition, PQ becomes the tangent at P when $Q \to P$ i.e. when $x_2 \to x_1$ and $y_2 \to y_1$.

Hence, the equation of tangent at P is

$y - y_1 = \frac{4a}{2y_1}(x - x_1)$

or, $yy_1 - y_1^2 = 2ax - 2ax_1$

or, $yy_1 - 4ax_1 = 2ax - 2ax_1$ [using (i)]

or, $yy_1 = 2a(x + x_1)$ which is the equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1).

The above equation can be written as

$y = \frac{2a}{y_1}x + \frac{2ax_1}{y_1}$ ...(i)

Writing $\frac{2a}{y_1} = m$, then $y_1 = \frac{2a}{m}$

Since $y_1^2 = 4ax_1$, so $\left(\frac{2a}{m}\right)^2 = 4ax_1$.

$\Rightarrow x_1 = \frac{a}{m^2}$

Then from (i)

$y = mx + 2a\frac{a/m^2}{2a/m}$

$y = mx + \frac{a}{m}$ which is the equation of tangent in slope form.

OR

What is a conic section? Find the equation of the parabola in the standard form. [4]

Solution

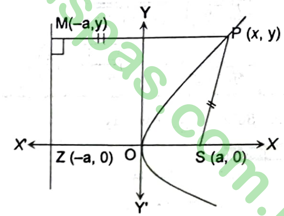
Conic Section

Please see Model Set I, Q.No. 8a

Second Part

Let S be the focus and ZM be the directrix of the parabola. Let SZ be drawn perpendicular to ZM. Let O be the middle point of SZ i.e. SO = OZ. Then O is the vertex of the parabola and SZ is the axis of the parabola. Let OS = a. Then the coordinates of S, O and Z be (a, 0), (0, 0) and (–a, 0) respectively.

Let P(x, y) be any point on the parabola. Join PS and draw PM perpendicular to ZM. Then by definition of parabola,



PS = PM

or, $PS^2 = PM^2$

or, $(x - a)^2 + (y - 0)^2 = (x + a)^2 + (y - y)^2$

or, $x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$\therefore y^2 = 4ax$ which is the required equation of parabola in standard form.

b. Find the equation of plane through the point (2, 1, 4) and perpendicular to each of the planes. $9x - 7y + 6z + 48 = 0$ and $x + y + z = 0$. [4]

Solution

Equation of any plane through the point (2, 1, 4) is

$a(x - 2) + b(y - 1) + c(z - 4) = 0$ ...(i)

Since (i) is perpendicular to the planes $9x - 7y + 6z + 48 = 0$ and $x + y + z = 0$, we have

$9a - 7b + 6c = 0$ ...(ii)

$a + b + c = 0$ ...(iii)

Solving by the method of cross multiplication, we have

$\frac{a}{-7-6} = \frac{b}{6-9} = \frac{c}{9+7}$

$\Rightarrow \frac{a}{-13} = \frac{b}{-3} = \frac{c}{16} = k$ (say)

$\therefore a = -13k, b = -3k, c = 16k$

Substituting these values in (i), we get

$-13(x - 2) - 3(y - 1) + 16(z - 4) = 0$

or, $13x - 26 + 3y - 3 - 16z + 64 = 0$

$\therefore 13x + 3y - 16z + 35 = 0$

7. a. Evaluate: $\int \frac{dx}{a + b\cos x}$ $(a > b > 0)$. [4]

Solution

$$\int \frac{dx}{a + b\cos x} = \int \frac{dx}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + b\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} = \int \frac{1}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{(a+b) + (a-b)\tan^2\frac{x}{2}}dx \quad ...(i)$$

Put $\sqrt{a-b}\tan\frac{x}{2} = y$

$\frac{1}{2}\sqrt{a-b}\sec^2\frac{x}{2}dx = dy$

Then the integral (i) becomes

$\frac{2}{\sqrt{a-b}}\int \frac{dy}{(a+b) + y^2} = \tan^{-1}\left(\frac{y}{\sqrt{a+b}}\right) + C$

$$=\frac{2}{\sqrt{a-b}}\int\frac{dy}{\sqrt{(a+b)^2+y^2}}$$

$$=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)+C$$

[4]

b. **Solve: $x^2\frac{dy}{dx}+y^2=xy$.**

Solution

Given equation is

$$x^2\frac{dy}{dx}+y^2=xy$$

or, $\frac{dy}{dx}=\frac{xy-y^2}{x^2}$ ...(i)

This is a homogeneous equation. So, put y = vx. Then,

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Hence (i) becomes

$$v+x\frac{dv}{dx}=\frac{x\cdot vx-v^2x^2}{x^2}$$

or, $v+x\frac{dv}{dx}=v-v^2$

or, $x\frac{dv}{dx}=-v^2$

or, $-v^{-2}\,dv=\frac{dx}{x}$

On integration, we have

$$\frac{1}{v}=\log x+c$$

or, $\frac{x}{y}=\log x+c$

∴ $x=y(\log x+c)$

OR

Solve: $(1-x^2)\frac{dy}{dx}-xy=1$ [4]

Solution

Here, $(1-x^2)\frac{dy}{dx}=1+xy$

or, $(1-x^2)\frac{dy}{dx}-xy=1$

or, $\frac{dy}{dx}-\frac{x}{1-x^2}y=\frac{1}{1-x^2}$ ...(i)

Comparing equation (i) with $\frac{dy}{dx}+Py=Q$, we get

$$P=\frac{-x}{1-x^2},\quad Q=\frac{1}{1-x^2}$$

$$I.F=e^{\int Pdx}=e^{\int\frac{-x}{1-x^2}dx}=e^{\frac{1}{2}\int\frac{-2x}{1-x^2}dx}=e^{\frac{1}{2}\log(1-x^2)}=e^{\log(1-x^2)^{1/2}}=\sqrt{1-x^2}$$

Multiplying both sides of equation (i) by $\sqrt{1-x^2}$, we get

$$\sqrt{1-x^2}\frac{dy}{dx}-\sqrt{1-x^2}\cdot\frac{x}{1-x^2}\cdot y=\frac{1}{\sqrt{1-x^2}}$$

or, $d\left(y\sqrt{1-x^2}\right)=\frac{1}{\sqrt{1-x^2}}dx$

Integrating, we have

$$y\sqrt{1-x^2}=\int\frac{1}{\sqrt{1-x^2}}dx$$

$$y\sqrt{1-x^2}=\sin^{-1}x+c$$

8. a. **Find Karl Pearson's coefficient of skewness from the following distribution.** [4]

Marks	Above 20	Above 30	Above 40	Above 50	Above 60
No. of students	50	46	30	24	8

Solution

Calculation of Karl Pearson's Coefficient of Skewness

Marks	Mid value (x)	f	c.f.	$d'=\frac{x-a(45)}{h(10)}$	fd'	fd'²
20-30	25	4	4	-2	-8	16
30-40	35	16	20	-1	-16	16
40-50	45	6	26	0	0	0
50-60	55	16	42	1	16	16
60-70	65	8	50	2	16	32
		N = 50			Σ fd' = 8	Σfd'² = 80

Mean $(\bar{x})=a+\frac{\Sigma fd'}{N}\times h=45+\frac{8}{50}\times10=46.6$

Median $=\left(\frac{N}{2}\right)^{th}$ item $=\left(\frac{50}{2}\right)^{th}$ item = 25th item

∴ Median lies in (40 – 50) class interval.

∴ $l=40$, f = 6, c.f = 20, h = 10

We have, Median $(M_d)=l+\frac{\frac{N}{2}-C.f}{f}\times h=40+\frac{25-20}{6}\times10=48.33$

$$S.D(\sigma)=h\times\sqrt{\frac{\Sigma fd'^2}{N}-\left(\frac{\Sigma fd'}{N}\right)^2}=10\times\sqrt{\frac{80}{50}-\left(\frac{8}{50}\right)^2}=10\times\sqrt{1.6-0.0256}$$

$$=10\times\sqrt{1.5744}=12.55$$

We know, Karl Pearson's coefficient of skewness

$$S_k(P)=\frac{3(\bar{x}-M_d)}{\sigma}=\frac{3(46.6-48.33)}{12.55}=-0.41$$

b. **The chance that A can solve a certain problem is ¼ and the chance that B can solve it is ⅔ Find the chance that (i) the problem will be solved if they both try (ii) A solves but B cannot.** [4]

Solution

i. The problem will be solved if A or B solve it.

Probability of solving the problem by A, $P(A)=\frac{1}{4}$

Probability of solving the problem by B, $P(B)=\frac{2}{3}$

Probability of solving the problem, P(A or B) = P(A∪B) = ?

We have,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$
$$=P(A)+P(B)-P(A)\cdot P(B)$$
$$=\frac{1}{4}+\frac{2}{3}-\frac{1}{4}\times\frac{2}{3}=\frac{1}{4}+\frac{2}{3}-\frac{1}{6}$$
$$=\frac{3+8-2}{12}=\frac{9}{12}=\frac{3}{4}$$

ii. $P(\bar{B})=1-P(B)=1-\frac{2}{3}=\frac{1}{3}$

The probability of solving by A but not by B is

$$P(A\cap\bar{B})=P(A)\cdot P(\bar{B})=\frac{1}{4}\times\frac{1}{3}=\frac{1}{12}$$

OR

Suppose that in certain city 60% of all the recorded births are male. Suppose we select 5 birth records from population. What is the probability that

i. exactly three of them are male?

ii. 4 or more are male? [4]

Solution

Given, $p = 60\% = \frac{60}{100} = \frac{3}{5}$

$\therefore\ q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$

$n = 5$

(i) $P(3) = ?$

We have, $P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(3) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^{5-3} = \frac{5\times4}{2} \times \frac{27}{125} \times \frac{4}{25} = \frac{216}{625}$

ii. $P(4 \text{ or more are male}) = P(r \geq 4)$

$= P(4) + P(5) = {}^5C_4 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right)^1 + {}^5C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^0 = 5 \times \frac{81}{625} \times \frac{2}{5} + 1 \times \frac{243}{3125} \times 1 = \frac{1053}{3125}$

9. Show that: $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!} = e - 1$. [6]

Solution

The n^{th} term of the series is

$t_n = \frac{n^2}{(n+1)!} = \frac{n^2 - 1 + 1}{(n+1)!} = \frac{(n+1)(n-1)}{(n+1)!} + \frac{1}{(n+1)!} = \frac{n-1}{n!} + \frac{1}{(n+1)!} = \frac{n}{n!} - \frac{1}{n!} + \frac{1}{(n+1)!}$

$= \frac{1}{(n-1)!} - \frac{1}{n!} + \frac{1}{(n+1)!}$

$\therefore\ t_1 = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} \quad t_2 = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} \quad t_3 = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}$

....

Now by addition, the sum of given series

$= \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots\right) - \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\right) = e - (e-1) + (e-2)$

$= e - e + 1 + e - 2 = e - 1$

10. Define scalar product of two vectors. Find the geometrical interpretation of scalar product of two vectors. Prove vectorically that

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

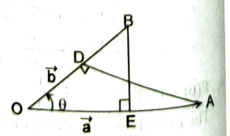
Solution

Scalar product of two vectors: Scalar product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \cdot \vec{b}$, is defined by $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta = ab\cos\theta$ where θ is the angle between the two vectors.

Geometrical Interpretation

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$. Let $\angle AOB = \theta$. Draw BE perpendicular to OA and AD perpendicular to OB.

Now,



$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$= ab\cos\theta$

$= (OA)(OB)\cos\theta$

$= (OA)(OB\cos\theta)$

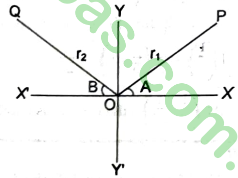
$= (OA)(OE)$

$=$ (magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$)

Similarly, $\vec{a} \cdot \vec{b}$ = (magnitude of $\vec{b}$) (projection of $\vec{a}$ on $\vec{b}$)

Next part

Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let $\angle XOP = A$ and $\angle QOX' = B$ so that $\angle POQ = \pi - (A + B)$. Also, let $OP = r_1$ and $OQ = r_2$. Then the coordinates of P and Q are $(r_1 \cos A, r_1 \sin A)$ and $(r_2\cos(\pi - B), r_2\sin(\pi - B)) = (-r_2\cos B, r_2 \sin B)$.



So $\vec{OP} = (r_1\cos A, r_1 \sin A)$

& $\vec{OQ} = (-r_2\cos B, r_2\sin B)$

Now, $\vec{OP} \cdot \vec{OQ} = (r_1\cos A, r_1\sin A) \cdot (-r_2\cos B, r_2\sin B)$

$= -r_1 r_2 \cos A\cos B + r_1 r_2 \sin A \sin B$

$= -r_1 r_2 (\cos A \cos B - \sin A \sin B)$

Since $\pi - (A + B)$ is the angle between $\vec{OP}$ and $\vec{OQ}$, so

$\cos[\pi - (A + B)] = \frac{\vec{OP} \cdot \vec{OQ}}{|\vec{OP}|\,|\vec{OQ}|}$

or, $-\cos(A + B) = \frac{-r_1 r_2 (\cos A\cos B - \sin A \sin B)}{r_1 r_2}$

$\therefore\ \cos(A + B) = \cos A\cos B - \sin A \sin B$

11. State Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function. $f(x) = x(x-1)^2$ in $[0, 1]$

Also, find the point on the curve where the tangent is parallel to the x-axis. [6]

Solution

First Part

Statement of Rolle's theorem:

If a function $f(x)$ is

(a) continuous in $[a, b]$

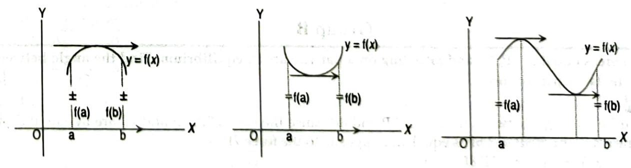
(b) differentiable in (a, b)

(c) $f(a) = f(b)$

then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Second Part

If all the conditions of Rolle's theorem are satisfied then there is at least one point $c \in (a, b)$ where the tangent is parallel to x-axis.



Last Part

Here, $f(x) = x(x-1)^2 = x^3 - 2x^2 + x$

Since $f(x)$ is a polynomial, it is continuous in $[0, 1]$.

$f'(x) = 3x^2 - 4x + 1$ which exists for all $x \in (0, 1)$

So, $f(x)$ is differentiable in $(0, 1)$

Also, $f(0) = 0$ and $f(1) = 0$

$\therefore\ f(0) = f(1)$

All the conditions of Rolle's theorem are satisfied. So, there exists a number $c \in (0, 1)$ such that $f'(c) = 0$

or, $3c^2 - 4c + 1 = 0$

or, $(c - 1)(3c - 1) = 0$

$\therefore$ $c = 1, \frac{1}{3}$

Clearly, $c = \frac{1}{3} \in (0, 1)$ but $c = 1 \notin (0, 1)$

Hence, Rolle's theorem is verified.

$f\left(\frac{1}{3}\right) = \frac{1}{3}\left(\frac{1}{3} - 1\right)^2 = \frac{4}{27}$

$\therefore$ At $\left(\frac{1}{3}, \frac{4}{27}\right)$, the tangent is parallel to x-axis.

OR

Find from first principle the derivative of $\ln \cos^{-1} x$. [6]

Solution

Let $f(x) = \ln \cos^{-1}x$

We have, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\therefore$ $\frac{d}{dx}(\ln \cos^{-1}x) = \lim_{h \to 0} \frac{\ln\cos^{-1}(x+h) - \ln\cos^{-1}x}{h}$ ...(i)

Put $\cos^{-1} x = y$ and $\cos^{-1}(x+h) = y + k$

then $x = \cos y$ and $x + h = \cos(y + k)$

$\therefore$ $h = \cos(y + k) - \cos y$

Also, as $h \to 0, k \to 0$

From (i)

$$\frac{d}{dx}(\ln\cos^{-1} x) = \lim_{h \to 0} \frac{\ln(y+k) - \ln y}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{y+k}{y}\right)}{h} = \lim_{k \to 0} \left\{ \frac{\ln\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \cdot \frac{\frac{k}{y}}{h} \right\}$$

$$= 1 \cdot \frac{1}{y} \lim_{k \to 0} \left\{ \frac{k}{\cos(y+k) - \cos y} \right\} = \frac{1}{y} \lim_{k \to 0} \left\{ \frac{k}{2\sin\left(\frac{y+k+y}{2}\right) \sin\left(\frac{y-y-k}{2}\right)} \right\}$$

$$= \frac{1}{y} \lim_{k \to 0} \left\{ \frac{-1}{\sin\left(y + \frac{k}{2}\right) \cdot \frac{\sin\frac{k}{2}}{\frac{k}{2}}} \right\} = \frac{1}{y} \cdot \frac{-1}{\sin y \cdot 1} = \frac{-1}{\cos^{-1}x\sqrt{1 - \cos^2 y}} = \frac{-1}{\cos^{-1}x\sqrt{1 - x^2}}$$

Group B

12. a. Forces equal to 7P, 5P and 8P acting on a particle are in equilibrium. Find the angle between the latter pair of forces. [2]

Solution

Let α be the angle between the forces 5P and 8P. Since the forces 5P, 8P and 7P are in equilibrium so the resultant of 5P and 5P is equal and opposite to the force 7P. So,

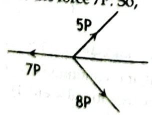
$(7P)^2 = (5P)^2 + (8P)^2 + 2 \cdot 5P \cdot 8P \cos\alpha$

or, $49P^2 = 25P^2 + 64P^2 + 80P^2 \cos\alpha$

or, $-40P^2 = 80P^2 \cos\alpha$

or, $\cos\alpha = \frac{-40P^2}{80P^2} = \frac{-1}{2} = \cos 120°$

$\therefore$ $\alpha = 120°$



b. A body is projected vertically upwards with a velocity of 19.6 m/s. How long will it take to reach a point 294m below the point of projection? (g = 9.8 m/s) [2]

Solution

Initial velocity (u) = 19.6 m/s

Weight (h) = 294 m

Time (t) = ?

Taking upward direction as positive, we have,

$-h = ut - \frac{1}{2}gt^2$

or, $-294 = 19.6t - \frac{1}{2} \times 9.8 \times t^2$

or, $4.9t^2 - 19.6t - 294 = 0$

or, $t^2 - 4t - 60 = 0$

or, $(t + 6)(t - 10) = 0$

Either t = –6 (not possible)

or t = 10

$\therefore$ t = 10 sec

c. A body of mass 50kg falling from a certain height is brought to rest after striking the ground with a speed of 5 m/s. If the resistance force of the ground is 500N, find the duration of the contract. [2]

Solution

Mass (m) = 50kg

Initial velocity (u) = 5 m/s

Final velocity (v) = 0

Resistance force (F) = –500 N

Duration of contact (t) = ?

We have,

$F = \frac{m(v - u)}{t}$

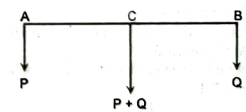
or, $-500 = \frac{50(0 - 5)}{t}$

$\therefore$ t = 0.5 sec

13. a. P and Q are two like parallel forces acting at A and B. Show that if they interchange positions, the point of application of the resultant is displaced by a distance $\frac{P - Q}{P + Q}$AB. [4]

Solution

Let P and Q be two like parallel forces acting at the points A and B, so their resultant P + Q acts at the point C as shown in the figure.

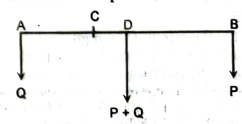


Since the forces are like parallel, so

$\frac{P}{BC} = \frac{Q}{AC} = \frac{P+Q}{AB}$

$\therefore$ $AC = \frac{Q}{P+Q} AB$ ...(i)

If forces P and Q be interchanged in their positions, let its resultant act at point D as shown in the figure below. Then,



$\frac{P}{AD} = \frac{Q}{BD} = \frac{P+Q}{AB}$

$\therefore$ $AD = \frac{P \cdot AB}{P+Q}$ ...(ii)

From (i) and (ii)

$AD - AC = \frac{P \cdot AB}{P+Q} - \frac{Q \cdot AB}{P+Q} = \left(\frac{P-Q}{P+Q}\right) AB$

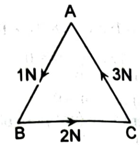
Hence, the line of action of the resultant is displaced along AB through a distance $\frac{P-Q}{P+Q}$AB where P > Q.

OR

Forces 1N, 2N and 3N act at a point in direction parallel to the sides of an equilateral triangle taken in order. Find their resultant. [4]

Solution

Let the forces 1N, 2N and 3N act along the sides AB, BC and CA of an equilateral triangle of side a, Now, resolving the forces along and perpendicular to BC, we have



$X = 2 \cdot \cos 0° + 3 \cos 120° + 1 \cos 240°$

$= 2 \cdot 1 + 3 \cdot \left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right) = 2 - \frac{3}{2} - \frac{1}{2} = 0$

$Y = 2 \sin 0° + 3 \sin 120° + \sin 240°$

$= 2 \times 0 + 3 \times \frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

Let R be the resultant. Then,

$R = \sqrt{X^2 + Y^2} = \sqrt{0^2 + (\sqrt{3})^2} = \sqrt{3}$

Let θ be the angle made by the resultant with BC. Then,

$\tan\theta = \frac{Y}{X} = \frac{\sqrt{3}}{0} = \infty = \tan 90°$

$\therefore \theta = 90°$

b. **Prove that the sum of the kinetic and potential energies of a freely falling body remains constant throughout the motion.** [4]

Solution

Suppose that a body of mass m is initially at the point A which is at a height h from the ground B. Let the body start falling from A and C be the position of the body at any instant such that AC = x. Then BC = $h-x$

At the point A

K.E. = 0

P.E. = mgh

$\therefore$ K.E + P.E = 0 + mgh = mgh

At the point C

Let v_1 be the initial velocity of the body at the point C. Then. $v_1^2 = 2gx$.

$K.E. = \frac{1}{2} m v_1^2 = \frac{1}{2} m \cdot 2gx = mgx$

$P.E. = mg(h - x)$

$\therefore K.E. + P.E = mgx + mg(h - x) = mgh$

At the point B

Let v be the velocity of the body when it reaches the ground. Then,

$v^2 = 2gh$

$K.E = \frac{1}{2} mv^2 = \frac{1}{2} m \cdot 2gh = mgh$

P.E = 0

$\therefore$ K.E + P.E = mgh + 0 = mgh

The sum of K.E and P.E of the freely falling body at any instant is same (i.e. mgh) and hence it is constant.

14. **The horizontal and the vertical components of the initial velocity of a projectile are U and V respectively. If R be the horizontal range and H, the greatest height attained, prove that**

i. $\frac{4H}{R} = \frac{V}{U}$ ii. $\left(\frac{R}{U}\right)^2 = \frac{8H}{g}$ [6]

Solution

Let u be the initial velocity of projectile and α of the angle of projection. Then,

Horizontal component (U) = u cos α

Vertical component (V) = u sin α

Also, we have

Greatest height (H) = $\frac{u^2 \sin^2\alpha}{2g}$ and

Horizontal range (R) = $\frac{u^2 \sin 2\alpha}{g}$

(a) L.H.S. $= \frac{4H}{R} = 4 \cdot \frac{\frac{u^2\sin^2\alpha}{2g}}{\frac{u^2 \sin 2\alpha}{g}}$

$= \frac{2u^2 \sin^2\alpha}{g} \times \frac{g}{u^2 \times 2 \sin\alpha \cos\alpha} = \frac{\sin\alpha}{\cos\alpha} = \frac{u \sin\alpha}{u\cos\alpha} = \frac{V}{U}$ = R.H.S.

(b) L.H.S $= \left(\frac{R}{U}\right)^2 = \left(\frac{\frac{u^2 \sin 2\alpha}{g}}{u\cos\alpha}\right)^2$

$= \left(\frac{u^2 \cdot 2\sin\alpha\cos\alpha}{g} \times \frac{1}{u\cos\alpha}\right)^2$

$= \left(\frac{2u\sin\alpha}{g}\right)^2 = \frac{4u^2\sin^2\alpha}{g^2}$

$= \frac{\frac{4u^2\sin^2\alpha}{g}}{g} = \frac{8\left(\frac{u^2\sin^2\alpha}{2g}\right)}{g} = \frac{8H}{g}$ = R.H.S.

OR

A cat seeing a mouse at a distance of 15m before it, starts from rest with an acceleration of 2 m/s² and pursues it. If the mouse be moving uniformly with a velocity of 14 m/s, find when and where the cat will catch the mouse. [6]

Solution

Let t be the time taken by the cat to catch the mouse and let the mouse run x m in t sec. Then the distance covered by the cat to catch the mouse =(15+ x) m.

Since the mouse runs with uniform velocity u = 14 m/s, using s = ut, we get

x = 14t ...(i)

For cat, using $s = ut + \frac{1}{2}at^2$, we get

$15 + x = 0 + \frac{1}{2} \times 2 \times t^2$

or, $t^2 = 15 + x$ ...(ii)

From (i) and (ii)

$t^2 = 15 + 14t$

or, $t^2 - 14t - 15 = 0$

or, $(t + 1)(t - 15) = 0$

Either t = –1 (impossible)

or t = 15 sec

Now, from (i)

x = 14 × 15= 210 m

So, the cat will catch the mouse in 15 sec after running a distance of 210 m.

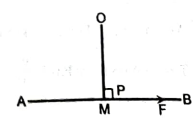
15. **Define the moment of a force about a point and interpret its geometrical meaning. Prove that the algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point.** [6]

Solution

Moment

The moment of a force about a given point is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force..

Thus if F be a force and p be the perpendicular distance of the point O from AB then the moment of F about O is F × OM = F × p



Geometric Meaning of the Moment of a Force

Let a force F be represented in magnitude and direction by AB and O be a point. Join OA and OB. Let us draw OD perpendicular to AB or AB produced.

The moment of the force F about O is F × OD = AB × OD = 2 (area of ΔOAB) numerically. Hence, the moment of the force F about O is represented numerically by two times the area of a triangle whose base is the line representing the force and vertex is the point about which moment is taken.

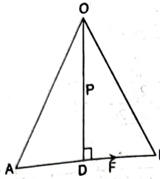


fig (i)

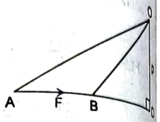


fig (ii)

Last Part

Let O be a given point. Let us draw OC parallel to P to meet the line of action of Q at C. Let AC represent Q in magnitude and AB represent P in magnitude. Now, complete the parallelogram ABDC.

Then AD represents the resultant R of P and Q.

Join OA and OB.

The point O may lie outside the ∠BAC in fig (i) and inside ∠BAC in fig (ii). The moment of P about O is 2 ΔOAB and is positive in both figures.

The moment of Q about O is 2ΔOAC which is positive in fig (i) and negative in fig (ii)

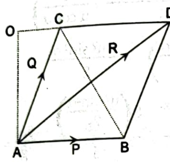


fig (i)

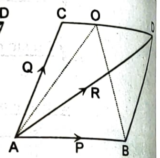


fig (ii)

In fig (i)

The sum of moments of P and Q about O

= 2ΔOAB + 2ΔOAC

= 2ΔADB + 2ΔOAC (∵ ΔOAB = ΔADB)

= 2ΔOAD

= moment of R about O

In fig (ii),

The moment of P and Q about O

= 2ΔOAB – 2ΔOAC

= 2ΔADB + 2ΔOAC (∵ ΔOAB = ΔADB)

= 2ΔOAD = moment of R about O.

Group C

16. a. If a man rides his car at 25 km/hr, he has to spend Rs. 2 per km on petrol. If he rides it at faster speed of 40 km/hr, the petrol cost increases to Rs. 5 per km. He has Rs. 100 to spend on petrol and wishes to find the maximum distance he can travel within one hour. Formulate the above problem as a linear programming problem. [2]

Solution

Let x km and y km be the distances covered at 25 km/hr and 40 km/hr respectively. Then, total distance covered = (x + y) km

When the speed is 25 km/hr, expenditure on petrol = Rs 2 per km

Hence, for x km, expenditure on petrol = Rs 2x

When the speed is 40 km/hr, expenditure on petrol = Rs 5 per km

Hence, for y km, expenditure on petrol = Rs 5 y

Therefore, total expenditure = Rs(2x + 5y)

But amount to spend on petrol = Rs 100

∴ $2x + 5y \le 100$

Again, time taken for x km is $\frac{x}{25}$

Time taken for y km is $\frac{y}{40}$

Total time = $\frac{x}{25} + \frac{y}{40}$

∴ $\frac{x}{25} + \frac{y}{40} \le 1$

or, $8x + 5y \le 200$.

Hence, the mathematical model of given problem is

maximize Z (x,y) = x + y subject to

$2x + 5y \le 100$

$8x + 5y \le 200$

$x, y \ge 0$

b. Convert the decimal number 2011 into octal form. [2]

Solution

		Remainder
8	2011	
8	251	3
8	31	3
8	3	7
	0	3

∴ $2011_{10} = 3733_8$

c. Is the following equations diagonally dominant:

$12x + 3y - 5z = 1$ $x + 5y + 3z = 28$ $3x + 7y + 13z = 1$? [2]

Solution

Here, |12| > |3|+ |-5| = 8

and |5| > |1| + |3| = 4

|13| > |3|+ |7| = 11

Hence, the given system of equations is diagonally dominant.

17. a. Using Gauss elimination method, solve the following system of equations:

$x + 3y - z = -2$ $3x + 2y - z = 3$ $-6x - 4y - 2z = 18$ [4]

Solution

$x + 3y - z = -2$ ...(i)

$3x + 2y - z = 3$ ...(ii)

$-6x - 4y - 2z = 18$ ...(iii)

Multiplying equation (i) by 3 and then subtracting from (ii)

$$\begin{array}{rrrr} 3x & +2y & -z & =3 \\ 3x & +9y & -3z & =-6 \\ - & - & + & + \\ \hline & -7y & +2z & =9 \end{array} \quad \text{(iv)}$$

Again, multiplying equation (i) by 6 and adding with equation (iii)

$$\begin{array}{rrrr} -6x & -4y & -2z & =18 \\ 6x & +18y & -6z & =-12 \\ \hline & 14y & -8z & =6 \end{array} \quad \text{...(v)}$$

Multiplying equation (iv) by 2 and adding with equation (v)

$$\begin{array}{rrr} 14y & -8z & =6 \\ -14y & +4z & =18 \\ \hline & -4z & =24 \end{array} \quad \text{... (vi)}$$

Now, we have the following three equations

$x + 3y - z = -2$ ...(i)

$-7y + 2z = 9$ ...(iv)

$-4z = 24$ ... (vi)

From equation (vi), we have z = -6.

Using z = -6, in equation (iv), we have

$-7y + 2 \times (-6) = 9$

or, $-7y = 21$

∴ $y = -3$

Again, using y = -3, and z = -6 in equation (i), we have

$x + 3(-3) - (-6) = -2$

or, $x = 1$

$\therefore$ The required solution is:
$x = 1, y = -3, z = -6.$

OR

Solve the following equations using Gauss-seidal method: [4]

$2x_1 - x_2 = 8;\ 3x_1 + 7x_2 = -5$

Solution

Given equations are:

$$x_1 = \frac{1}{2}(8 + x_2)$$

$$x_2 = -\frac{1}{7}(5 + 3x_1)$$

1st iteration:

$$x_1 = \frac{1}{2}(8 + 0) = 4$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 4) = -2.429$$

2nd iteration:

$$x_1 = \frac{1}{2}(8 - 2.429) = 2.786$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 2.786) = -1.908$$

3rd iteration:

$$x_1 = \frac{1}{2}(8 - 1.908) = 3.046$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 3.046) = -2.020$$

4th iteration:

$$x_1 = \frac{1}{2}(8 - 2.020) = 2.990$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 2.990) = -1.996$$

5th iteration:

$$x_1 = \frac{1}{2}(8 - 1.996) = 3.002$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 3.002) = -2.001$$

6th iteration:

$$x_1 = \frac{1}{2}(8 - 2.001) = 3.000$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 3.000) = -2$$

7th iteration:

$$x_1 = \frac{1}{2}(8 - 2) = 3$$

$$x_2 = \frac{-1}{7}(5 + 3 \times 3) = -2$$

From 6th and 7th iterations, the values of x_1 and x_2 are equal.

$\therefore$ $x_1 = 3, x_2 = -2$

b. Evaluate the following integral using Simpson's rule:

$$\int_0^1 \frac{dx}{1 + x^2}$$ **taking 4 equal intervals (i.e. n = 4).** [4]

Solution

Given, $y = f(x) = \frac{1}{1 + x^2}$

$a = 0, b = 1, n = 4$

We have, $h = \frac{b - a}{n} = \frac{1 - 0}{4} = 0.25$

The five points to be considered are $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1.$

The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \frac{1}{1 + x^2}$	1	0.9412	0.8	0.64	0.5

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \frac{1}{1 + x^2} dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] = \frac{0.25}{3}[1 + 4 \times 0.9412 + 2 \times 0.8 + 4 \times 0.64 + 0.5]$$

$$= \frac{0.25}{3}(9.4254) = 0.7854$$

18. Using Simplex method, maximize $Z = 5x_1 + 7x_2$ subject to: $2x_1 + 3x_2 \le 13$, $3x_1 + 2x_2 \le 12$, $x_1, x_2 \ge 0$. [6]

Solution

Introducing the non-negative slack variables x_3 and x_4, then given LPP can be written as

$2x_1 + 3x_2 + x_3 = 13$

$3x_1 + 2x_2 + x_4 = 12$

$Z = 5x_1 + 7x_2$

$\Rightarrow$ $2x_1 + 3x_2 + x_3 + 0 \cdot x_4 + 0 \cdot Z = 13$

$3x_1 + 2x_2 + 0 \cdot x_3 + x_4 + 0 \cdot Z = 12$

$-5x_1 - 7x2 + 0 \cdot x_3 + 0 \cdot x_4 + Z = 0$

Initial tableau

x_1	x_2	x_3	x_4	Z	R.H.S
2	[3]	1	0	0	13
3	2	0	1	0	12
-5	-7	0	0	1	0
	↑				

Here, -7 is the most negative entry in the last row, so second column is the pivot column.

Since $\frac{13}{3} = 4.33$, $\frac{12}{2} = 6$ and $4.33 < 6$, so 3 is the pivot element.

Applying $R_1 \to \frac{1}{3}R_1$

x_1	x_2	x_3	x_4	Z	R.H.S
2/3	1	1/3	0	0	13/3
3	2	0	1	0	12
-5	-7	0	0	1	0

Again, applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 7R_1$

x_1	x_2	x_3	x_4	Z	R.H.S
2/3	1	1/3	0	0	13/3
[5/3]	0	-2/3	1	0	10/3
-1/3	0	7/3	0	1	91/3
↑					

This is not the optimal solution as the last row contains negative entry.

Again, first column is the pivot column. Since $\frac{13/3}{2/3} = 6.5$, $\frac{10/3}{5/3} = 2$ and $2 < 6.5$, so 5/3 is the pivot element.

Applying $R_2 \to \frac{3}{5} R_2$

x_1	x_2	x_3	x_4	Z	R.H.S
2/3	1	1/3	0	0	13/3
1	0	-2/5	3/5	0	2
-1/3	0	7/3	0	1	91/3

Applying $R_1 \to R_1 - \frac{2}{3} R_2$ and $R_3 \to R_3 + \frac{1}{3} R_2$

x_1	x_2	x_3	x_4	Z	R.H.S
0	1	3/5	-6/5	0	3
1	0	-2/5	3/5	0	2
0	0	7/3	1/5	1	31

This is the optimal solution as the last row contains all non-negative entry.
So, max. value of Z = 31 at $x_1 = 2$ and $x_2 = 3$.

19. Show that the equation $f(x) = x^3 - 18 = 0$ has only one positive root. Using bisection method, find the positive root correct to 3 places of decimal in the interval (2, 3). [6]

Solution

Here, $f(x) = x^3 - 18$
Since f(x) has one change in sign, so f(x) has only one positive root.
Given, a = 2, b = 3
$f(2) = 2^3 - 18 = -10$
$f(3) = 3^3 - 18 = 9$
Since f(2) and f(3) have opposite signs, so a root lies between 2 and 3.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
2	3	2.5	-10	9	-2.375
2.5	3	2.75	-2.375	9	2.79687
2.5	2.75	2.625	-2.375	2.79687	0.08789
2.5	2.625	2.5625	-2.375	0.08789	-1.17358
2.5625	2.625	2.59375	-1.17358	0.08789	-0.55045
2.59375	2.625	2.60938	-0.55045	0.08789	-0.23319
2.60938	2.625	2.61719	-0.23319	0.08789	-0.07308
2.61719	2.625	2.62110	-0.07308	0.08789	-0.00729
2.61719	2.62110	2.61915	-0.07308	0.00729	-0.03287
2.61915	2.62110	2.62013	-0.03287	0.00729	-0.01270
2.62013	2.62110	2.62062	0.01270	0.00729	-0.00260
2.62062	2.62110	2.62086	-0.00260	0.00729	0.00244
2.62062	2.62086		-0.00260	0.00244	

Since a and b have same value to three places of decimal, the required root is 2.620.

OR

Use Newton-Raphson method to find the positive root of $x^3 + 3x - 5 = 0$ lying between 1 and 2 correct to three places of decimals. [6]

Solution

Let, $f(x) = x^3 + 3x - 5$
$f'(x) = 3x^2 + 3$
Let us make an initial guess $x_0 = 1$
By Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{9}{15} = 1.4$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.4 - \frac{1.944}{8.88} = 1.1811$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.1811 - \frac{0.1909}{7.185} = 1.1545$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.1545 - \frac{0.0023}{6.9986} = 1.1542$$

Comparing the values of x_3 and x_4 we find the digits on first three places of decimal are same.
Hence, required root is 1.154.

UNIT 1
PERMUTATION AND COMBINATION

A. PERMUTATION

2 MARKS QUESTIONS

1. 2058 Q.No. 2 a

In a certain election, there are three candidates for president, five for secretary and only two for the treasurer. Find in how many ways the election may turn out. [2]

SOLUTION

There are 3 choices for president, 5 for secretary and 2 for treasurer.

Since they are all independent of one another, the total number of ways in which the election may turn out $= 3 \times 5 \times 2 = 30$

2. 2059 Q.No. 1 b

How many permutations are there of the letters of the word 'mathematics' taken all together? [2]

SOLUTION

There are 11 letters in the word 'MATHEMATICS' in which 'M' comes twice, 'A' comes twice, 'T' comes twice and the rest are single. So
n = 11, p = 2, q = 2, r = 2
Total number of arrangements

$$= \frac{n!}{p!\,q!\,r!} = \frac{11!}{2!\,2!\,2!} = 4989600$$

3. 2060 Q.No. 2 a

Find the numbers of permutation of the letters of the word 'MATHEMATICS'. [2]

SOLUTION

Please refer to 2059 Q.No. 1b

4. 2061 Q.No. 2 a

In how many ways can 6 different beads be strung on a necklace? [2]

SOLUTION

Here, n = 6
The clockwise and anticlockwise arrangements are same in the necklace.

So, total number of arrangements $= \frac{1}{2}(n-1)!$

$$= \frac{1}{2}(6-1)! = \frac{1}{2}(5 \times 4 \times 3 \times 2 \times 1) = 60$$

5. 2063 Q.No. 2 a

How many numbers of three different digits less than 500 can be formed from the integers 1, 2, 3, 4, 5, 6? [2]

SOLUTION

Given integers are 1, 2, 3, 4, 5, 6
Since the numbers should be made from three different digits less than 500, the hundred's place has either 1 or 2 or 3 or 4. So, there are 4 choices for hundred's place.
When the hundred's place has been filled up, there will be 5 digits left. So, there are 5 choices for ten's place. In the same way, there are 4 choices for unit's place.
By basic principle of counting, total number of ways $= 4 \times 5 \times 4 = 80$

6. 2065 Q.No 2 a

How many numbers are there between 100 and 1000 such that every digit is either 2 or 9? [2]

SOLUTION

We know that the numbers between 100 and 1000 are 3 digit numbers. Since each digit is either 2 or 9, so there are 2 choices for unit's, 2 choices for ten's and 2 choices for hundred's place.
By basic principle of counting, total number of arrangements $= 2 \times 2 \times 2 = 8$
Note: Formed numbers are 222, 229, 292, 299, 922, 929, 992, 999

7. 2066 C Q.No. 2 a

How many four digits odd numbers can be formed using the digits 0, 1, 2, 3, 4, 5 no digit being repeated? [2]

SOLUTION

Given digits are 0, 1, 2, 3, 4, 5
For odd numbers there are 3 choices for unit's place either 1 or 3 or 5. Also for 4 digit number starting digit cannot be 0. So, there are 4 choices for thousand's place. And for hundred's place there are 4 and for ten's place there are 3 choices.
So, by basic principle of counting, total number of ways $= 4 \times 4 \times 3 \times 3 = 144$

8. 2067 Q.No. 2a

In how many ways letters of the word PRECARIOUS can be arranged so that all the vowels are always together? [2]

SOLUTION

Given word is 'PRECARIOUS'

Since the vowels are always together consider the vowels E, A, I, O, U as a single letter. Then there will be 6 letters in which 'R' comes twice. So the number of arrangements

$= \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$

Also, the 5 vowels can be arranged among themselves in 5! ways

$= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

Required number of arrangements $= 360 \times 120$
$= 43,200$

9. 2068 Q.No. 2a

How many license plates consisting of 3 different digits can be made out of given integers 3, 4, 5, 6, 7? [2]

SOLUTION

Given integers are 3, 4, 5, 6, 7

$\therefore$ n = 5

& r = 3

Total number of license plates that can be made $= P(5, 3) = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$

10. 2069 (Set A) Q.No. 1a

In how many ways can four boys and three girls be seated in a row containing seven seats if they may sit any where? [2]

SOLUTION

Number of boys = 4

Number of girls = 3

If they may sit anywhere, then there is no difference between the position of girls and boys

$\therefore$ n = 4 + 3 = 7, r = 7

$\therefore$ Total number of arrangements = P(7, 7) ways
= 7! ways

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ ways

11. 2070 Set D Q.No. 1 a

In how many ways can the letters of the word "ELEMENT" be arranged? [2]

SOLUTION

There are 7 letters in the word 'ELEMENT'. Also there are 3 E's and the rest are single.

$\therefore$ n = 7, p = 3

Total number of arrangements $= \frac{n!}{p!} = \frac{7!}{3!}$

$= \frac{7 \times 6 \times 5 \times 4 \times 3!}{3!} = 840$

12. 2070 (Old) Q.No. 1 b

How many numbers of three different digits less than 500 can be formed from the integers 1, 2, 3, 4, 5, 6, 7? [2]

SOLUTION

Given integers are 1, 2, 3, 4, 5, 6, 7.

Since the formed numbers should be less than 500, so for hundred's place there may be 1 or 2 or 3 or 4. So there are 4 choices for hundred's place. To fill up the ten's place there are 6 choices and for unit's place there are 5 choices. Then by basic principle of counting, total number of three different digit numbers less that 500 that can be formed $= 4 \times 6 \times 5 = 120$

13. 2070 Supp. Q.No. 1 a

Six children are to be seated on a bench. How many arrangements are possible if the youngest child sits at the left end of the bench. [2]

SOLUTION

Fix youngest child at the left end of the bench then remaining 5 children can be arranged in P(5, 5) ways

= 5! ways

$= 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways

14. 2071 Old Q.No. 2 a

How many permutations are there of the letters of the word "SAARC". [2]

SOLUTION

There are 5 letters in the word SAARC. Also there are two A's and the rest are single.

$\therefore$ n = 5, p = 2

Total number of arrangements $= \frac{n!}{P!} = \frac{5!}{2!}$

$= \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$

15. 2071 Supp. Q.No. 1a

In how many ways can eight different colour heads be made into a bracelet? [2]

SOLUTION

Here, number of beads = 8

8 beads can be arranged on a circle (8 – 1)! = 7! ways

In this case, clockwise and anticlockwise arrangements are same.

$\therefore$ Total number of arrangements $= \frac{7!}{2}$

$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2} = 2520$

16. 2072 Set C Q.No. 1a

In how many ways can 7 students be seated in a circle? [2]

SOLUTION

Here, n = 7

The required number of ways 7 students can be seated in a circle

$= (n-1)! = (7-1)! = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
= 720 ways

17. 2072 Set D Q.No. 1a

In how many ways the letters of the word ELEMENT can be arranged so that vowels are always together? [2]

SOLUTION

The word 'ELEMENT' consists of 3 vowels E, E, E. Consider these 3 vowels as a single letter. Then we have to arrange 5 different letters L, M, N, T, (E,E,E).

$\therefore$ Required number of ways in which all the vowels are always together

$= 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

18. 2072 Set E Q.No. 1a

Find the number of ways in which 4 men and 3 women can be seated in a row having seven seats so that the men and the women must alternate. [2]

SOLUTION

Here, number of men = 4

Number of women = 3

Seats

1	2	3	4	5	6	7
M	W	M	W	M	W	M

If men and women sit alternatively, then men should be in odd places (i.e. 1st, 3rd, 5th and 7th) and women should be in even place (i.e. 2nd, 4th and 6th).

$\therefore$ 4 men in 4 seats can be arranged in P(4,4) ways

3 women in 3 seats can be arranged in P(3,3) ways

Total number of arrangements

$= P(4,4) \times P(3,3)$

$= 4! \times 3!$ $(\because P(n,n) = n!)$

$= 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 = 144$

19. 2072 Supp Q.No. 1a

How many numbers between 3000 and 4000 can be formed with the digits 2, 3, 4, 5, 6, 7? [2]

SOLUTION

Given digits are 2, 3, 4, 5, 6 and 7. Since the numbers lie between 3000 and 4000, there must be 3 at first place (thousand's place) and remaining 3 places can be filled by the remaining 5 digits 2, 4, 5, 6 and 7. So, fix 3 in the thousand's place, then the number of arrangements of 5 digits in 3 places can be done in P(5, 3) ways

$= \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$

20. 2073 Set C Q.No. 1a

How many different numbers of five digits can be formed with the digits 0, 1, 2, 3, 4 ? [2]

SOLUTION

The number of 5 digits numbers formed from given digits $= P(5,5) = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

But, some of these numbers have 0 in the first place, which are not 5-digit significant numbers. So, we fix 0 in first place, then remaining 4 digits in 4 places can be arranged in P(4,4) ways $= 4! = 4 \times 3 \times 2 \times 1 = 24$ ways

These are the numbers with 0 in first place.

Hence, the required number of 5-digit significant numbers = 120 – 24 = 96.

21. 2073 Supp Q.No. 1a

In how many ways the letters of the word HEXAGON can be arranged so that vowels are always together? [2]

SOLUTION

Since the vowels are always together, so consider (E,A,O) as a single letter. Then, there will be 5 letters namely (E,A,O), H, X, G, N.

These letters can be arranged in 5! ways.

Again, 3 vowels among themselves can be arranged in 3! ways.

$\therefore$ Total no. of arrangements $= 5! \times 3!$

$= 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 = 720$

22. 2074 Set B Q.No. 1a

In how many ways the letters of the word COMPLETE can be arranged so that the repeated letters are always together? [2]

SOLUTION

There are 8 letters in the word COMPLETE in which E repeats 2 times. So. consider 2 E's as a single letter, then there are 7 letters C, O, M, P, L, T (E, E). The total no. of arrangement in which 2 E's always come together

$= P(7, 7) = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

23. 2075 Set A Q.No. 1a

In how many ways can eight people be seated in a row of eight seats so that two particular persons are always together? [2]

SOLUTION

Suppose two particular persons who are always together as one and their two seats as one seat. Then there are 7 persons and 7 seats. Now, 7 persons in 7 seats can be arranged in P(7, 7) ways.

= 7! ways.

Also, 2 persons who are always together can be arranged among themselves in 2! ways.

Hence,

Total No. of arrangements $= 7! \times 2!$

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 \times 1 = 10080$

4 MARKS QUESTIONS

24. 2057 Q.No. 7 b

In how many ways can the letters of the word ARRANGE be arranged so that no. two R's come together. [4]

SOLUTION

Given word is 'ARRANGE'

Total number of letters (n) = 7

Number of A's (p) = 2

Number of R's (q) = 2

And the rest are single.

Total number of arrangements $= \frac{n!}{p!\,q!} = \frac{7!}{2!\,2!}$

$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2!}{2! \times 2 \times 1} = 1260$

To find the number of arrangements in which no two 'R' come together, we first find the number of arrangements in which two R's come together. For this, consider two R's as a single letter, then the number of letters will be 6.

Total number of arrangements in which two R's come together

$= \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$

∴ Required number of arrangements in which no two 'R' come together

= 1260 – 360 = 900

25. 2059 Q.No. 7 b

Prove that the total number of permutations of a set of n objects taken r at a time is given by $P(n, r) = \frac{n!}{(n-r)!}$. [4]

SOLUTION

The number of permutations of a set of n objects taken r at a time is equivalent to the number of ways in which r places can be filled up by n objects. Now, there are n choices to fill up the first place. When first place has been filled up, there will be left n – 1 objects to fill up the second place. So, there are (n –1) choices to fill up the second place. Similarly, there are (n –2) choices to fill up the third place and so on. Finally, to fill up the r^{th} place, there are n – (r –1) = n – r + 1 choices.

Then by basic principle of counting, total number of ways = n (n –1). (n – 2) ... (n – r + 1)

∴ P(n,r) = n (n – 1) (n – 2) ...(n – r + 1)

$= \frac{n(n-1)(n-2)\ldots(n-r+1)(n-r)\ldots 3 \cdot 2 \cdot 1}{(n-r)\ldots 3 \cdot 2 \cdot 1}$

$= \frac{n!}{(n-r)!}$, n ≥ r

26. 2061 Q.No. 7 b

In how many ways can the letters of the word "MONDAY" be arranged? How many of these arrangements do not begin with M? How many begin with M and don't end with Y? [4]

SOLUTION

Total number of letters in the word 'MONDAY' is 6.

∴ Number of arrangements = 6!

= 6 × 5 × 4 × 3 × 2 × 1 = 720

To find the number of arrangements that do not begin with M, firstly we find the number of arrangements that begin with 'M'. For this fix 'M' at first place, then remaining 5 letters can be arranged in P (5, 5) ways

= 5 × 4 × 3 × 2 × 1 = 120 ways

∴ Required number of arrangements that do not begin with 'M' = 720 – 120 = 600

Again, fix 'M' at first place and 'Y' at last, the remaining 4 letters can be arranged in P(4, 4) ways = 4 × 3 × 2 × 1 = 24

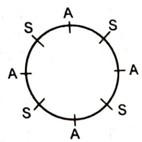
∴ Required number of arrangements that begin with 'M' and do not end with 'Y' = 120 –24 = 96

27. 2062 Q.No. 7 b

In how many ways can 4 Art students and 4 Science students be arranged alternately at a round table? [4]

SOLUTION

4 Art students at a round table can be arranged in (4 – 1)! = 3! ways = 3 × 2 × 1 = 6 ways



Since Art and Science students must be arranged alternatively, so 4 Science Students in 4 seats can be arranged in P(4, 4) ways = 4! ways = 4 × 3 × 2 × 1 = 24 ways

Total number of arrangements = 6 × 24 = 144

28. 2064 Q.No. 7 b

Show that the number of ways in which the letters of the word "arrange" can be arranged so that two r's do not come together is 900. [4]

➥ Please refer to 2057 Q.No. 7b

29. 2066 Q.No 7 b

Prove that the total no. of permutations of a set of n objects taken r at a time is given by P (n, r) = n(n – 1) (n – 2) ... (n – r + 1), n ≥ r.

SOLUTION

The number of permutations of a set of n objects taken r at a time is equivalent to the number of ways in which r places can be filled

up by n objects. Now, there are n choices to fill up the first place. When first place has been filled up, there will be left n – 1 objects to fill up the second place. So, there are (n –1) choices to fill up the second place. Similarly, there are (n –2) choices to fill up the third place and so on. Finally, to fill up the r^{th} place, there are n – (r –1) = n – r + 1 choices.

Then by basic principle of counting, total number of ways = n (n –1). (n – 2)... (n – r + 1)

∴ P(n,r) = n (n – 1) (n – 2) ...(n – r + 1)

30. 2069 (Set A) Q.No. 5a

In how many ways can the letters of the world "ARRANGE" be arranged so that no two R is come together? [4]

➥ Please refer to 2057 Q.No 7b

31. 2069 (Set A) Old Q.No. 7b

In how many ways can the letters of the word "MONDAY" be arranged? How many of these arrangements do not begin with M? How many begins with M and does not end with Y? [4]

➥ Please refer to 2061 Q.No. 7b

32. 2069 (Set B) Q.No. 5a

In how many ways can the letters of the word "MONDAY" be arranged? How many of these arrangements do not begin with M? How many begin with M and do not end with N? [4]

➥ Please refer to 2061 Q.No 7b

33. 2069 Old (Set B) Q.No. 7b

In how many ways can ten people be seated in a round table if two people insists on sitting next to each other? [4]

SOLUTION

Consider two people who insist on sitting next to each other as one, then there will be 9 people. Then, 9 people in a round table can be arranged in (9 – 1)! ways = 8! ways

Also two people sitting next to each other among themselves can be arranged in P(2, 2) ways = 2! ways

∴ Total number of arrangements

= 8! × 2! = 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1 × 2 × 1

= 80640

34. 2070 Set C Q.No. 5 a

In how many ways can the letters of the word, 'CALCULUS' be arranged so that the two L's do not come together? [4]

SOLUTION

There are 8 letters in the word 'CALCULUS'. Also there are 2 C's, 2 U's and 2 L's and the rest are different.

Total number of arrangements $= \frac{8!}{2!\,2!\,2!} = 5040$

First we find the arrangements which contain 2 C's together. For, consider 2 C's as one. Now there are 7 letters in which there are 2 U's and 2 L's.

The number of arrangements in which 2 C's are always together $= \frac{7!}{2!\,2!} = 1260$

Required number of arrangements in which 2 C's do not come together

= 5040 –1260 = 3780

35. 2070 (Old) Q.No. 7 a

In how many ways can the letter of the word 'Sunday' be arranged? How many of these arrangements do not begin with S? How many begin with S and do not end with a? [4]

SOLUTION

There are 6 letters in the word 'Sunday'. Total number of arrangements

= P(6, 6) ways = 6! ways

= 6 × 5 × 4 × 3 × 2 × 1

= 720 ways

To find the number of arrangements which do not begin with 'S', we first find the number of arrangements that begin with 'S'. For, fix 'S' at first place, then remaining 5 letters can be arranged in

= P(5, 5) ways = 5! ways = 5 × 4 × 3 × 2 × 1

= 120 ways

The number of arrangements that do not begin with 'S'

= 720 – 120 = 600

Again, fix 'S' at first place and 'a' at last, then remaining 4 letters can be arranged in P(4,4) ways = 4! ways = 4 × 3 × 2 × 1 = 24 ways

∴ Total number of arrangements that begin with S and do not end with a = 120 – 24 = 96

36. 2071 Set C Q.No. 5 a

In how many ways can the letters of the word "TUESDAY" be arranged? How many of these arrangements do not begin with T? How many begin with T and do not end with Y? [4]

SOLUTION

There are 7 letters in the word 'TUESDAY'.

So total no. of arrangements = P(7,7) = 7!

= 7 × 6 × 5 × 4 × 3 × 2 × 1 = 5040

To find the arrangements that do not begin with 'T', we find the arrangements that begin with 'T'. For, fix 'T' at first place.

Then, remaining 6 letters can be arranged in P(6, 6) ways = 6! = 6 × 5 × 4 × 3 × 2 × 1 = 720

∴ Number of arrangements that do not begin with 'T' = 5040 – 720 = 4320

Again, fix 'T' at first place and 'Y' at last place. Then remaining 5 letters can be arranged in P(5, 5) ways = 5! = 5 × 4 × 3 × 2 × 1 = 120 ways

∴ The number of arrangements begin with 'T' and do not end with 'Y' = 720 – 120 = 600

37. 2071 Set D Q.No. 5 a

In how many ways can the letter of the word "COMPUTER" be arranged so that i) all vowels are always together? ii) the relative positions of the vowels and consonants are not changed? [4]

SOLUTION

There are 8 letters in the word 'COMPUTER'

(i) There are 3 vowels in the word 'COMPUTER'. Consider 3 vowels as one letter. Then 6 letters namely C, M, P, T, R, (O, U, E) can be arranged in P(6, 6) ways = 6! ways

Also, 3 vowels among themselves can be arranged in P(3, 3) ways = 3! ways

∴ Total number of arrangements = 6! × 3!

= 6 × 5 × 4 × 3 × 2 × 1 × 3 × 2 × 1 = 4320 ways

(ii) There are 3 vowels and 5 consonants in the word 'COMPUTER'.

The 3 vowels in 3 places can be arranged in P(3,3) ways = 3!

And 5 consonants in 5 places can be arranged in P(5,5) ways = 5!

∴ Total number of arrangements = 3! × 5!

= 3 × 2 × 1 × 5 × 4 × 3 × 2 × 1 = 720 words

38. 2072 Supp Q.No. 5a

Prove that $P(n, r) = \frac{n!}{(n-r)!}$ where the symbols have their usual meanings. [4]

➥ Please refer to 2059 Q.No. 7b

39. 2073 Set D Q.No. 5a

In how many ways can the letters of the word "COMPUTER" be arranged so that (i) all the vowels are always together (ii) the vowels may occupy only odd positions. [4]

➥ Please refer to Model Set II, Q.No. 5a

40. 2074 Set A Q.No. 5a

Prove that the number of permutations of n distinct objects taken r at a time is given by: $p(n, r) = \frac{n!}{(n-r)!} \cdot (n \geq r)$ [4]

➥ Please refer to 2059 Q.No. 7b

41. 2074 Set A Q.No. 6a OR

How many numbers of 4 different digits can be formed from the digits 4, 5, 6, 7, 8? How many of these numbers are divisible by 5? How many of these numbers are not divisible by 5? [4]

SOLUTION

Given digits are 4, 5, 6, 7, 8

So, n = 5, r = 4

The number of 4 different digits numbers that can be formed = P(n, r)

$= P(5, 4) = \frac{5!}{(5-4)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{1!} = 120$

For the numbers which are divisible by 5, we must have 5 in unit's place. So, fix 5 in unit's place, then remaining 3 places by remaining digits can be filled up by P(4, 3) ways

$= \frac{4!}{(4-3)!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24$

∴ The number of 4 digit numbers which are divisible by 5 = 24.

Hence, the number of 4 different digits numbers which are not divisible by 5

= total numbers – numbers which are divisible by 5

= 120 – 24 = 96

42. 2074 Supp Q.No. 5a

In how many ways can the letters of the word "ARRANGE" be arranged so that no two R come together? [4]

➥ Please refer to 2057 Q.N. 7b

43. 2075 Set B Q.No. 5a

How many words can be formed from the letters of the word 'ENGLISH'? How many of these do not begin with E? How many of these begin with E and do not end with H? [4]

SOLUTION

There are 7 letters in the work ENGLISH

So, the letters can be arranged in 7! ways

= 7 × 6 × 5 × 4 × 3×2×1

= 5040

Next, fix E at first place, then remaining letters can be arranged in 6! ways

= 6 × 5 × 4 × 3×2×1

= 720

Hence, the no. of arrangements that do not begin with E = 5040 – 720 = 4320

Again, fix E at first place and H at last, then remaining 5 letters can be arranged in 5! ways

= 5 × 4 × 3 × 2 × 1 = 120

Hence, the no. of arrangements that begin with E and do not end with H = 720 – 120 = 600

B. COMBINATION

2 MARKS QUESTIONS

44. 2057 Q.No. 2 a

A committee is to be chosen from 12 men and 8 women and is to consist of 3 men and 2 women. How many such committee can be formed?

SOLUTION

Selection of men

3 men out of 12 can be selected in

$C(12, 3)$ ways $= \frac{12!}{9!\,3!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$

Selection of women

2 women out of 8 can be selected in

$C(8, 2)$ ways $= \frac{8!}{6!\,2!} = \frac{8 \times 7}{2 \times 1} = 28$

Total number of committees = 220 × 28 = 6160

45. 2062 Q.No. 2 a

From 10 persons, in how many ways can a committee of 4 be made when one particular person is always included? [2]

SOLUTION

Total number of players = 10

Number of players to be selected = 4

When one particular player is always included, then we have to select 4 –1= 3 players out of 10 –1 = 9

Total number of selections

$= C(9, 3) = \frac{9!}{6!\,3!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$

46. 2064 Q.No. 2 a

A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them may be relatives? [2]

SOLUTION

Total number of acquaintances = 12

Number of relatives = 8

Number of non-relatives = 12 –8 = 4

Relatives (8)	Non-relatives (4)	Selection
5	7 –5 = 2	C(8, 5) × C(4, 2)

Total number of selections

= C(8, 5) × C(4, 2)

$= \frac{8!}{3!\,5!} \times \frac{4!}{2!\,2!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 56 \times 6$

= 336

47. 2066 Q.No. 2 a

From 10 persons in how many ways can a selection of 4 be made if two particular persons are always excluded? [2]

SOLUTION

Total number of players = 10

Number of players to be selected = 4

When two particular players are excluded, then we have to select 4 players out of 10–2 = 8

Required number of selections

$= C(8, 4) = \frac{8!}{4!\,4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$

48. 2069 (Set A) Old Q.No. 2a

In an examination paper containing 10 questions, a candidate has to answer 7 questions only, in how many ways can he choose the questions? [2]

SOLUTION

Here, n = 10, r = 7

Required number of selections

$= C(10, 7) = \frac{10!}{3!\,7!} = \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$

49. 2069 (Set B) Q.No. 1a

From 10 persons, in how many ways can a selection of 4 be made if two particular persons are always excluded. [2]

➥ Please refer to 2066 Q.No. 2a

50. 2069 Old (Set B) Q.No. 1b

How many different sums of money can be made from 4 coins of different denominations? [2]

SOLUTION

Required number of different sums that can be made

= C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4)

$= \frac{4!}{3!\,1!} + \frac{4!}{2!\,2!} + \frac{4!}{1!\,3!} + \frac{4!}{0!\,4!} = 4 + 6 + 4 + 1 = 15$

51. 2070 Set C Q.No. 1 a

From 10 persons, in how many ways can a selection of 4 be made when two particular persons are always included? [2]

SOLUTION

When two particular persons are always included, then we have to select 4 – 2 = 2 persons out of 10 –2 = 8.

Required number of selections = C(8, 2)

$= \frac{8!}{6!\,2!} = \frac{8 \times 7}{2 \times 1} = 28$

52. 2070 Set D Q.No. 5 a

From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least two gentlemen? [2]

SOLUTION

The selection of the members in the committee can be made as follow

Gentlemen (6)	Ladies (4)	Selection
2	3	C(6, 2) × C(4, 3)
3	2	C(6, 3) × C(4, 2)
4	1	C(6, 4) × C(4, 1)
5	0	C(6, 5) × C(4, 0)

Required number of committees

= C(6, 2) × C(4, 3) + C(6, 3) × C(4, 2) + C(6,4) × C(4, 1) + C(6, 5) × C(4, 0)

$= \frac{6!}{4!\,2!} \times \frac{4!}{1!\,3!} + \frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!} + \frac{6!}{2!\,4!} \times \frac{4!}{3!\,1!} + \frac{6!}{1!\,5!} \times \frac{4!}{4!\,0!}$

= 60 + 120 + 60 + 6 = 246

53. 2071 Set C Q.No. 1 a

Find the number of ways in which 5 courses out of 8 can be selected when 3 courses are compulsory. **[2]**

SOLUTION

When 3 courses are compulsory, then 5 - 3 = 2 courses out of 8 - 3 = 5 can be selected in

$C(5, 2)$ ways $= \frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3!}{3! \times 2 \times 1} = 10$

54. 2071 Set D Q.No. 1 a

A man has 5 friends. In how many ways can he invite one or more of them to a dinner? **[2]**

SOLUTION

Total number of friends = 5

Required number of ways

$= C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5,4) + C(5, 5)$

$= \frac{5!}{4!\,1!} + \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} + \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!}$

$= 5 + 10 + 10 + 5 + 1 = 31$

55. 2073 Set D Q.No. 1a

From 10 persons, in how many ways can a selection of 4 be made when two particular persons are always excluded? **[2]**

➥ Please refer to 2058 Q.No. 7b

56. 2074 Set A Q.No. 1a

If there are 10 persons in a party and each two of them shakes hands with each other, how many hand shakes happen in the party? **[2]**

SOLUTION

Here, n = 10, r = 2

Total number of hand shakes = C(n,r)

$= C(10, 2) = \frac{10!}{(10-2)!\,2!} = \frac{10 \times 9 \times 8!}{8! \times 2 \times 1} = 45$

57. 2074 Supp Q.No. 1a

In an examination, a candidate has to pass in each of the four subjects. In how many ways can the candidate fail? **[2]**

SOLUTION

A candidate can fail in an examination if he fails either in 1 or 2 or 3 or 4 subjects.

∴ Total no. of ways by which he can fail

$= C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4)$

$= \frac{4!}{3!\,1!} + \frac{4!}{2!\,2!} + \frac{4!}{1!\,3!} + \frac{4!}{0!\,4!}$

$= 4 + 6 + 4 + 1 = 15$

58. 2075 Set B Q.No. 1a

Find the value of r if $^9C_{2r} = {}^9C_{3r-1}$. **[2]**

SOLUTION

Given,

$^9C_{2r} = {}^9C_{3r-1}$

We know that if $^nC_{r_1} = {}^nC_{r_2}$,

then $r_1 = r_2$ or $r_1 + r_2 = n$

Let $r_1 = r_2$.

Then,

$2r = 3r - 1$

or, $1 = 3r - 2r$

∴ $r = 1$

Again, let $r_1 + r_2 = n$

Then, $2r + (3r - 1) = 9$

or, $5r = 9 + 1$

or, $r = 2$

∴ $r = 1, 2.$

59. 2075 Set C Q.No. 1a

In an examination, an examinee has to secure A+ grade in each of the five subjects. In how many ways can the examinee fail to secure A+ grade? **[2]**

SOLUTION

The candidate fails to secure A+ grade in each of the five subject if he can not secure A+ either in 1 or 2 or 3 or 4 or 5 subjects.

Total no. of ways by which the candidate fail to secure A + grade

$= C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5)$

$= \frac{5!}{4!\,1!} + \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} + \frac{5!}{1!\,4!} + \frac{5!}{0!\,5!}$

$= 5 + 10 + 10 + 5 + 1 = 31$

4 MARKS QUESTIONS

60. 2058 Q.No. 7 b

From 10 football players in how many ways can a selection of a 4 be made (i) when one particular player is always included (ii) when two particular players are always excluded? **[4]**

SOLUTION

Total number of players = 10

Number of players to be selected = 4

When one particular player is always included, then we have to select 4 - 1 = 3 players out of 10 - 1 = 9

Total number of selections

$= C(9, 3) = \frac{9!}{6!\,3!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$

When two particular players are excluded, then we have to select 4 players out of 10 - 2 = 8

Required number of selections

$= C(8, 4) = \frac{8!}{4!\,4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$

61. 2060 Q.No. 7 b

From 6 gentlemen and 4 ladies a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady? **[4]**

SOLUTION

The selections of the members in the committee can be made as follows:

Gentlemen (6)	Ladies (4)	Selection
4	1	C(6, 4) × C(4, 1)
3	2	C(6, 3) × C(4, 2)
2	3	C(6, 2) × C(4, 3)
1	4	C(6, 1) × C(4, 4)

The required number of committees

$= C(6, 4) \times C(4, 1) + C(6, 3) \times C(4, 2) + C(6, 2) \times C(4, 3) + C(6, 1) \times C(4, 4)$

$= \frac{6!}{2!\,4!} \times \frac{4!}{3!\,1!} + \frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!} \times \frac{6!}{4!\,2!} \times \frac{4!}{1!\,3!} + \frac{6!}{5!\,1!} \times \frac{4!}{0!\,4!}$

$= 60 + 120 + 60 + 6 = 246$

62. 2063 Q.No. 7 b

A candidate is required to answer 6 out of 10 questions which are divided into two groups each containing 5 questions and he is not permitted to attempt more than 4 from any group. In how many different ways can he make up his choice? **[4]**

The selection of questions can be made as follows

1st group (5)	2nd group (5)	Selection
4	2	C(5, 4) × C(5, 2)
3	3	C(5, 3) × C(5, 3)
2	4	C(5, 2) × C(5, 4)

Total number of selections

$= C(5, 4) \times C(5, 2) + C(5, 3) \times C(5, 3) + C(5, 2) \times C(5, 4)$

$= \frac{5!}{1!\,4!} \times \frac{5!}{3!\,2!} + \frac{5!}{2!\,3!} \times \frac{5!}{2!\,3!} + \frac{5!}{3!\,2!} \times \frac{5!}{1!\,4!}$

$= 5 \times 10 + 10 \times 10 + 10 \times 5 = 200$

63. 2066 Q.No. 7 b

A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite seven guests so that 5 of them may be relatives? **[4]**

➥ Please refer to 2064 Q.No. 2a

64. 2066 C Q.No. 7 b

From 10 players in how many ways can a selection of 4 be made, when one particular player is always included, when two particular players are excluded? **[4]**

➥ Please refer to 2058 Q.No. 7b

65. 2067Q.No. 7b

A committee of five is to be constituted from six boys and five girls. In how many ways can this be done so as to include at least one boy and one girl? **[4]**

SOLUTION

The selection of the members in the committee can be made as follows

Boys (6)	Girls (5)	Selection
1	4	C(6, 1) × C(5, 4)
2	3	C(6, 2) × C(5, 3)
3	2	C(6, 3) × C(5, 2)
4	1	C(6, 4) × C(5, 1)

Required number of selections

$= C(6, 1) \times C(5, 4) + C(6, 2) \times C(5, 3) + C(6, 3) \times C(5, 2) + C(6, 4) \times C(5,1)$

$= \frac{6!}{5!\,1!} \times \frac{5!}{1!\,4!} + \frac{6!}{4!\,2!} \times \frac{5!}{2!\,3!} + \frac{6!}{3!\,3!} \times \frac{5!}{3!\,2!} + \frac{6!}{2!\,4!} \times \frac{5!}{4!\,1!}$

$= 6 \times 5 + 15 \times 10 + 20 \times 10 + 15 \times 5$

$= 30 + 150 + 200 + 75 = 455$

66. 2068 Q.No. 7b

A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them may be relatives? **[4]**

➥ Please refer to 2064 Q.No. 2a

67. 2069 (Set B) Q.No. 5a Or

From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least 2 ladies? **[4]**

SOLUTION

The selection of the members in the committee can be made as follows

Gents (6)	Ladies (4)	Selection
3	2	C(6,3) × C(4, 2)
2	3	C(6,2) × C(4,3)
1	4	C(6, 1) × C(4, 4)

The required number of committees

$= C(6, 3) \times C(4,2) + C(6, 2) \times C(4, 3) + C(6, 1) \times C(4, 4)$

$= \frac{6!}{3!\,3!} \times \frac{4!}{2!\,2!} + \frac{6!}{4!\,2!} \times \frac{4!}{1!\,3!} + \frac{6!}{5!\,1!} \times \frac{4!}{0!\,4!}$

$= 20 \times 6 + 15 \times 4 + 6 \times 1 = 120 + 60 + 6 = 186$

68. 2070 Supp. Q.No. 5 a

A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways can this be done when at least two ladies are to be included? **[4]**

➥ Please refer to 2069 (Set B) Q.No. 5a OR

69. 2071 Old Q.No. 8 b

In how many ways a committee of 8 members be selected from 8 gentlemen and 6 ladies, if the committee is to include not more than three ladies. **[4]**

SOLUTION

The selection of the members in the committee can be made as follows

Gentlemen (8)	Ladies (6)	Selection
5	3	C(8,5) × C(6, 3)
6	2	C(8, 6) × C(6, 2)
7	1	C(8, 7) × C(6, 1)
8	0	C(8, 8) × C(6, 0)

The required number of committees

$= C(8, 5) \times C(6, 3) + C(8, 6) \times C(6, 2) + C(8, 7) \times C(6, 1) + C(8, 8) \times C(6, 0)$

$= \frac{8!}{3!5!} \times \frac{6!}{3!3!} + \frac{8!}{2!6!} \times \frac{6!}{4!2!} + \frac{8!}{1!7!} \times \frac{6!}{5!1!} + \frac{8!}{0!8!} \times \frac{6!}{6!0!}$

$= 1120 + 420 + 48 + 1 = 1589$

70. 2071 Supp. Q.No. 5a

An examination paper consists of 12 questions divided into two parts A and B. Part A contains 7 questions, part B contains remaining questions. A candidate is required to attempt 8 questions selecting at least 3 from each part. In how many ways can the candidates select the questions? [4]

SOLUTION

Total number of questions = 12

Number of questions in part A = 7

∴ Number of questions in part B = 12 – 7 = 5

The selection of 8 questions selecting at least 3 from each part can be made in the following ways.

Part A (7)	Part B (5)	Selection
3	5	C(7, 3) × C(5, 5)
4	4	C(7, 4) × C(5, 4)
5	3	C(7, 5) × C(5, 3)

Required number of ways

= C(7,3) × C(5,5) + C(7,4) × C(5,4) + C(7,5) × C(5,3)

$= \frac{7!}{4!\,3!} \times \frac{5!}{0!\,5!} + \frac{7!}{3!\,4!} \times \frac{5!}{1!\,4!} + \frac{7!}{2!\,5!} \times \frac{5!}{2!\,3!}$

$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times 1 + \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{5}{1} + \frac{7 \times 6}{2 \times 1} \times \frac{5 \times 4}{2 \times 1}$

= 35 + 175 + 210 = 420

71. 2072 Set C Q.No. 5a

A committee of five persons is to be selected from 5 men and 4 ladies. In how many ways can this be done so that at least two ladies are always included? [4]

SOLUTION

The selection of the members in the committee can be made as follows

Ladies (4)	Men (5)	Selection
2	3	C(4, 2) × C(5, 3)
3	2	C(4, 3) × C(5, 2)
4	1	C(4, 4) × C(5, 1)

∴ The required number of committees

= C(4,2) × C(5,3) + C(4,3) × C(5,2) + C(4,4) × C(5,1)

$= \frac{4!}{2!\,2!} \times \frac{5!}{2!\,3!} + \frac{4!}{1!\,3!} \times \frac{5!}{3!\,2!} + \frac{4!}{0!\,4!} \times \frac{5!}{4!\,1!}$

$= \frac{4 \times 3}{2 \times 1} \times \frac{5 \times 4}{2 \times 1} + \frac{4}{1} \times \frac{5 \times 4}{2 \times 1} + 1 \times 5$

= 60 + 40 + 5 = 105

72. 2072 Set D Q.No. 5a

A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them may be relatives? [4]

➥ Please refer to 2064 Q.No. 2a

73. 2072 Set E Q.No. 5a

In a group of 10 students, 6 are boys. In how many ways can 4 students be selected for mathematical competition so as to include atmost two girls? [4]

SOLUTION

Here, total number of students = 10

Number of boys = 6

∴ Number of girls = 10 – 6 = 4

Now, the selection of the members for mathematical competition can be made as follows

Girls (4)	Boys (6)	Selection
2	2	C(4, 2) × C(6, 2)
1	3	C(4, 1) × C(6, 3)
0	4	C(4, 0) × C(6, 4)

∴ The required number of selections

= C(4,2) × C(6,2) + C(4,1) × C(6,3) + C(4,0) × C(6,4)

$= \frac{4!}{2!\,2!} \times \frac{6!}{4!\,2!} + \frac{4!}{3!\,1!} \times \frac{6!}{3!\,3!} + \frac{4!}{4!\,0!} \times \frac{6!}{2!\,4!}$

$= \frac{4 \times 3}{2 \times 1} \times \frac{6 \times 5}{2 \times 1} + \frac{4}{1} \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1} + 1 \times \frac{6 \times 5}{2 \times 1}$

= 90 + 80 + 15 = 185

74. 2072 Supp Q.No. 5a OR

Prove that C(n, r) + C(n, r– 1) = C(n + 1,r), when C(n, r) is the combination of n things taken r at a time. [4]

SOLUTION

L.H.S.= C(n,r) + C(n, r – 1)

$= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$

$= \frac{n!}{(n-r)!\,r \cdot (r-1)!} + \frac{n!}{(n-r+1) \cdot (n-r)!\,(r-1)!}$

$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$

$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$

$= \frac{(n+1) \cdot n!}{(n-r+1) \cdot (n-r)! \cdot r \cdot (r-1)!}$

$= \frac{(n+1)!}{(n+1-r)!\,r!}$ = C(n + 1, r) = R.H.S.

75. 2073 Set C Q.No. 5a

There are ten electric bulbs in the stock of a shop out of which four are defectives. In how many ways can a selection of 6 bulbs be made so that 4 of them may be good bulbs? [4]

SOLUTION

Here, number of bulbs = 10

number of defective bulbs = 4

∴ number of good bulbs = 10 – 4 = 6

The selection of bulbs can be made as follows

Good bulbs (6)	Defective bulbs (4)	Selection
4	6 – 4 = 2	C(6, 4) × C(4, 2)

∴ Required number of selections

$= C(6, 4) \times C(4, 2) = \frac{6!}{2!\,4!} \times \frac{4!}{2!\,2!}$

$= \frac{6 \times 5 \times 4!}{4! \times 2 \times 1} \times \frac{4 \times 3 \times 2!}{2! \times 2 \times 1}$ = 15 × 6 = 90

76. 2073 Supp Q.No. 5a

In how many ways can a committee of three can be formed out of 5 men and 2 women so that it always consists at least one women? [4]

SOLUTION

The selection of the members in the committees can be made as follows:

Men (5)	Women (2)	Selection
2	1	C(5, 2) × C(2, 1)
1	2	C(5, 1) × C(2, 2)

The required no. of committees

= C(5, 2) × C(2, 1) + C(5, 1) × C(2, 2)

$= \frac{5!}{3!\,2!} \times \frac{2!}{1!\,1!} + \frac{5!}{4!\,1!} \times \frac{2!}{0!\,2!}$

$= \frac{5 \times 4 \times 3!}{3! \times 2 \times 1} \times 2 \times 1 + \frac{5 \times 4!}{4!} \times 1$ = 20 + 5 = 25

77. 2074 Set B Q.No. 5a

From 3 men and 7 women a committee of 5 is to be formed. In how many ways can this be done so as to include at least one man? [4]

SOLUTION

The selection of the members in the committee can be made as

Men (3)	Women (7)	Selection
1	4	C(3,1) × C(7,4)
2	3	C(3,2) × C(7,3)
3	2	C(3,3) × C(7,2)

∴ The required no. of committees

= C(3, 1) × C(7, 4) + C(3, 2) × C(7, 3) + C(3, 3) × C(7, 2)

$= \frac{3!}{2!\,1!} \times \frac{7!}{3!\,4!} + \frac{3!}{1!\,2!} \times \frac{7!}{4!\,3!} + \frac{3!}{0!\,3!} \times \frac{7!}{5!\,2!}$

= 105 + 105 + 21 = 231

78. 2075 Set A Q.No. 5a

An examination paper consisting of 10 questions, is divided into two groups A and B. Group A contains 6 questions. In how many ways can an examinee attempt 7 questions selecting at least two questions from each group. [4]

SOLUTION

Total no. of questions = 10

No. of questions in group A = 6

No. of questions in group B = 10 – 6 = 4

The selection of questions from two groups can be made as follows:

Group A (6)	Group B (4)	Selection
5	2	$^6C_5 \times {^4C_2}$
4	3	$^6C_4 \times {^4C_3}$
3	4	$^6C_3 \times {^4C_4}$

Total no. of selections

$= {^6C_5} \times {^4C_2} + {^6C_4} \times {^4C_3} + {^6C_3} \times {^4C_4}$

$= \frac{6!}{1!\,5!} \times \frac{4!}{2!\,2!} + \frac{6!}{2!\,4!} \times \frac{4!}{1!\,3!} + \frac{6!}{3!\,3!} \times \frac{4!}{0!\,4!}$

= 36 + 60 + 20 = 116

79. 2075 Set C Q.No. 5a

If C(n, r – 1) = 36, C(n, r) = 84 and C(n, r + 1) = 126, find the value of r and n. [4]

SOLUTION

Given,

C(n, r – 1) = 36 … (i)

C(n, r) = 84 … (ii)

C(n, r + 1) = 126 … (iii)

Dividing equation (ii) by (i),

$\frac{C(n, r)}{C(n, r-1)} = \frac{84}{36}$

or, $\frac{\frac{n!}{(n-r)!\,r!}}{\frac{n!}{(n-r+1)!\,(r-1)!}} = \frac{7}{3}$

or, $\frac{(n-r+1)!\,(r-1)!}{(n-r)!\,r!} = \frac{7}{3}$

or, $\frac{(n-r+1).\,(n-r)!\,(r-1)!}{(n-r)!\,.r.\,(r-1)!} = \frac{7}{3}$

or, $\frac{n-r+1}{r} = \frac{7}{3}$

or, 3n – 3r + 3 = 7r

or, 3n – 10r = – 3 … (iv)

Again, dividing equation (iii) by (ii)

$\frac{C(n, r+1)}{C(n, r)} = \frac{126}{84}$

or, $\frac{\frac{n!}{(n-r-1)!\,(r+1)!}}{\frac{n!}{(n-r)!\,r!}} = \frac{3}{2}$

or, $\frac{(n-r)!\,r!}{(n-r-1)!\,(r+1)!} = \frac{3}{2}$

or, $\frac{(n-r)\,.\,(n-r-1)!\,r!}{(n-r-1)!\,(r+1)\,.\,r!} = \frac{3}{2}$

or, $\frac{n-r}{r+1} = \frac{3}{2}$

or, 2n – 2r = 3r + 3

or, 2n – 5r = 3 …(v)

Multiplying (v) by 2 and subtracting from (iv)

3n – 10r = – 3

4n – 10r = 6
– + –

–n = – 9

∴ n = 9

Putting the value of n in equation (v), we get,

2 × 9 – 5r = 3

or, 18 – 3 = 5r

or, r = 3

∴ r = 3, n = 9

UNIT 2

BINOMIAL THEOREM

A. BINOMIAL THEOREM

2 MARKS QUESTIONS

1. 2057 Q.No. 1 b

Write the middle terms in the expansion of $(a + x)^n$ when n is odd. [2]

SOLUTION

When n is odd, the number of terms after expansion of $(a + x)^n$ is n + 1 which is even. So there will be two middle terms namely

$t_{\frac{n+1}{2}}$ and $t_{\frac{n+1}{2}+1}$

$$t_{\frac{n+1}{2}} = t_{\frac{n-1}{2}+1}$$

$$= C\left(n, \frac{n-1}{2}\right) a^{n-\left(\frac{n-1}{2}\right)} x^{\left(\frac{n-1}{2}\right)}$$

$$= C\left(n, \frac{n-1}{2}\right) a^{\frac{n+1}{2}} x^{\frac{n-1}{2}}$$

& $$t_{\frac{n+1}{2}+1} = C\left(n, \frac{n+1}{2}\right) a^{n-\frac{n+1}{2}} x^{\frac{n+1}{2}}$$

$$= C\left(n, \frac{n+1}{2}\right) a^{\frac{n-1}{2}} x^{\frac{n+1}{2}}$$

2. 2058 Q.No. 1 b

Find the seventh term of $(2x + y)^{12}$. [2]

SOLUTION

Seventh term $= t_7 = t_{6+1} = C(12, 6)\,(2x)^{12-6}\,y^6$

$$= \frac{12!}{6!\,6!}\,2^6 x^6 y^6 = 59136\,x^6 y^6$$

3. 2060 Q.No. 1 b

Find the coefficient of x^5 in $\left(x + \frac{1}{2x}\right)^7$. [2]

SOLUTION

The general term in the expansion of $\left(x + \frac{1}{2x}\right)^7$ is

$$t_{r+1} = C(7, r)\,x^{7-r}\left(\frac{1}{2x}\right)^r$$

$$= C(7, r)\,\frac{1}{2^r}\,.\,x^{7-r}\,.\,\frac{1}{x^r} = \frac{C(7, r)}{2^r}\,.\,x^{7-2r}$$

For the coefficient of x^5, we have

$5 = 7 - 2r$

$\therefore$ r = 1

$\therefore$ Coefficient of $x^5 = \frac{C(7, 1)}{2^1} = \frac{7!}{6!\,1!} \times \frac{1}{2} = \frac{7}{2}$

4. 2061 Q.No. 1 b

Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$. [2]

SOLUTION

Let t_{r+1} be the term independent of x.

Then,

$$t_{r+1} = C(12, r)\,(x^2)^{12-r}\left(\frac{1}{x}\right)^r = C(12, r)\,x^{24-3r}$$

For the term independent of x, we have

$24 - 3r = 0$

$\therefore$ r = 8

$\therefore$ t_9 is the term independent of x.

And, $t_9 = t_{8+1} = C(12, 8)$

$$= \frac{12!}{4!\,8!} = \frac{12 \times 11 \times 10 \times 9 \times 8!}{8! \times 4 \times 3 \times 2 \times 1} = 495$$

5. 2062 Q.No. 1 b

If $C_0, C_1, C_2, ..., C_n$ are the binomial coefficients in the expansion of $(1 + x)^n$, show that:

$C_0 + C_2 + C_4 + ... = 2^{n-1}$

SOLUTION

We have,

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_n x^n$ ...(i)

Putting $x = -1$ in (i), we get

$(1 - 1)^n = C_0 - C_1 + C_2 - C_3 + C_4 - C_5 + \ldots$

or, $0 = (C_0 + C_2 + C_4 + \ldots) - (C_1 + C_3 + C_5 + \ldots)$

$\therefore$ $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots$

Again putting x = 1 in (i), we get

$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + \ldots = (1 + 1)^n = 2^n$

$\therefore$ $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = \frac{2^n}{2} = 2^{n-1}$

6. 2063 Q.No. 1 b

Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$ [2]

SOLUTION

Please refer to 2061 Q.No. 1b

7. 2064 Q.No. 1 b

Find the middle term in the expansion of $\left(x + \frac{1}{x}\right)^{18}$. [2]

SOLUTION

Here, the numbers of terms in the expansion of $\left(x + \frac{1}{x}\right)^{18}$ is 18 + 1 = 19 which is odd. So there is only one middle term. The middle term is given by $t_{\frac{18}{2}+1}$

$$= t_9 + 1 = C(18, 9)\,x^{18-9}\left(\frac{1}{x}\right)^9 = \frac{18!}{9!\,9!}$$

8. 2065 Q.No 1 b

Find the term free from x in the expansion of $\left(\frac{3x^2}{2} + \frac{1}{3x}\right)^9$ [2]

SOLUTION

Let t_{r+1} be the general term.

Then,

$$t_{r+1} = C(9, r)\left(\frac{3x^2}{2}\right)^{9-r}\left(\frac{1}{3x}\right)^r$$

$$= C(9, r)\,\frac{3^{9-2r}}{2^{9-r}}\,x^{18-3r}$$

For the term free from x, we have

$18 - 3r = 0$

$\therefore$ r = 6

$\therefore$ t_7 is the term free from x.

and $t_7 = t_{6+1} = C(9, 6)\,\frac{3^{9-12}}{2^{9-6}} = \frac{9!}{3!\,6!} \cdot \frac{1}{6^3} = \frac{7}{18}$

9. 2066 C Q.No. 1 b

Find the middle term in the expansion of $\left(x + \frac{1}{x}\right)^{18}$ [2]

➥ Please refer to 2064 Q.No. 1b

10. 2067 Q.No. 1b

Find the middle term in the expansion of $\left(x + \frac{1}{x}\right)^{18}$. [2]

➥ Please refer to 2064 Q.No. 1b

11. 2068 Q.No. 1b

Find the coefficient of x^5 in the expansion of $\left(x + \frac{1}{2x}\right)^7$ [2]

➥ Please refer to 2060 Q.No. 1b

12. 2069 (Set A) Q.No. 1b

Which term is free from x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{15}$? [2]

SOLUTION

Let t_{r+1} be the term free from x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{15}$

Then,

$$t_{r+1} = C(15, r)\,(x^2)^{15-r}\left(\frac{1}{x}\right)^r = C(15, r)\,x^{30-2r}\,\frac{1}{x^r}$$

$$= C(15, r)\,x^{30-3r}$$

For the term free from x, we have

$30 - 3r = 0$

$\therefore$ r = 10

$\therefore$ t_{11} is the term free from x.

And, $t_{11} = t_{10+1} = C(15, 10) = \frac{15!}{5!\,10!} = 3003$

13. 2069 (Set A) Old Q.No. 1b

Find the coefficient of x^5 in the expansion of $\left(x + \frac{1}{2x}\right)^7$. [2]

➥ Please refer to 2060 Q.No. 1b

14. 2070 (Old) Q.No. 2 a

Find the term independent of x in the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$ [2]

➥ Please refer to 2061 Q.No. 1b

15. 2071 Set C Q.No. 1 b

Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{3x^2}\right)^{12}$. [2]

SOLUTION

Let t_{r+1} be the term independent of x.

Them $t_{r+1} = C(12, r)\,(x^2)^{12-r}\left(\frac{-1}{3x^2}\right)^r$

$$= (-1)^r\,C(12, r)\,x^{24-2r}\,.\,\frac{1}{3^r}\,.\,\frac{1}{x^{2r}}$$

$$= (-1)^r\,\frac{C(12, r)}{3^r}\,.\,x^{24-4r}$$

For the term independent of x, we have

$24 - 4r = 0$, r = 6

$\therefore$ t_7 is the term independent of x

and $t_7 = (-1)^6\,\frac{C(12,6)}{3^6} = \frac{12!}{6!\,6!} \times \frac{1}{3^6} = \frac{308}{243}$

16. 2071 Set D Q.No. 1 b

Find the coefficient of x in the expansion of $\left(x^2 + \frac{a^2}{x}\right)^5$. [2]

SOLUTION

We have the general term of $(a + x)^n$ is

$$t_{r+1} = C(n, r)\,a^{n-r}\,x^r$$

$\therefore$ The general term of $\left(x^2 + \frac{a^2}{x}\right)^5$ is

$$t_{r+1} = C(5, r)\,(x^2)^{5-r}\left(\frac{a^2}{x}\right)^r$$

$$= C(5, r)\,x^{10-2r}\,\frac{a^{2r}}{x^r} = C(5, r)\,a^{2r}\,x^{10-3r}$$

Suppose the coefficient of x occurs in $(r + 1)^{th}$ term. Then,

$10 - 3r = 1$

$r = 3$

∴ t_4 will contain the coefficient of x

Coefficient of $x = C(5, 3)\,a^6 = \frac{5!}{3!\,2!}a^6 = 10a^6$

17. 2071 Old Q.No. 1 b

Find middle term or terms in the expansion of $\left(ax + \frac{1}{ax}\right)^{16}$. [2]

SOLUTION

Here, the number of terms in the expansion of $\left(ax + \frac{1}{ax}\right)^{16}$ is 16 + 1 = 17 which is odd. So there is only one middle term.

Middle term $= t_{\frac{16}{2}+1} = t_{8+1}$

$$= C(16, 8)\,(ax)^{16-8}\left(\frac{1}{ax}\right)^8 = \frac{16!}{8!\,8!}$$

18. 2072 Set C Q.No. 1b

Find the coefficient of the term containing x^2 in the expansion of $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$. [2]

➥ Please refer to Model Set I, Q.No. 9

19. 2072 Supp Q.No. 1b

Find the middle term in the expansion of $\left(x + \frac{1}{2x}\right)^{18}$. [2]

SOLUTION

Here, the number of terms in the expansion of $\left(x + \frac{1}{2x}\right)^{18}$ is 18 + 1 = 19. So, there is only one middle term. It is given by $t_{\frac{18}{2}+1}$ i.e. t_{10}.

We have,

$t_{r+1} = C(n,r)\,a^{n-r}\,x^r$

$$\therefore\ t_{10} = t_{9+1} = C(18,9)\,x^{18-9}\left(\frac{1}{2x}\right)^9 = \left(\frac{18!}{9!\,9!}\right)\frac{1}{2^9} = \frac{12,155}{128}$$

20. 2073 Set C Q.No. 1b

Find middle term in the expansion of $\left(x - \frac{1}{3x^2}\right)^{12}$. [2]

SOLUTION

Here, the number of terms in the expansion of $\left(x - \frac{1}{3x^2}\right)^{12}$ is 12 + 1 = 13 which is odd. So, there is only one middle term. The middle term is given by $t_{\frac{12}{2}+1}$ i.e. t_7.

We have,

$t_{r+1} = C(n, r)\,a^{n-r}\,x^r$

$$\therefore\ t_7 = t_{6+1} = C(12, 6)\,x^{12-6}\left(-\frac{1}{3x^2}\right)^6$$

$$= \frac{12!}{6!\,6!}x^6\left(-\frac{1}{3}\right)^6\cdot\frac{1}{x^{12}} = \frac{308}{243\,x^6}$$

21. 2074 Set B Q.No. 1b

Find the coefficient of x^6 in the expansion of $\left(x^3 + \frac{1}{x}\right)^{10}$. [2]

SOLUTION

Let t_{r+1} be the general term in the expansion of $\left(x^3 + \frac{1}{x}\right)^{10}$.

Then, $t_{r+1} = C(n, r)\,a^{n-r}.x^r$

$$\therefore\ t_{r+1} = C(10, r)\,(x^3)^{10-r}\left(\frac{1}{x}\right)^r$$

$= C(10, r)\,x^{30-3r}.\,x^{-r}$

$= C(10, r)\,x^{30-4r}$

For the coefficient of x^6, we have

$6 = 30 - 4r$

or, $4r = 30 - 6$

or, $4r = 24$

∴ $r = 6$

t_7 contains the coefficient of x^6.

And, the coefficient of x^6

$$= C(10, 6) = \frac{10!}{(10-6)!\,6!} = \frac{10\times9\times8\times7\times6!}{4\times3\times2\times1\times6!} = 210$$

22. 2075 Set A Q.No. 1b

Find the coefficient of x in the expansion of $\left(x^2 + \frac{a^2}{x}\right)^5$. [2]

➥ Please refer to 2071 Set D Q. No. 1b

4 MARKS QUESTIONS

23. 2058 Q.No. 8 b

Find the middle term in the expansion of $(1 + x)^{2n}$, where n is a positive integer. [4]

SOLUTION

Here the number of terms in the expansion of $(1 + x)^{2n}$ is 2n + 1 which is odd. So, there is only one middle term. It is given by $t_{\frac{2n}{2}+1}$

$$= t_{n+1} = C(2n, n)\,(1)^{2n-n}\,x^n = \frac{2n!}{n!\,n!}x^n$$

$$= \frac{2n\,(2n-1)\,(2n-2)\ldots6\cdot5\cdot4\cdot3\cdot2\cdot1}{n!\,n!}x^n$$

$$= \frac{\{2n\,(2n-2)\ldots6\cdot4\cdot2\}\{(2n-1)\ldots5\cdot3\cdot1\}\,x^n}{n!\,n!}$$

$$= \frac{2^n\,\{n\cdot(n-1)\ldots3\cdot2\cdot1\}\{(2n-1)\ldots5\cdot3\cdot1\}\,x^n}{n!\,n!}$$

$$= \frac{n!\,\{(2n-1)\ldots5\cdot3\cdot1\}\,2^n\,x^n}{n!\,n!}$$

$$= \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}(2x)^n$$

24. 2059 Q.No. 8 b

If $C_0, C_1, C_2, \ldots, C_n$ are the binomial coefficients in the expansion of $(1 + x)^n$ then prove that:

$C_0C_n + C_1C_{n-1} + \ldots + C_nC_0 = \frac{2n!}{n!\,n!}$. [4]

SOLUTION

Here,

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_{n-1}x^{n-1} + C_nx^n$ …(i)

Also,

$(1 + x)^n = C_nx^n + C_{n-1}x^{n-1} + \ldots + C_1x + C_0$ … (ii)

Multiplying (i) and (ii),

$(1 + x)^{2n} = (C_0C_n + C_1C_{n-1} + \ldots C_nC_0)\,x^n + \ldots$ …(iii)

Since (iii) is an identity, the coefficient of any power of x of the LHS should be equal to the coefficient of the same power of x of the R.H.S.

The coefficient of x^n in the L.H.S of (iii)

$$= C(2n, n) = \frac{2n!}{n!\,n!}$$

Again, the coefficient of x^n in the R.H.S of (iii)

$C_0C_n + C_1C_{n-1} + \ldots C_nC_0$

$$\therefore\ C_0C_n + C_1C_{n-1} + \ldots + C_nC_0 = \frac{2n!}{n!\,n!}$$

25. 2061 Q.No. 8 b

If the three consecutive coefficients in the expansion of $(1 + x)^n$ be 165, 330, 462, find n. [4]

SOLUTION

Let C(n, r − 1), C(n, r) and C(n, r + 1) be three consecutive coefficients in the expansion of $(1 + x)^n$. By given,

$C(n, r - 1) = 165$ …(i)

$C(n, r) = 330$ …(ii)

$C(n, r + 1) = 462$ … (iii)

Dividing (ii) by (i)

$$\frac{C(n,r)}{C(n,r-1)} = \frac{330}{165}$$

$$\text{or, } \frac{\frac{n!}{(n-r)!\,r!}}{\frac{n!}{(n-r+1)!\,(r-1)!}} = 2$$

$$\text{or, } \frac{(n-r+1)!\,(r-1)!}{(n-r)!\,r!} = 2$$

$$\text{or, } \frac{(n-r+1)\,(n-r)!\,(r-1)!}{(n-r)!\,r\,(r-1)!} = 2$$

$$\text{or, } \frac{n-r+1}{r} = 2$$

or, $n - r + 1 = 2r$

or $3r = n + 1$ … (iv)

Again dividing (iii) by (ii)

$$\frac{C(n,r+1)}{C(n,r)} = \frac{462}{330}$$

$$\text{or, } \frac{\frac{n!}{(n-r-1)!\,(r+1)!}}{\frac{n!}{(n-r)!\,r!}} = \frac{7}{5}$$

$$\text{or, } \frac{(n-r)!\,r!}{(n-r-1)!\,(r+1)!} = \frac{7}{5}$$

$$\text{or, } \frac{(n-r)\,(n-r-1)!\,r!}{(n-r-1)!\,(r+1)\cdot r!} = \frac{7}{5}$$

$$\text{or, } \frac{n-r}{r+1} = \frac{7}{5}$$

or, $5n - 5r = 7r + 7$

or, $5n - 12r = 7$

or, $5n - 4\,(3r) = 7$

or, $5n - 4\,(n + 1) = 7$ [using (iv)]

∴ $n = 11$

26. 2066 Q.No. 8 b

If three consecutive coefficients in the expansion of $(1 + x)^n$ be 165, 330 and 462, find n. [4]

➥ Please refer to 2061 Q.No. 8b

27. 2069 Old (Set B) Q.No. 8b

If: $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$. Prove that:

$C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{(2n)!}{(n!)^2}$. [4]

SOLUTION

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ …(i)

Also,

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \ldots + C_n$ …(ii)

Multiplying (i) and (ii),

$(1 + x)^{2n} = (C_0^2 + C_1^2 + C_2^2 + \ldots C_n^2)\,x^n + \ldots$ … (iii)

Since (iii) is an identity, the coefficient of any power of x of the LHS should be equal to the coefficient of the same power of x of the RHS.

The coefficient of x^n in the LHS of (iii)

$$= C(2n, n) = \frac{2n!}{n!\,n!} = \frac{2n!}{(n!)^2}$$

And, the coefficient of x^n in the RHS of (iii)

$C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2$

$$\therefore\ C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = \frac{2n!}{(n!)^2}$$

28. 2070 Supp. Q.No. 5 b

If the coefficient of x in the expansion of $\left(x^2 + \frac{k}{x}\right)^5$ is 270, find k. [4]

SOLUTION

The general term in the expansion of $\left(x^2 + \frac{k}{x}\right)^5$ is

$t_{r+1} = C(5,r)(x^2)^{5-r}\left(\frac{k}{x}\right)^r = C(5,r)\,k^r x^{10-3r}$

For the coefficient of x, we must have

$10 - 3r = 1$

$\therefore r = 3$

Coefficient of $x = C(5,3)\,k^3$

By question, coefficient of $x = 270$

$\therefore C(5,3)\,k^3 = 270$

or, $k^3 \frac{5!}{2!\,3!} = 270$

or, $k^3 = 27$

$\therefore k = 3$

29. 2071 Old Q.No. 7 b

Define the general term of the binomial expansion of $(x + a)^n$. In the expansion of $(1 + x)^n$ prove that the sum of the coefficients of even terms is equal to the sum of the coefficients of odd terms and each is equal to 2^{n-1}. **[4]**

SOLUTION

General term: The $(r + 1)^{th}$ term in the expansion of $(a + x)^n$ is called the general term. It is denoted by t_{r+1}. it is given by

$t_{r+1} = C(n, r)\, a^{n-r} x^r$

Next part

Please refer to 2062 Q.No. 1b

30. 2071 Supp. Q.No. 5b

If the three successive coefficient in the expansion of $(1+x)^n$ are 28, 56 and 70, find n. **[4]**

SOLUTION

If t_{r+1}, t_{r+2} and t_{r+3} be three consecutive terms in the expansion of $(1 + x)^n$, then their respective coefficients are $C(n,r)$, $C(n,r+1)$, $c(n, r+2)$ respectively. By given,

$C(n,r) = 28$ ...(i)

$C(n,r+1) = 56$ ...(ii)

$C(n,r+2) = 70$ ...(iii)

Dividing (ii) by (i), we have

$\frac{C(n, r+1)}{C(n,r)} = \frac{56}{28}$

or, $\frac{\frac{n!}{(n-r-1)!\,(r+1)!}}{\frac{n!}{(n-r)!\,r!}} = 2$

or, $\frac{(n-r)!\,r!}{(n-r-1)!\,(r+1)!} = 2$

or, $\frac{(n-r)\cdot(n-r-1)!\,r!}{(n-r-1)!\cdot(r+1)\cdot r!} = 2$

or, $\frac{n-r}{r+1} = 2$

or, $n - r = 2r + 2$

or, $n - 3r = 2$ ...(iv)

Again, dividing (iii) by (ii), we get

$\frac{C(n, r+2)}{C(n,r+1)} = \frac{70}{56}$

or, $\frac{\frac{n!}{(n-r-2)!\,(r+2)!}}{\frac{n!}{(n-r-1)!\,(r+1)!}} = \frac{5}{4}$

or, $\frac{(n-r-1)!\,(r+1)!}{(n-r-2)!\,(r+2)!} = \frac{5}{4}$

or, $\frac{(n-r-1)\cdot(n-r-2)!\,(r+1)!}{(n-r-2)!\,(r+2)\cdot(r+1)!} = \frac{5}{4}$

or, $\frac{n-r-1}{r+2} = \frac{5}{4}$

or, $4n - 4r - 4 = 5r + 10$

or, $4n - 9r = 14$ ...(v)

Multiplying (iv) by 3 and subtracting from (v) we get

$4n - 9r = 14$

$3n - 9r = 6$

$-\quad+\quad-$

$n = 8$

$\therefore n = 8$

31. 2074 Set A Q.No. 5b

Show that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1\cdot3\cdot5\cdots(2n-1)}{n!}2^n x^n$. **[4]**

SOLUTION

Here, the number of terms in the expansion of $(1 + x)^{2n}$ is $2n + 1$, which is odd. So, there is only one middle term, which is given by

$t_{\frac{2n}{2}+1}$ i.e. $= t_{n+1}$.

Middle term $= t_{n+1} = C(2n,n)x^n$

$= \frac{2n!}{(2n-n)!\,n!}.x^n$

$= \frac{\{1\cdot2\cdot3\cdot4\cdots(2n-2)\,(2n-1).\,2n\}.x^n}{n!\,n!}$

$= \frac{\{1\cdot3\cdot5\cdots(2n-1)\}\{2\cdot4\cdot6\cdots(2n-2).\,2n\}x^n}{n!\,n!}$

$= \frac{\{1\cdot3\cdot5\cdots(2n-1)\}\,2^n\,\{1\cdot2\cdot3\cdots(n-1).\,2n\}x^n}{n!\,n!}$

$= \frac{\{1\cdot3\cdot5\cdots(2n-1)\}.\,2^n\cdot n!\cdot x^n}{n!\,n!}$

$= \frac{1\cdot3\cdot5\cdots(2n-1)}{n!}2^n x^n$

32. 2075 Set B Q.No. 5b

If $C_0, C_1, C_2 \ldots, C_n$ are binomial coefficients in the expansion of $(1 + x)^n$, prove that $C_0 + 4C_1 + 7C_2 + 10C_3 + \ldots + (3n + 1)C_n = (3n + 2)2^{n-1}$. **[4]**

SOLUTION

L.H.S

$= C_0 + 4C_1 + 7C_2 + 10C_3 + \ldots + (3n + 1)\,C_n$

$= C_0 + (C_1 + 3C_1) + (C_2 + 6C_2) + (C_3 + 9C_3) + \ldots + (3n\,C_n + C_n)$

$= (C_0 + C_1 + C_2 + C_3 + \ldots + C_n) + (3C_1 + 6C_2 + 9C_3 + \ldots + 3nC_n)$

$= 2^n + 3(C_1 + 2C_2 + 3C_3 + \ldots + nC_n)$

$[\because C_0 + C_1 + C_2 + \ldots + C_n = 2^n]$

$= 2^n + 3\left\{\frac{n!}{(n-1)!\,1!} + 2.\frac{n!}{(n-2)!\,2!} + 3.\frac{n!}{(n-3)!\,3!} + \ldots + n.\frac{n!}{(n-n)!\,n!}\right\}$

$= 2^n + 3\left\{n + 2n.\frac{(n-1)}{2!} + \frac{3n(n-1)(n-2)}{3!} + \ldots + n\right\}$

$= 2^n + 3n\left\{1 + \frac{n-1}{1!} + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right\}$

$= 2^n + 3n({}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1})$

$= 2^n + 3n\,(1 + 1)^{n-1}$

$= 2^n + 3n.2^{n-1}$

$= 2.2^{n-1} + 3n.2^{n-1}$

$= (3n + 2)\,2^{n-1} =$ R.H.S.

6 MARKS QUESTIONS

33. 2069 (Set B) Q.No. 9

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots = C_nx^n$, prove that: $C_0C_n + C_1C_{n-1} + \ldots + C_nC_0 = \frac{2n!}{n!n!}$ **[6]**

➥ Please refer to 2059 Q.No. 8b

34. 2070 Set C Q.No. 9

Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is $\frac{1.3.5 \ldots\ldots (2n-1)}{n!}(-2)^n$ **[6]**

SOLUTION

Here, the number of terms in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is $2n + 1$ which is odd. So there is only one middle term. The middle term is given by,

$t_{\frac{2n}{2}+1}$

$= t_{n+1} = C(2n, n)\,(x)^{2n-n}\left(\frac{-1}{x}\right)^n$

$= (-1)^n\,C(2n, n) = (-1)^n\frac{2n!}{n!\,n!}$

$= (-1)^n\frac{2n\cdot(2n-1)\cdot(2n-2)\ldots\ldots6\cdot5\cdot4\cdot3\cdot2\cdot1}{n!\,n!}$

$= \frac{(-1)^n\{(2n\cdot(2n-2)\ldots6\cdot4\cdot2\}\{(2n-1)\ldots5\cdot3\cdot1)\}}{n!\,n!}$

$= \frac{(-1)^n\,2^n\,\{n\cdot(n-1)\ldots3\cdot2\cdot1\}\{(2n-1)\ldots5\cdot3\cdot1\}}{n!n!}$

$= \frac{(-1)^n\,2^n\cdot n!\,\{(2n-1)\ldots5\cdot3\cdot1\}}{n!\,n!}$

$= \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}(-1)^n.\,2^n$

$= \frac{1\cdot3\cdot5\ldots(2n-1)}{n!}(-2)^n$

35. 2070 Set D Q.No. 9

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that: $C_0\,C_n + C_1\,C_{n-1} + \ldots + C_n\,C_0 = \frac{2n!}{(n!)^2}$ **[6]**

➥ Please refer to 2059 Q.No. 8b

36. 2072 Set D Q.No. 9

State Binomial theorem. In the expansion of $(1 + x)^n$ prove that the sum of the coefficients of the odd terms is equal to the sum of coefficients of the even terms and each equals to 2^{n-1}. **[6]**

SOLUTION

Bionomical Theorem

For any positive integer n,

$(a + x)^n = C(n,0)a^n + C(n,1)\,a^{n-1}x + C(n, 2)\,a^{n-2}x^2 + \ldots C(n,r)\,a^{n-r}x^r + \ldots + C(n,n)\,x^n$

Next part: Please refer to 2062 Q.No. 1b

37. 2072 Set E Q.No. 9

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that: $C_0C_n + C_1C_{n-1} + C_2C_{n-2} + \ldots + C_nC_0 = \frac{2n!}{n!n!}$ **[6]**

➥ Please refer to 2059 Q.No 8b

38. 2073 Set D Q.No. 9

If $(1 + x)^n = C_0 + C_1x + C_2x^2 \ldots + C_nx^n$, prove that: $C_0C_n + C_1C_{n-1} + \ldots + C_nC_0 = \frac{2n!}{n!n!}$ **[6]**

➥ Please refer to 2059 Q.No. 8b

39. 2073 Supp Q.No. 9

Prove that $C_1 - 2C_2 + 3C_3 - 4C_4 + \ldots + n(-1)^{n-1}C_n = 0$, where $C_0, C_1, \ldots\ldots, C_n$ are the binomial coefficients. **[6]**

SOLUTION

L.H.S $= C_1 - 2C_2 + 3C_3 - \ldots\ldots + (-1)^{n-1}.\,nC_n$

$= \frac{n!}{(n-1)!\,1!} - 2.\frac{n!}{(n-2)!\,2!} + 3.\frac{n!}{(n-3)!\,3!} - \ldots + (-1)^{n-1}.n\frac{n!}{(n-n)!\,n!}$

$= \frac{n.(n-1)!}{(n-1)!} - 2.\frac{n(n-1).(n-2)!}{(n-2)!.2.1!} + 3.\frac{n(n-1)(n-2)(n-3)!}{(n-3)!.\,3.2!} - \ldots + (-1)^{n-1}.n$

$= n - \frac{n(n-1)}{1!} + \frac{n(n-1)(n-2)}{2!} - \ldots + (-1)^{n-1}n$

$= n\left\{1 - \frac{(n-1)}{1!} + \frac{(n-1)(n-2)}{2!} - \ldots + (-1)^{n-1}\right\}$

$= n\{C(n-1,0) - C(n-1,1) + C(n-1,2) - \ldots + (-1)^{n-1}C(n-1, n-1)\}$

$= n(1-1)^{n-1}$

$[\because$ n is replaced by $n - 1$ & $x = -1$ in $(1 + x)^n)]$

$= n \times 0 = 0 =$ R.H.S proved.

40. 2074 Supp Q.No. 9

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, prove that $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \ldots + \frac{nC_n}{C_{n-1}} = \frac{n(n+1)}{2}$ **[6]**

SOLUTION

L.H.S.

$= \frac{C_1}{C_0} + \frac{2.C_2}{C_1} + \frac{3.C_3}{C_2} + \ldots + \frac{n.C_n}{C_{n-1}}$

$$= \frac{\frac{n!}{(n-1)!\,1!}}{1} + 2\cdot\frac{\frac{n!}{(n-2)!\,2!}}{\frac{n!}{(n-1)!\,1!}} + 3\cdot\frac{\frac{n!}{(n-3)!\,3!}}{\frac{n!}{(n-2)!\,2!}} + \ldots$$

$$+ n\cdot\frac{\frac{n!}{(n-n)!\,n!}}{\frac{n!}{(n-n+1)!\,(n-1)!}}$$

$$= \frac{n!}{(n-1)!} + \frac{(n-1)!}{(n-2)!} + \frac{(n-2)!}{(n-3)!} + \ldots + \frac{n.(n-1)!}{n!}$$

$$= \frac{n.(n-1)!}{(n-1)!} + \frac{(n-1)(n-2)!}{(n-2)!} + \frac{(n-2)(n-3)!}{(n-3)!} + \ldots + \frac{n!}{n!}$$

$= n + (n-1) + (n-2) + \ldots + 1$

= sum of 1st n natural numbers

$= \frac{n(n+1)}{2}$ = R.H.S.

B. EXPONENTIAL AND LOGARITHMIC SERIES

2 MARKS QUESTIONS

41. 2059 Q.No. 3 a

Prove that: $\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$ [2]

SOLUTION

We have

$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \ldots\ldots$

Putting $x = 1$, we have

$\log_e(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

$\therefore \log_e 2 = \left(1-\frac{1}{2}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{5}-\frac{1}{6}\right) + \ldots$

$= \frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \frac{1}{5\cdot6} + \ldots$

42. 2066 Q.No. 1 b

Prove that $\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \ldots$ [2]

➥ Please refer to 2059 Q.No. 3a

43. 2069 (Set B) Q.No. 1b

Prove that $\log_e 2 = \frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \frac{1}{5\cdot6} + \ldots$ [2]

➥ Please refer to 2059 Q.No. 3a

44. 2069 Old (Set B) Q.No. 3a

Find the value of $\frac{1}{2}(e + e^{-1})$. [2]

SOLUTION

We have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

Putting $x = 1$ and -1, we have

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$

Adding, we get

$e + \frac{1}{e} = 2 + \frac{2}{2!} + \frac{2}{4!} + \ldots$

$= 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right)$

$\therefore \frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$

45. 2070 Set C Q.No. 1 b

Show that $\frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$ [2]

SOLUTION

We have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

Putting $x = 1$ and -1, we have

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$

Adding, we get

$e + \frac{1}{e} = 2 + \frac{2}{2!} + \frac{2}{4!} + \ldots = 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right)$

$\therefore \frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$

46. 2070 Set D Q.No. 1 b

Show that $\log_e 2 = \frac{1}{1\cdot2} + \frac{1}{3\cdot4} + \frac{1}{5\cdot6} + \ldots$ [2]

➥ Please refer to 2059 Q.No. 3a

47. 2070 Supp. Q.No. 1 b

Prove that $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \ldots = 2(1 - \ln 2)$ [2]

➥ Please refer to Model Set I, Q.No. 1b

48. 2071 Supp. Q.No. 1b

If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$, show that:

$y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ [2]

SOLUTION

Given, $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$

or, $x = \log_e(1+y)$

or, $1 + y = e^x$

or, $1 + y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$\therefore y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

49. 2072 Set D Q.No. 1b

Prove that $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots = e.$ [2]

SOLUTION

L.H.S. $= \frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots$

$= \frac{1+1}{1!} + \frac{3+1}{3!} + \frac{5+1}{5!} + \ldots$

$= \frac{1}{1!} + \frac{1}{1!} + \frac{3}{3!} + \frac{1}{3!} + \frac{5}{5!} + \frac{1}{5!} + \ldots$

$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots = e =$ R.H.S.

50. 2072 Set E Q.No. 1b

Show that: $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots$ **to ∞ = e.** [2]

➥ Please refer to 2072 Set D Q.No. 1b

51. 2073 Set D Q.No. 1b

Show that $\frac{2}{1!} + \frac{4}{3!} + \frac{6}{5!} + \ldots\ldots\ldots$ **to ∞ = e** [2]

➥ Please refer to 2072 Set D Q.No. 1b

52. 2073 Supp Q.No. 1b

If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ **show that:** $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$ [2]

SOLUTION

Here, $y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots$

or, $y = \log_e(1 + x)$

or, $1 + x = e^y$

or, $1 + x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$

$\therefore x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$

53. 2074 Set A Q.No. 1b

If $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$, **show that:**

$x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$ [2]

SOLUTION

Here, $y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$

or, $-y = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots\ldots$

or, $-y = \log_e(1 - x)$

or, $1 - x = e^{-y}$

or, $1 - x = 1 - \frac{y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \ldots$

or, $-x = \frac{-y}{1!} + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \ldots$

$\therefore x = y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \ldots$

54. 2074 Supp Q.No. 1b

Prove that: $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \ldots$ [2]

SOLUTION

We have

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

Putting $x = 1$ & $x = -1$, we get

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$ (i)

& $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots$ (ii)

Subtracting (ii) from (i)

$e - e^{-1} = 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\right)$

or, $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \ldots$

55. 2075 Set B Q.No. 1b

If $y = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$,

show that $x = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots$ [2]

SOLUTION

Given,

$y = \frac{x}{1!} - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots$

or, $-y = \frac{-x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$

Adding 1 to both sides,

$1 - y = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$

or, $1 - y = e^{-x}$

or, $-x = \log_e(1 - y)$

or, $-x = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots$

$\therefore x = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \ldots$

56. 2075 Set C Q.No. 1b

Prove that: $\frac{1}{1.3} + \frac{1}{2.5} + \frac{1}{3.7} + \frac{1}{4.9} + \ldots = 2(1 - \ln 2)$. [2]

➥ Please refer to Model Set I Q.No. 1b

4 MARKS QUESTIONS

57. 2057 Q.No. 8 a

Show that: $\dfrac{\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\ldots\ldots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\ldots\ldots} = \frac{e-1}{e+1}$ [4]

SOLUTION

We have

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots$

Putting $x = 1$ and -1, we get

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots$

and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} + \ldots$

Now,

$e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \ldots$

$= 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots\right)$

$\therefore \frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \ldots$

$$= \left[\frac{1}{(1-2)!}+\frac{1}{(2-2)!}+\frac{1}{(3-2)!}+\frac{1}{(4-2)!}+\dots\right]+\left[\frac{1}{(1-1)!}+\frac{1}{(2-1)!}+\frac{1}{(3-1)!}+\frac{1}{(4-1)!}+\dots\right]$$

$$= \left[\frac{1}{(-1)!}+\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots\right]+\left[\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right]$$

$$= \left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)+\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right) \qquad [\because (-1)! = \infty]$$

$= e + e = 2e$

72. 2072 Supp Q.No. 9

Show that: $\sum_{n=1}^{\infty} \frac{n^2}{(n+1)!} = e - 1.$ [6]

➥ Please refer to Model Set II, Q.No. 9

73. 2073 Set C Q.No. 9

Prove that : $\dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots} = \dfrac{e-1}{e+1}.$ [6]

➥ Please refer to 2057 Q.No. 8a

74. 2074 Set B Q.No. 9

Sum to infinity the series: $\frac{1^2}{1!}+\frac{2^2}{2!}+\frac{3^2}{3!}+\dots$ [6]

SOLUTION

Let t_n be the n^{th} term of given series.

Then, $t_n = \frac{n^2}{n!} = \frac{n^2}{n.(n-1)!}$

$$= \frac{n}{(n-1)!} = \frac{n-1+1}{(n-1)!} = \frac{n-1}{(n-1)!} + \frac{1}{(n-1)!}$$

$$= \frac{n-1}{(n-1)(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{1}{(n-2)!} + \frac{1}{(n-1)!}$$

Putting n = 1, 2, 3, 4, ..., we get,

$t_1 = 0 + \frac{1}{0!}$ $\left[\because \frac{1}{(-1)!} = 0\right]$

$t_2 = \frac{1}{0!} + \frac{1}{1!}$

$t_3 = \frac{1}{1!} + \frac{1}{2!}$

$t_4 = \frac{1}{2!} + \frac{1}{3!}$

⋮ ⋮ ⋮

Adding all the terms, we get,

$$= \left(0+\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots\dots\right)+\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\dots\dots\right)$$

$$= \left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)+\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\dots\dots\right)$$

$= e + e = 2e$

75. 2075 Set A Q.No. 9

Sum to infinity the series: $1+\frac{3}{1!}+\frac{5}{2!}+\frac{7}{3!}+\dots$ [6]

SOLUTION

The n^{th} term of given series

$$t_n = \frac{2n-1}{(n-1)!} = \frac{2n-2+1}{(n-1)!} = \frac{2n-2}{(n-1)!} + \frac{1}{(n-1)!}$$

$$= \frac{2(n-1)}{(n-1)(n-2)!} + \frac{1}{(n-1)!}$$

$$= \frac{2}{(n-2)!} + \frac{1}{(n-1)!}$$

Putting, n = 1, 2, 3 ..., we get,

$t_1 = 0 + \frac{1}{0!}$

$t_2 = \frac{2}{0!} + \frac{1}{1!}$

$t_3 = \frac{2}{1!} + \frac{1}{2!}$

$t_4 = \frac{2}{2!} + \frac{1}{3!}$

.. ..

Adding all the terms, we get,

$$\left(\frac{2}{0!}+\frac{2}{1!}+\frac{2}{2!}+\dots\right)+\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right)$$

$$= 2\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)+\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right)$$

= 2e + e

= 3e

76. 2075 Set C Q.No. 9

Write e^x in the series form. Using it show that

$$\dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots} = \frac{1-e^{-1}}{1+e^{-1}}.$$ [6]

SOLUTION

For all values of x, the series form of e^x is

$$e^x = 1+\frac{x}{11}+\frac{x^2}{21}+\frac{x^2}{31}+\dots$$

Putting x = 1 and –1, we get

$$e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\dots$$

and

$$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}+\dots$$

Now,

$$e+e^{-1} = 2+\frac{2}{2!}+\frac{2}{4!}+\frac{2}{6!}+\dots$$

$$= 2\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots\right)$$

$$\therefore \frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots$$

Also,

$$e-e^{-1} = \frac{2}{1!}+\frac{2}{3!}+\frac{2}{5!}+\dots$$

$$= 2\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots\right)$$

$$\therefore \frac{e-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots$$

Now,

L.H.S.

$$= \dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots}$$

$$= \dfrac{\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\dots\right)-1}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\dots}$$

$$= \dfrac{\frac{e+e^{-1}}{2}-1}{\frac{e-e^{-1}}{2}}$$

$$= \frac{e+e^{-1}-2}{e-e^{-1}}$$

$$= \dfrac{e+\frac{1}{e}-2}{e-\frac{1}{e}}$$

$$= \frac{e^2-2e+1}{e^2-1}$$

$$= \frac{(e-1)^2}{(e+1)(e-1)}$$

$$= \frac{e-1}{e+1}$$

$$= \dfrac{\frac{e}{e}-\frac{1}{e}}{\frac{e}{e}+\frac{1}{e}} \qquad \left[\text{Dividing numerator and denominator by e}\right]$$

$$= \frac{1-e^{-1}}{1+e^{-1}} \text{ R.H.S}$$

UNIT 3

ELEMENTARY GROUP THEORY

2 MARKS QUESTIONS

1. 2069 (Set A) Q.No. 1c

Let S = {-1, 1} and * denote the usual operation of multiplication. Represent it by Cayley's table. Show that the multiplication is a binary operation on S. [2]

SOLUTION

Please refer to Model Set II, Q.No. 1c

2. 2069 (Set B) Q.No. 1c

If the binary operation * on Q, the set of rational numbers is defined by a*b = a + b + ab for every a, b ∈ Q show that * satisfies associative property. [2]

SOLUTION

Here, a*b = a + b + ab for every a, b ∈ Q

Let a, b, c ∈ Q.

Now,

a*(b*c) = a* (b + c + bc)= a + (b + c + bc) + a (b + c + bc)
= a + b + c + bc +ab + ca + abc

And,

(a*b)*c = (a + b + ab)*c = (a + b + ab) + c + (a + b + ab) c
= a + b + ab + c + ac + bc + abc
= a + b + c + ab + bc + ca + abc

∴ a*(b*c) = (a*b)*c for all a, b, c∈ Q.

This shows that * satisfies associative property.

3. 2070 Set C Q.No. 1 c

Let G = {0, 1, 2}. Form a composition table for G under addition modulo 3. Find the identity element of 1. [2]

SOLUTION

Here, G = {0, 1, 2}

Composition table for G under addition modulo 3

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

From above table, $1 +_3 0 = 0 +_3 1 = 1$

0 is the identity element of 1.

4. 2070 Set D Q.No. 1 c

Show that the multiplication is a binary operation on the set S = { -1, 0, 1} [2]

SOLUTION

Given, S = {-1, 0, 1}

Cayley's table

×	-1	0	1
-1	1	0	-1
0	0	0	0
1	-1	0	0

From above Cayley's table, we see that product of any two elements of S is unique belong to S. So, × is a binary operation on S.

5. 2070 Supp. Q.No. 1 c

Let $m*n = \sqrt{mn}$ for m, n ∈ Z. Verify that operation * is not a binary operation on Z.

SOLUTION

We have, $m*n = \sqrt{mn}$ for m, n ∈ Z

We have to show * is not a binary opera[tion] on Z.

For, take m = 1, n = 2. Clearly 1, 2 ∈ Z

Now, $1*2 = \sqrt{2} \notin Z$

So, * is not a binary operation on Z.

6. 2071 Set C Q.No. 1 c

Let G = {0, 1, 2}. Form a composition table [for G] under the multiplication modulo 3. Find [the] identity element of 2.

SOLUTION

Composition table for G under multipli[cation] modulo 3

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Here, $2 \times_3 1 = 1 \times_3 2 = 2$

∴ 1 is the identity element of 2.

7. 2071 Set D Q.No. 1 c

Show that the multiplication is a binary ope[ration] on the set S = {-1, 0, 1}

SOLUTION

Please refer to 2070 Set D Q.No. 1c

8. 2071 Supp. Q.No. 1c

If possible, solve 2x + 1 = 6 in Z_7

SOLUTION

Given, 2x + 1 = 6

2x + 1 = 6 in Z_7

⇒ $2 \times_7 x +_7 1 = 6$

⇒ $2 \times_7 x +_7 1 +_7 6 = 6 +_7 6$

⇒ $2 \times_7 x = 5$

⇒ $4 \times_7 (2 \times_7 x) = 4 \times_7 5$

⇒ $(4 \times_7 2) \times_7 x = 6$

⇒ $1 \times_7 x = 6$

⇒ $x = 6$

9. 2072 Set C Q.No. 1c

If a * b = 3a + 2b for a, b ∈ Z, the set of integers, show that * is a binary operation on Z. [2]

SOLUTION

If a and b are any two integers then 3a and 2b are also integers and their sum 3a + 2b is again an integer.

Thus, for all a, b ∈ Z, a*b = 3a + 2b ∈ Z uniquely.

Hence * is a binary operation on Z.

10. 2072 Set D Q.No. 1c

In a Caley's table for a finite group, why does each element occur exactly once in each row and exactly once in each column? [2]

SOLUTION

Let $a_1, a_2, \ldots, a_n$ be the distinct element of a finite group (G,*). If possible,

let $a_i * a_j$ and $a_i * a_k$ be the same in the Cayley's table, then $a_i * a_j = a_i * a_k$

By left cancellation law, we have

$a_j = a_k$ which is a contradiction as $a_j \neq a_k$.

Hence, in a Cayley's table for a finite group, each element occurs exactly once in each row and exactly once in each column.

11. 2072 Set E Q.No. 1c

Test the commutative property for the operation * defined by m * n = n, m, n ∈ Z. [2]

SOLUTION

For any m, n ∈ Z

m * n = n

n * m = m

So, m * n ≠ n * m

Hence, * is not commutative.

12. 2072 Supp Q.No. 1c

Let G = {0, 1, 2). Form a composition table for G under multiplication modulo 3. Find the inverse element of 2. [2]

SOLUTION

Here, G = {0, 1, 2}

Composition table for G under $\times_3$

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

From above Cayley's table, 1 is the identity element.

Here, $2 \times_3 2 = 1$

So, 2 is the inverse element of 2.

13. 2073 Set C Q.No. 1c

Find the inverses of the elements of G = {-1,1} under multiplications, if exist. [2]

SOLUTION

Cayley's table

×	-1	1
-1	1	-1
1	-1	1

Clearly, 1 is the identity element.

From table,

-1 × (-1) = 1

1 × 1 = 1

∴ -1 and 1 are the inverses of -1 and 1 respectively.

14. 2073 Set D Q.No. 1c

Let G = {0, 1, 2}. Form a composition table for G under addition modulo 3. Find the inverse element of 2. [2]

15. 2073 Supp Q.No. 1c

Show that the set G = {-1, 1, -i, i}, the fourth roots of unity satisfies the binary operation of multiplication. [2]

SOLUTION

Here, G = {-1, 1, -i, i}

Cayle's table

×	-1	1	-i	i
-1	1	-1	i	-i
1	-1	1	-i	i
-i	i	-i	-1	1
i	-i	i	1	-1

From above Cayley's table, we see that the product of any two elements of G is unique and belongs to G. So, × is a binary operation on G.

16. 2074 Set A Q.No. 1c

Let a∗b = a - b on Z. Show that "∗" is not an associative binary operation. [2]

SOLUTION

Here,

a∗b = a – b ∈ Z for all a, b ∈ Z.

So, ∗ is a binary operation on Z.

Again, let a, b, c ∈ Z.

Then a∗(b∗c) = a∗ (b – c) = a – (b – c) = a – b + c

And, (a∗b)∗ c = (a∗b) – c = (a – b) – c = a – b – c

∴ a∗(b∗c) ≠ (a∗b)∗c for some a, b, c ∈ Z.

i.e. ∗ is not associative

Hence, ∗ is not an associative binary operation.

17. 2074 Set B Q.No. 1c

Prepare Cayley's table for the set S = {0,1, 2} under the operation multiplication modulo 3. [2]

SOLUTION

Here, G = {0, 1, 2}

Composition table for G under $\times_3$

$\times_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

18. 2074 Supp Q.No. 1c

If the binary operation * on Z, the set of integers is defined by m * n = m + n + 1 for every m, n∈Z, show that * satisfies associative property. [2]

SOLUTION

Given, m * n = m + n + 1 for all m, n ∈ Z

Let m, n, p ∈ Z

m*(n * p) = m * (n + p + 1)
= m + (n + p + 1) + 1
= m + n + p + 2

& $(m * n) * p = (m + n + 1) * p$
$= (m + n + 1) + p + 1$
$= m + n + p + 2$

$\therefore$ $m * (n * p) = (m * n) * p$ for all $m, n, p \in Z$

$*$ satisfies the associative property on Z.

19. 2075 Set A Q.No. 1c

If a and b are the elements of a group (G, *) and if a * b = e, prove that $b = a^{-1}$. **[2]**

➥ Please refer to 2070 Supp. Q.No. 6a (ii)

20. 2075 Set B Q.No. 1c **[2]**

Solve 3x + 6 = 5, in Z_7.

SOLUTION

Here,

$3x + 6 = 5$ in Z_7

$\Rightarrow 3 \times_7 x +_7 6 = 5$

$\Rightarrow 3 \times_7 x +_7 6 +_7 1 = 5 +_7 1$

$\Rightarrow 3 \times_7 x = 6$

$\Rightarrow 5 \times_7 (3 \times_7 x) = 5 \times_7 6$

$\Rightarrow (5 \times_7 3) \times_7 x = 2$

$\Rightarrow 1 \times_7 x = 2$

$\Rightarrow x = 2$

21. 2075 Set C Q.No. 1c

Prove that multiplication on the set Z^- of all negative integers is not a binary operation on Z^-. **[2]**

SOLUTION

Let Z^- be the set of all negative integers.

Consider $-1, -2 \in Z^-$

Then, $(-1).(-2) = 2 \notin Z^-$

So, multiplication on Z^- is not a binary operation on Z^-.

4 MARKS QUESTIONS

22. 2069 (Set A) Q.No. 5b

Define group. Let G = {1, –1, i, – i} where i is an imaginary unit and * stands for the binary operation of multiplication. Show that (G, *) forms a group. **[4]**

SOLUTION

Group: Let G be a non empty set and * is an operation defined on G. Then, (G, *) is said to be a group if * satisfies the following axioms.

1. **Closure axiom:** G is closed under the operation *.
   i.e. $a*b \in G$ for all $a, b \in G$
2. **Associative axiom**
   $a * (b*c) = (a*b)*c$ for all $a, b, c, \in G$
3. **Identity axiom:** For all $a \in G$ there exists an element $e \in G$ such that
   $a*e = e*a = a$
   The element e is called the identity element.
4. **Inverse axiom:** Each element of G possesses inverse i.e. for all $a \in G$, there exists $b \in G$ such that $a*b = b*a = e$.
   The element b is called the inverse of a. We write $b = a^{-1}$.

Next part:

Here, G = {1, –1, i, – i}

a) Since the product of any two elements of G is also an element of G, so G is closed under x.

b) $1 * (-1 * i) = i * (-i) = -i$
$(1 * (-1)) * i = -1 * i = -i$

$\therefore$ $1 * (-1 * i) = (1 * (-1)) * i$

This result is true for every element of G.

$\therefore$ Associativity is satisfied.

c) Here,
$1 * 1 = 1$
$(-1) * 1 = 1 * (-1) = -1$
$i * 1 = 1 * i = i$
$-i * 1 = 1 * (-i) = -i$

$\therefore$ 1 is the identity element.

d) $1 * 1 = 1$
$(-1) * (-1) = 1$
$i * (-i) = -i^2 = -(-1) = 1$
& $(-i) * i = -i^2 = 1$

$\therefore$ 1, –1, –i and i are the inverse elements of 1, –1, i and –i respectively.

Hence (G,*) is a group.

23. 2069 (Set A) Q.No. 5b or

If a and b are the elements of a group (G, o) prove that: $(a o b)^{-1} = b^{-1} o a^{-1}$

SOLUTION

Let G be a group and $a, b \in (G, 0)$.

Now, $(a_o b)_o (b^{-1} {}_o a^{-1})$

$= ((a_o b)_o b^{-1})_o a^{-1}$ (by associativity)

$= (a_o (b_o b^{-1}))_o a^{-1}$ (by associativity)

$= (a_o e)_o a^{-1}$ (as $b_o b^{-1} = e$)

$= a_o a^{-1}$ (as e is the identity element)

$= e$

Similarly, $(b^{-1} {}_o a^{-1})_o (a_o b) = e$

$\therefore$ $(a_o b)_o (b^{-1} {}_o a^{-1}) = (b^{-1} {}_o a^{-1})_o (a_o b) = e$

Hence $b^{-1} {}_o a^{-1}$ is the inverse of $a_o b$.

$\therefore$ $(a_o b)^{-1} = b^{-1} {}_o a^{-1}$

24. 2069 (Set B) Q.No. 5b

Given the algebraic structure (G, *) G = {1, ω, ω²} where ω represents the imaginary cube root of unity and * stands for the binary operation of multiplication, show that (G, *) is a group.

SOLUTION

Please refer to Model Set II, Q.No. 5b

25. 2070 Set C Q.No. 5 b

Show that the set T = {-1, 1} forms a group under multiplication operation.

SOLUTION

Here, T = {-1, 1}

i. **Closure axiom:**
$(-1) \times (-1) = 1 \in T$
$(-1) \times 1 = -1 \in T$
$1 \times (-1) = -1 \in T$
$1 \times 1 = 1 \in T$
Hence T is closed under multiplication

ii. **Associative axiom:**
Here, $-1, 1, 1 \in T$
$-1 \times (1 \times 1) = -1 \times 1 = -1$
and $(-1 \times 1) \times 1 = -1 \times 1 = -1$
$\therefore$ $-1 \times (1 \times 1) = (-1 \times 1) \times 1$
This result is true for all elements of T.
$\therefore$ Associativity is satisfied.

iii. **Identity axiom**
Here, $(-1) \times 1 = 1 \times (-1) = -1$
$1 \times 1 = 1$
$\therefore$ 1 is the identity element.

iv. **Inverse axiom:**
$(-1) \times (-1) = 1$
and $1 \times 1 = 1$
So, –1 and 1 are the inverse elements of –1 and 1 respectively.

From (i) – (iv), we conclude that T is a group under multiplication.

26. 2070 Set C Q.No. 5 b Or

a, b, c are the elements of a group (G,o)

i. if a o b = a o c prove that b = c.

i. if b o a = c o a prove that b = c. **[4]**

SOLUTION

Sine $a \in G$, has the inverse a^{-1} such that

$a o a^{-1} = a^{-1} o a = e$

We have, $a o b = a o c$ ...(i)

Operating both sides of (i) by a^{-1} on the left, we have

$a^{-1} o (a o b) = a^{-1} o (a o c)$

or, $(a^{-1} o a) o b = (a^{-1} o a) o c$ (by associativity)

or, $e o b = e o c$ (as $a^{-1} o a = e$)

$\therefore$ $b = c$ (as e is the identity element.)

Again, we have $b o a = c o a$ ...(ii)

Operating both sides of (ii) by a^{-1} on the right, we have

$(boa) o a^{-1} = (c o a) o a^{-1}$

or, $b o (a o a^{-1}) = co (a o a^{-1})$ (by associativity)

or, $b o e = c o e$ (as $a o a^{-1} = e$)

$\therefore$ $b = c$ (as e is the identity element)

27. 2070 Set D Q.No. 5 b

Show that the set of integers Z forms a group under the operation of addition. **[4]**

SOLUTION

We know,

Z = { ......, –3, –2, –1, 0, 1, 2, 3, ...}

i. Closure axiom: Since the addition of two integers is again an integer, so Z is closed under addition.

ii. Associative axiom
Let $a, b, c \in Z$
Now, $a + (b + c) = a + b + c$
and $(a + b) + c = a + b + c$
$\therefore$ $a + (b + c) = (a + b) + c$ for all $a, b, c \in Z$
Hence, associativity is satisfied.

iii. Identity axiom: Let $a \in Z$. Also let e be the identity element of a. Then,
$a + e = a$
$\Rightarrow e = 0 \in Z$
$\therefore$ 0 is the identity element.

iv. Inverse axiom: Let $a \in Z$. Also let b be the inverse element of a.
Then,
$a + b = e$
or, $a + b = 0$ ($\because e = 0$)
$\Rightarrow b = -a$
i.e. –a is the inverse element of a.

So every element of Z possesses an inverse.

From (i) – (iv), Z is a group under addition.

28. 2070 Set D Q.No. 5 b Or

If a and b are the elements of a group (G, 0) prove that the equation $a o x = b$ has a unique solution in (G, o). **[4]**

SOLUTION

Since a belongs to group G, so $a^{-1} \in G$ such that $a_o a^{-1} = a^{-1} {}_o a = e$

Now, $a_o x = b$

Operating both sides by a^{-1} on the left

$a^{-1} {}_o (a_o x) = a^{-1} {}_o b$

$\Rightarrow (a^{-1} {}_o a)_o x = a^{-1} {}_o b$ (by associavity)

$\Rightarrow e_o x = a^{-1} {}_o b$ (as $a^{-1} {}_o a = e$)

$\Rightarrow x = a^{-1} {}_o b$ (as e is the identity element)

This is the required solution.

To show the **uniqueness of the solution,** let x_1 and x_2 be the solutions, then

$a_o x_1 = b$ and $a_o x_2 = b$

$\therefore$ $a_o x_1 = a_o x_2$

$\Rightarrow a^{-1} {}_o (a_o x_1) = a^{-1} {}_o (a_o x_2)$

$\Rightarrow (a^{-1} {}_o a)_o x_1 = (a^{-1} {}_o a)_o x_2$ (by associativity)

$\Rightarrow e_o x_1 = e_o x_2$ (as $a^{-1} {}_o a = e$)

$\Rightarrow x_1 = x_2$ (as e is the identity element)

So, the solution is unique.

29. 2070 Supp. Q.No. 6 a

If a and b are the elements of a group (G, *) such that

i. a * b = b, prove that a = e

ii. a * b = e, prove that $b = a^{-1}$ **[4]**

SOLUTION

i. Since b is in group G, $b^{-1} \in G$.
We have,
$a*b = b$
$\Rightarrow (a*b)* b^{-1} = b*b^{-1}$
$\Rightarrow a* (b*b^{-1}) = b*b^{-1}$ (by associativity)
$\Rightarrow a*e = e$ (as $b*b^{-1} = e$)
$\Rightarrow a = e$ (as e is the identity element)

ii. Since a belongs to group G, $a^{-1} \in G$.
We have,
$a*b = e$
$\Rightarrow a^{-1} * (a*b) = a^{-1} * e$
$\Rightarrow (a^{-1}*a) *b = a^{-1}*e$ (by associativity)
$\Rightarrow e*b = a^{-1} *e$ (as $a^{-1}*a = e$)
$\Rightarrow b = a^{-1}$ (as e is the identity element)

30. 2070 Supp. Q.No. 6 a OR

Let G = {0, 1, 2, 3, 4}. Construct cayley's table for G under the multiplication modulo 5. Find the inverse of each element of G. **[4]**

SOLUTION

We have, G = {0, 1, 2, 3, 4}.

Cayley's table for G under $\times_5$

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

From above table, we see that 1 is the identity element.
The inverse of 0 does not exist.
From table,
$1 \times_5 1 = 1$
$2 \times_5 3 = 1$
$3 \times_5 2 = 1$
$4 \times_5 4 = 1$
$\therefore$ 1, 3, 2 and 4 are inverse elements of 1, 2, 3 and 4 respectively.

31. 2071 Set C Q.No. 5 b

Given the algebraic structure (G, *) with $G = \{1, \omega, \omega^2\}$ where ω represents the cube root of unity and * stands for the binary operation of ordinary multiplication of complex numbers, show that (G,*) is a group. [4]

SOLUTION

Please refer to Model Set II, Q.No. 5b

32. 2071 Set C Q.No. 5 b OR

If a, b, c, are the elements of a group (G, *), prove that:
$a * b = a * c \Rightarrow b = c$ and $b * a = c * a \Rightarrow b = c$. [4]

SOLUTION

Sine $a \in G$, has the inverse a^{-1} such that $a * a^{-1} = a^{-1} * a = e$
We have, $a * b = a * c$ ...(i)
Operating both sides of (i) by a^{-1} on the left, we have
$a^{-1} * (a*b) = a^{-1} * (a * c)$
or, $(a^{-1} * a) * b = (a^{-1} * a) * c$ (by associativity)
or, $e*b = e*c$ (as $a^{-1}*a = e$)
$\therefore$ $b = c$ (as e is the identity element.)
Again, we have $b*a = c*a$ ...(ii)
Operating both sides of (ii) by a^{-1} on the right, we have
$(b*a) * a^{-1} = (c*a) * a^{-1}$
or, $b * (a*a^{-1}) = c* (a*a^{-1})$ (by associativity)
or, $b*e = c*e$ (as $a*a^{-1} = e$)
$\therefore$ $b = c$ (as e is the identity element)

33. 2071 Set D Q.No. 5 b

A binary operation * defined on the set S = {a, b, c} is presented in the following Cayley's table.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Show that: (S, *) forms a group. [4]

SOLUTION

Please refer to Model Set I, Q.No. 5b

34. 2071 Set D Q.No. 5 b OR

Let a, b, c be the elements of a group (G, *)
i. If $a* b = b$, prove that: $a = e$
ii. If $a*b = e$, prove that: $b = a^{-1}$ [4]

SOLUTION

Please refer to 2070 Supp Q.No. 6a

35. 2071 Supp. Q.No. 6a

Define group. Verify that $\{2^m : m \in Z\}$ is a group with respect to multiplication. [4]

SOLUTION

First Part: Please refer to 2069 Set A Q.No. 5b

Next Part

Let $G = \{2^m : m \in Z\}$

i. **Closure property:** Let $2^m, 2^n \in G$. Then $2^m \cdot 2^n = 2^{m+n} \in G$ for all $m, n \in Z$
Hence, G is closed under multiplication.

ii. **Associative property:** Let $2^m, 2^n, 2^p \in G$.
Then, $2^m \cdot (2^n \cdot 2^p) = 2^m \cdot 2^{n+p} = 2^{m+n+p}$
and $(2^m \cdot 2^n) \cdot 2^p = 2^{m+n} \cdot 2^p = 2^{m+n+p}$
$\therefore$ $2^m \cdot (2^n \cdot 2^p) = (2^m \cdot 2^n) \cdot 2^p$ for all $m, n, p \in Z$
So, associativity is satisfied,

iii. **Existence of identity:** Let $2^m \in G$. Then there is $0 \in Z$ such that $2^0 \in G$.
Now, $2^m \cdot 2^0 = 2^{m+0} = 2^m$
and $2^0 \cdot 2^m = 2^{0+m} = 2^m$
$\therefore$ 2^0 is the identity element in G.

iv. **Existence of inverse:** Let $2^m \in G$. Since $m \in Z$, $-m \in Z$ such that $2^{-m} \in Z$.
Now, $2^m \cdot 2^{-m} = 2^{m-m} = 2^0$ (identity element)
Hence, 2^{-m} is the inverse of 2^m.
From (i) - (iv) we conclude that G is a group under multiplication.

36. 2072 Set C Q.No. 5b

Show that the set of all vectors in space under addition is a group.

SOLUTION

Let V be the set of all vectors in space. Also, let the operation of addition of vectors be denoted by +.

(i) **Closure property:** Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be any two vectors in V.
Now, $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in V$
$\therefore$ $\vec{u} + \vec{v} \in V$ for all $\vec{u}, \vec{v} \in v$
Hence V is closed under addition.

(ii) **Associative property:** Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be any three vectors in V.
Now,

$$\begin{aligned}\vec{u} + (\vec{v} + \vec{w}) &= (u_1, u_2, u_3) + ((v_1, v_2, v_3) + (w_1, w_2, w_3))\\ &= (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3)\\ &= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3)\\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) + (w_1, w_2, w_3)\\ &= ((u_1, u_2, u_3) + (v_1, v_2, v_3)) + (w_1, w_2, w_3)\\ &= (\vec{u} + \vec{v}) + \vec{w}\end{aligned}$$

$\therefore$ $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ for all $\vec{u}, \vec{v}, \vec{w} \in V$
Hence, associativity is satisfied.

(iii) **Existence of element:** Let $\vec{u} = (u_1, u_2, u_3) \in V$. There is a vector $\vec{0} = (0, 0, 0)$ in V such that $\vec{u} + \vec{0} = (u_1, u_2, u_3) + (0, 0, 0) = (u_1, u_2, u_3) = \vec{u}$
Similarly, $\vec{0} + \vec{u} = \vec{u}$
Thus, $\vec{0} \in V$ is the identity element.

(iv) **Existence of inverse element:**
Let $\vec{u} = (u_1, u_2, u_3) \in V$. Let $\vec{v}$ be the inverse of $\vec{u}$. Then
$\vec{u} + \vec{v} = 0$
or, $(u_1, u_2, u_3) + \vec{v} = (0, 0, 0)$
$\therefore$ $\vec{v} = -(u_1, u_2, u_3) = -\vec{u} \in V$ for all $\vec{u} \in V$.
So, each elements of v possesses an inverse.
Hence (V, +) is a group.

37. 2072 Set C Q.No. 5b OR

If a, b $\in$ (G, 0) where G is a group, prove (i) $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$ (ii) $(a^{-1})^{-1} = a$ [4]

SOLUTION

Let G be a group and $a, b \in (G, 0)$

i. $(a \circ b) \circ (b^{-1} \circ a^{-1}) = ((a \circ b) \circ b^{-1}) \circ a^{-1}$ (by associativity)
$= (a \circ (b \circ b^{-1})) \circ a^{-1}$ (by associativity)
$= (a \circ e) \circ a^{-1}$ (as $b \circ b^{-1} = e$)
$= a \circ a^{-1}$ (as e is the identity element)
$= e$

Similarly, $(b^{-1} \circ a^{-1}) \circ (a \circ b) = e$
$\therefore$ $(a \circ b) \circ (b^{-1} \circ a^{-1}) = (b^{-1} \circ a^{-1}) \circ (a \circ b) = e$
Hence $b^{-1} \circ a^{-1}$ is the inverse of $a \circ b$.
$\therefore$ $(a \circ b)^{-1} = b^{-1} \circ a^{-1}$

ii. We have, $a^{-1} \circ a = e$
Operating both sides on left by $(a^{-1})^{-1}$, we get
$(a^{-1})^{-1} \circ (a^{-1} \circ a) = (a^{-1})^{-1} \circ e$
or $((a^{-1})^{-1} \circ a^{-1}) \circ a = (a^{-1})^{-1}$ (by associative law)
or, $e \circ a = (a^{-1})^{-1}$
$\therefore$ $a = (a^{-1})^{-1}$

38. 2072 Set D Q.No. 5a

Let (G, *) be a group. If a, b $\in$ G, then prove that (i) $(a * b)^{-1} = b^{-1} * a^{-1}$ and (ii) $(a^{-1})^{-1} = a$ [4]

➥ Please refer to 2072 'C' Q.No. 5b

39. 2072 Set D Q.No. 5a OR

Define a group. Let a, b, c and x be elements of a group G. Solve the following for x: $x^2 = a^2$ and $x^5 = e$ [4]

SOLUTION

First Part: Please refer to 2069 Set A Q.No. 5b

Next Part

Given equations are:
$x^2 = a^2$ ...(i)
and $x^5 = e$ ...(ii)
Let '.' be the binary operation on G. From (ii),
$x^5 = e$
or, $x \cdot (x^2 \cdot x^2) = e$
or, $x \cdot (a^2 \cdot a^2) = e$ [using (i)]
or, $x \cdot a^4 = e$ ...(iii)
Since $a \in G$, $a \cdot a \cdot a \cdot a = a^4 \in G$. Since G is a group, each element possesses an inverse, so $(a^4)^{-1} \in G$ such that $(a^4) \cdot (a^4)^{-1} = e$. Operating both sides of (iii) by $(a^4)^{-1}$ on right.
$(x \cdot a^4) \cdot (a^4)^{-1} = e \cdot (a^4)^{-1}$
or, $x \cdot (a^4 \cdot (a^4)^{-1}) = (a^4)^{-1}$ (by associativity law)
or, $x \cdot e = a^{-4}$
$\therefore$ $x = a^{-4}$ (as e is the identity element)

40. 2072 Set E Q.No. 5b

Show that the set T = {-1, 1} forms a group under multiplication operation. [4]

➥ Please refer to 2070 Set C Q.No. 5b

41. 2072 Set E Q.No. 5b OR

Prove that every element in a group (G, o) has unique inverse. [4]

SOLUTION

We know that an element a in a group (G, o) has an inverse a^{-1} such that $a \circ a^{-1} = a^{-1} \circ a = e$ where e is the identity of element of G.
If possible, let a' be another inverse of a.
Then, $a \circ a' = a' \circ a = e$
Now,
$a' = a' \circ e$ (as e is the identity element)
$= a' \circ (a \circ a^{-1})$ (as $e = a \circ a^{-1}$)
$= (a' \circ a) \circ a^{-1}$ (by associativity)
$= e \circ a^{-1}$ (as $a' \circ a = e$)
$= a^{-1}$ (as e is the identity element)
$\therefore$ $a' = a^{-1}$
This shows that the inverse elements is unique.

42. 2072 Supp Q.No. 5b

Given that the algebraic structures (G, *) with $G = \{1, \omega, \omega^2\}$ where ω represents an imaginary cube root of unity and * stands for the binary operation of multiplication, show that (G, *) is a group. [4]

➥ Please refer to Model Set II, Q.No. 5b

43. 2073 Set C Q.No. 5b

Define Abelian group. Prove that a group G is Abelian if and only if $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$ for all a, b$\in$G. [4]

SOLUTION

Abelian Group: A group (G, 0) is said to be an abelian group if $a \circ b = b \circ a$ for all $a, b \in G$.

Second part:
Suppose (G, 0) is abelian.
Then, $a \circ b = b \circ a$, for all $a, b \in G$.
Now, $(a \circ b)^{-1} = (b \circ a)^{-1}$ ($\because a \circ b = b \circ a$)
$= a^{-1} \circ b^{-1}$ [$\because (x \circ y)^{-1} = y^{-1} \circ x^{-1}$ for all x, y in a group G]
$\therefore$ $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$
Conversely, suppose that $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$
or, $b^{-1} \circ a^{-1} = a^{-1} \circ b^{-1}$
or, $b \circ (b^{-1} \circ a^{-1}) \circ b = b \circ (a^{-1} \circ b^{-1}) \circ b$
or, $(b \circ b^{-1}) \circ (a^{-1} \circ b) = (b \circ a^{-1}) \circ (b^{-1} \circ b)$ (Associativity)
or, $e \circ (a^{-1} \circ b) = (b \circ a^{-1}) \circ e$ (Existence of inverse)
or, $a^{-1} \circ b = b \circ a^{-1}$ (Existence of identity)
or, $a \circ (a^{-1} \circ b) \circ a = a \circ (b \circ a^{-1}) \circ a$
or, $(a \circ a^{-1}) \circ (b \circ a) = (a \circ b) \circ (a^{-1} \circ a)$ (Associativity)
or, $e \circ (b \circ a) = (a \circ b) \circ e$ (Existence of inverse)
or, $b \circ a = a \circ b$ for all $a, b \in G$(Existence of identity)
$\therefore$ G is abelian.

44. 2073 Set C Q.No. 5b OR

Show that the set of all positive rational numbers under the composition defined by $a*b = \frac{ab}{5}$ forms a group. [4]

SOLUTION

Let Q^+ denote the set of all positive rational numbers. Here, the operation * on Q^+ is defined by $a * b = \frac{ab}{5}$

i. **Closure property:** Since a, b ∈ Q^+, so a * b = $\frac{ab}{5}$ ∈ Q^+. Hence, Q^+ is closed under *.

ii. **Associativity:** Suppose a, b, c ∈ Q^+.

Then, $a * (b * c) = a * \frac{bc}{5} = \frac{a\left(\frac{bc}{5}\right)}{5} = \frac{abc}{25}$

and $(a*b)*c = \frac{ab}{5}*c = \frac{\left(\frac{ab}{5}\right)c}{5} = \frac{abc}{25}$

∴ a * (b * c) = (a * b) * c for all a, b, c ∈ Q^+. So, associativity is satisfied.

iii. **Identity element:** Let e be an identity element. If e ∈ Q^+, then

a * e = a

or, $\frac{ae}{5} = a$ $\left(\because a * b = \frac{ab}{5}\right)$

⇒ e = 5

∴ 5 is the identity element.

iv. **Inverse element:** Let a' be an inverse element of a. If a' ∈ Q^+,

then a * a' = e = 5

or, $\frac{aa'}{5} = 5$

or, $a' = \frac{25}{a}$

∴ $\frac{25}{a}$ is the inverse element of a.

Hence, from (i) – (iv), (Q^+, *) is a group.

Here, G = {0, 1, 2}

Composition table for G under $+_3$

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

From above table, 0 is the identity element.

Here, $2 +_3 1 = 1 +_3 2 = 0$

∴ 1 is the inverse element of 2.

45. 2073 Set D Q.No. 5b

Given the algebraic structure (G,*) with G = {1, ω, ω²} where ω represents the cube roots of unity and * stand for the binary operation of ordinary multiplication of complex numbers, show that (G, *) is a group. [4]

➥ Please refer to Model Set II, Q.No. 5b

46. 2073 Set D Q.No. 5b OR

If a, b ∈ (G, o), prove that $(a\,o\,b)^{-1} = b^{-1}\,o\,a^{-1}$. [4]

SOLUTION

$(a\,o\,b)\,o\,(b^{-1}oa^{-1})$ = $((aob)\,ob^{-1})\,oa^{-1}$ (by associative law)
= $(ao(bob^{-1}))\,oa^{-1}$ (by associative law)
= $(aoe)\,oa^{-1}$ (as $bob^{-1} = e$)
= aoa^{-1} (e is the identity element)
= e

Also,

$(b^{-1}oa^{-1})\,o\,(aob)$ = $((b^{-1}oa^{-1})\,oa)\,ob$
= $(b^{-1}\,o\,(a^{-1}oa))\,ob$
= $(b^{-1}oe)\,ob = b^{-1}ob = e$

$(aob)\,o\,(b^{-1}oa^{-1}) = (b^{-1}oa^{-1})\,o\,(aob) = e$

∴ $b^{-1}oa^{-1}$ is the inverse of aob

i.e. $(aob)^{-1} = b^{-1}oa^{-1}$.

47. 2073 Supp Q.No. 5b

Define abelian group. If (G, *) is an abelian group, show that $(a*b)^{-1} = a^{-1}*b^{-1}$ a, b ∈G. [4]

➥ Please refer to 2073 Set C Q.No. 5b

48. 2073 Supp Q.No. 5b or

Verify that {2^m : m ∈ Z} is an abelian group with respect to multiplication, where Z is the set of integers. [4]

SOLUTION

For a group.

Please refer to 2071 Supp Q.No. 6a

For abelian group,

Let 2^m, 2^n ∈G. Then

$2^m.2^n = 2^{m+n}$

and $2^n.2^m = 2^{n+m} = 2^{m+n}$

Hence, $2^m.2^n = 2^n.2^m$ for all m, n ∈Z

Hence G is an abelian group under multiplication

49. 2074 Set A Q.No. 6a

If a and b are the elements of group (G,*), and

i. If a * b = e, prove that b = a^{-1}

ii. if a * b = b, prove that a = e. [4]

➥ Please refer to 2070 Supp Q.No. 6a

50. 2074 Set B Q.No. 5b

Show that {2^n: n∈Z} is an Abelian group with respect to multiplication. [4]

➥ Please refer to 2071 Supp Q.No. 6a

51. 2074 Set B Q.No. 5b OR

If a, b ∈ (G, o) where G is a group. Prove that: [4]

i. $(aob)^{-1} = b^{-1}\,o\,a^{-1}$

ii. $(a^{-1})^{-1}$ = a.

➥ Please refer to 2072 Set C Q.No. 5b OR

52. 2074 Supp Q.No. 5b

Let G = {1, –1, i, –i}. Show that G forms a group under the operation of multiplication. [4]

➥ Please refer to 2069 Set A Q.N. 5b

53. 2074 Supp Q.No. 5b OR

Let G be a group. If a, b∈(G,∗), prove that $(a∗b)^{-1} = b^{-1}∗a^{-1}$. [4]

➥ Please refer to 2069 Set A Q.N. 5b OR

54. 2075 Set A Q.No. 5b

Show that the set of integers Z forms a group under the operation of addition. [4]

➥ Please refer to 2070 Set D Q.No. 5b

55. 2075 Set A Q.No. 5b OR

If a and b are the elements of a group (G, *), prove that a * x = b has a unique solution in (G, *). [4]

➥ Please refer to 2070 Set D Q.No. 5b OR

56. 2075 Set B Q.No. 6a

Let ∗ be defined on Q^+ by a ∗ b = $\frac{ab}{2}$. Show that (Q^+, ∗) is an Abelian group. [4]

SOLUTION

Let Q^+ denote the set of all positive rational numbers. Here, the operation * on Q^+ is defined by a * b = $\frac{ab}{2}$

i. **Closure property:** Since a, b ∈ Q^+, so a * b = $\frac{ab}{2}$ ∈ Q^+. Hence, Q^+ is closed under *.

ii. **Associativity:** Suppose a, b, c ∈ Q^+.

Then, $a * (b * c) = a * \frac{bc}{2} = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$

and $(a*b)*c = \frac{ab}{2}*c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$

∴ a * (b * c) = (a * b) * c for all a, b, c ∈ Q^+. So, associativity is satisfied.

iii. **Identity element:** Let e be an identity element. If e ∈ Q^+, then

a * e = a

or, $\frac{ae}{2} = a$ $\left(\because a * b = \frac{ab}{2}\right)$

⇒ e = 2

∴ 2 is the identity element.

iv. **Inverse element:** Let a' be an inverse element of a. If a' ∈ Q^+,

then a * a' = e = 2

or, $\frac{aa'}{2} = 2$

or, $a' = \frac{4}{a}$

∴ $\frac{4}{a}$ is the inverse element of a.

Hence, from (i) – (iv), (Q^+, *) is a group.

Also, for commutative property,

Let a * b = $\frac{ab}{2}$

and b * a = $\frac{ba}{2} = \frac{ab}{2}$

∴ a * b = b * a for all a, b ∈ Q^+.

Hence, (Q^+, *) is an abelian group.

57. 2075 Set C Q.No. 5b

Define a group. Prove that the set of all three dimensional vectors form an infinite Abelian group under vector addition. [4]

SOLUTION

First Part: Please refer to 2069 Set A Q.No. 5b

Next Part: Let V be the set of all vectors in space. Also, let the operation of addition of vectors be denoted by +.

i. **Closure property:** Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be any two vectors in V.

Now, $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in V$

∴ $\vec{u} + \vec{v} \in V$ for all $\vec{u}$, $\vec{v} \in V$

Hence V is closed under addition.

ii. **Associative property:** Let $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be any three vectors in V.

Now,

$\vec{u} + (\vec{v} + \vec{w})$ = $(u_1, u_2, u_3) + ((v_1, v_2, v_3) + (w_1, w_2, w_3))$
= $(u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
= $(u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3)$
= $((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3)$
= $(u_1 + v_1, u_2 + v_2, u_3 + v_3) + (w_1, w_2, w_3)$
= $((u_1, u_2, u_3) + (v_1, v_2, v_3)) + (w_1, w_2, w_3)$
= $(\vec{u} + \vec{v}) + \vec{w}$

∴ $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ for all $\vec{u}$, $\vec{v}$, $\vec{w} \in V$

Hence, associativity is satisfied.

iii. **Existence of element:** Let $\vec{u} = (u_1, u_2, u_3) \in V$. There is a vector $\vec{0} = (0, 0, 0)$ in V such that

$\vec{u} + \vec{0} = (u_1, u_2, u_3) + (0, 0, 0) = (u_1, u_2, u_3) = \vec{u}$

Similarly, $\vec{0} + \vec{u} = \vec{u}$

Thus, $\vec{0} \in V$ is the identity element.

iv. **Existence of inverse element:**

Let $\vec{u} = (u_1, u_2, u_3) \in V$. Let $\vec{v}$ be the inverse of $\vec{u}$.

Then, $\vec{u} + \vec{v} = 0$

or, $(u_1, u_2, u_3) + \vec{v} = (0, 0, 0)$

∴ $\vec{v} = -(u_1, u_2, u_3) = -\vec{u} \in V$ for all $\vec{u} \in V$.

So, each elements of v possesses an inverse.

v. **Commutative property:**

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ be any two vectors in V.

Now, $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$\vec{v} + \vec{u}$ = $(v_1 + u_1, v_2 + u_2, v_3 + u_3)$
= $(u_1 + v_1, u_2 + v_2, u_3 + v_3)$

∴ $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}$, $\vec{v} \in V$

Hence, commutativity property is satisfied. Clearly V is infinite set. Hence, (V, +) is an infinite Abelian group.

58. 2075 Set C Q.No. 5b OR

If a and b are the elements of group (G, ∗) such that

i. a * b = b prove that a = e.

ii. a * b = e, prove that b = a^{-1}.

➥ Please refer to 2070 Supp. Q.No. 6a

UNIT 4

CONIC SECTIONS

A. PARABOLA

2 MARKS QUESTIONS

1. 2057 Q.No. 2 c

Find the focus and directrix of the parabola $y^2 - 4y - 8x - 20 = 0$ [2]

SOLUTION

Given parabola is

$y^2 - 4y - 8x - 20 = 0$

or, $y^2 - 4y + 4 = 8x + 20 + 4$

or, $(y-2)^2 = 8x + 24$

or, $(y-2)^2 = 8(x+3)$

Comparing this equation with

$(y-k)^2 = 4a(x-h)$,

we have, $h = -3$, $k = 2$, $4a = 8$

$\therefore$ $a = 2$

Focus $= (h + a, k) = (-3 + 2, 2) = (-1, 2)$

Equation of directrix is

$x = h - a$

or, $x = -3 - 2$

$\therefore$ $x + 5 = 0$

2. 2058 Q.No. 2 c

Find the equation of the tangent to the parabola $y^2 = 16x$ at the point (4, 8). [2]

SOLUTION

Comparing the equation of parabola $y^2 = 16x$ with $y^2 = 4ax$, we have

$4a = 16$

$\therefore$ $a = 4$

The equation of tangent to the parabola at $(x_1, y_1) = (4, 8)$ is

$yy_1 = 2a(x + x_1)$

$y \cdot 8 = 2 \cdot 4\,(x + 4)$

$\therefore$ $y = x + 4$

3. 2059 Q.No. 2 c

Find the focus and directrix of the parabola $x^2 = 12y$. [2]

SOLUTION

Comparing $x^2 = 12y$ with $x^2 = 4ay$, we have

$4a = 12$

$\therefore$ $a = 3$

Focus $= (0, a) = (0, 3)$

The equation of directrix is

$y = -a$

or, $y = -3$

$\therefore$ $y + 3 = 0$

4. 2060 Q.No. 2 c

Find the equation of the normal to the parabola $y^2 = 5x$ perpendicular to the line $x + 2y = 7$. [2]

SOLUTION

Comparing $y^2 = 5x$ with $y^2 = 4ax$, we have

$4a = 5$

$\therefore$ $a = \frac{5}{4}$

Slope of the line $x + 2y = 7$ is $(m_1) = \frac{-1}{2}$

Since the normal is perpendicular to the given line, so its slope $(m) = \frac{-1}{m_1} = 2$

Now, the equation of normal is

$y = mx - 2am - am^3$

or, $y = 2.x - 2.\frac{5}{4}.2 - \frac{5}{4}.2^3$

or, $y = 2x - 5 - 10$

$\therefore$ $y = 2x - 15$

5. 2062 Q.No. 5 c

Determine the equation of the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$. [2]

SOLUTION

The equation of chord joining the point t_1 and t_2 i.e. $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ of the parabola $y^2 = 4ax$ is

$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$

or, $y - 2at_1 = \frac{2a(t_2 - t_1)}{a(t_2 - t_1)(t_2 + t_1)}(x - at_1^2)$

or, $(y - 2at_1)(t_2 + t_1) = 2(x - at_1^2)$

or, $(t_1 + t_2)y - 2at_1t_2 - 2at_1^2 = 2x - 2at_1^2$

$\therefore$ $2x - (t_1 + t_2)y + 2at_1t_2 = 0$

6. 2063 Q.No. 5 c

Find the coordinates of the vertex and the focus of the parabola whose equation is $y^2 = 6y - 12x + 45$. [2]

SOLUTION

Given equation of parabola is

$y^2 = 6y - 12x + 45$

or, $y^2 - 6y + 9 = -12x + 45 + 9$

or, $(y-3)^2 = -12x + 54$

or, $(y-3)^2 = -12\left(x - \frac{9}{2}\right)$ ...(i)

Comparing (i) with $(y-k)^2 = 4a(x-h)$, we get

$h = \frac{9}{2}, k = 3$

$4a = -12$

$\therefore$ $a = -3$

Focus of the parabola $= (h + a, k)$

$= \left(\frac{9}{2} - 3, 3\right) = \left(\frac{3}{2}, 3\right)$

Vertex of the parabola $= (h, k) = \left(\frac{9}{2}, 3\right)$

7. 2064 Q.No. 5 c

Find the equation of the parabola in which the ends of the latus rectum have the coordinates (–1, 5) and (–1, –11) and the vertex is (–5, –3). [2]

SOLUTION

Since the end points of latus rectum are (–1, 5) and (–1, –11), so the equation of latus rectum is $x = -1$. So it is parallel to x-axis.

Here, vertex $(h, k) = (-5, -3)$

We know that the equation of parabola is

$(y-k)^2 = 4a(x-h)$

or, $(y+3)^2 = 4a(x+5)$ ...(i)

If (i) passes through the point (–1, 5), then

$(5+3)^2 = 4a(-1+5)$

$64 = 16a$

$\therefore$ $a = 4$

Putting the value of a in (i), we get

$(y+3)^2 = 4 \times 4(x+5)$

or, $y^2 + 6y + 9 = 16x + 80$

$\therefore$ $y^2 + 6y - 16x - 71 = 0$

8. 2066 Q.No. 5 c

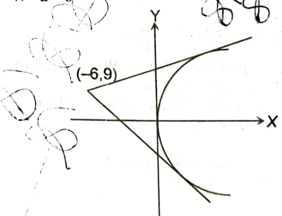
Find the equations of the tangents from the point (–6, 9) to the parabola $y^2 = 24x$. [2]

SOLUTION

Comparing $y^2 = 24x$ ... (i) with $y^2 = 4ax$, we get

$4a = 24$

$\therefore$ $a = 6$



The equation of line through the point (–6,9) is

$y - 9 = m(x + 6)$

or, $y = mx + 6m + 9$ ...(ii)

The line (ii) will be tangent to the parabola (i) if

$6m + 9 = \frac{6}{m}\left(\because c = \frac{a}{m}\right)$

or, $6m^2 + 9m = 6$

or, $2m^2 + 3m - 2 = 0$

or, $2m^2 + 4m - m - 2 = 0$

or, $2m(m + 2) - 1(m + 2) = 0$

or, $(m + 2)(2m - 1) = 0$

$\therefore$ $m = -2, \frac{1}{2}$

Putting $m = -2$ in (i),

$y = -2x - 12 + 9$

or, $2x + y + 3 = 0$

Again, putting $m = \frac{1}{2}$ in (i)

$y = \frac{1}{2}x + 6 \times \frac{1}{2} + 9$

or, $2y = x + 24$

or, $x - 2y + 24 = 0$

$\therefore$ Required equations of tangents are $2x + y + 3 = 0$ and $x - 2y + 24 = 0$.

9. 2068 Q.No. 2c

Prove that the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ if $ln = am^2$. [2]

SOLUTION

Given parabola is $y^2 = 4ax$

Equation of given line is $lx + my + n = 0$

or, $my = lx - n$

or, $y = \frac{-l}{m}x - \frac{n}{m}$

We know that the line $y = mx + c$ will be a tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$

So, the given line will be a tangent to the given parabola if

$\frac{-n}{m} = \frac{a}{-l/m}$

or, $\frac{ln}{m^2} = a$

$\therefore$ $ln = am^2$

10. 2069 (Set A) Old Q.No. 5c

Find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1). [2]

SOLUTION

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$

So, slope of the tangent $= \frac{2a}{y_1}$

The slope of the normal $= \frac{-1}{\text{slope of tangent}} = \frac{-y_1}{2a}$

Hence the equation of the normal is

$y - y_1 = \frac{-y_1}{2a}(x - x_1)$

11. 2069 Old (Set B) Q.No. 2c

Find the equation of the tangent to the parabola $y^2 = 16x$ at the point (4, 8). [2]

➥ Please refer to 2058Q.No. 2c

12. 2071 Supp. Q.No. 2a

Find the equation of the tangent to the parabola $y^2 = 9x$ at (4, –6). [2]

SOLUTION

Given equation of parabola is $y^2 = 9x$

Comparing it with $y^2 = 4ax$, we have

$4a = 9$

$\therefore a = \frac{9}{4}$

The equation of tangent to the given parabola at $(x_1, y_1) = (4, -6)$ is

$y \cdot y_1 = 2a(x + x_1)$

$y(-6) = 2 \times \frac{9}{4}(x + 4)$

or, $-6y = \frac{9}{2}(x + 4)$

or, $-2y = \frac{3}{2}(x + 4)$

or, $-4y = 3x + 12$

$\therefore 3x + 4y + 12 = 0$

13. 2075 Set B Q.No. 2a

Determine the equation of the chord joining the points t_1 and t_2 on the parabola $y^2 = 4ax$. [2]

➡ Please refer to 2062 Q.No. 5c

4 MARKS QUESTIONS

14. 2057 Q.No. 9 b

Find the equation of the parabola in the standard form $y^2 = 4ax$. [4]

➡ Please refer to Model Set II, Q.No. 6a or,

15. 2058 Q.No. 9 b

Show that the normal to the parabola $y^2 = 8x$ at (2, 4) meets the parabola again in (18, – 12). [4]

SOLUTION

The equation of tangent to the parabola $y^2 = 8x$ ...(i) at (2,4) is

$y \cdot 4 = 4(x + 2)$

$y = x + 2$

Slope of tangent = 1

Slope of normal $= \frac{-1}{\text{slope of tangent}} = -\frac{1}{1} = -1$

The equation of normal at (2, 4) is

$y - 4 = -1(x - 2)$

or, $y - 4 = -x + 2$

$\therefore x + y = 6$ (ii)

Solving (i) and (ii), we get

$y^2 = 8(6 - y)$

or, $y^2 + 8y - 48 = 0$

or, $(y + 12)(y - 4) = 0$

$y = 4, -12$

When $y = 4$, $x = 6 - y = 6 - 4 = 2$

When $y = -12$, $x = 6 - y = 6 - (-12) = 18$

$\therefore$ The two points of intersection of the normal and parabola are (2, 4) and (18, –22). Hence, if the normal is drawn at (2, 4), it again meets the parabola (i) at (18, –12).

16. 2059 Q.No. 9 b

Find the condition that the line $y = mx + c$ may be a tangent to the parabola $y^2 = 4ax$. [4]

SOLUTION

Let $y = mx + c$ ...(i) be a straight line and $y^2 = 4ax$ ...(ii) be a parabola.

From (i) and (ii)

$(mx + c)^2 = 4ax$

or, $m^2x^2 + 2(mc - 2a)x + c^2 = 0$ ...(iii)

The straight line will be a tangent to the parabola if the two values of x obtained from equation (iii) are equal. This will be so if discriminant of (iii) = 0

i.e. $\{2(mc - 2a)\}^2 - 4 \cdot m^2 \cdot c^2 = 0$

or, $4(mc - 2a)^2 - 4m^2c^2 = 0$

or, $m^2c^2 - 4amc + 4a^2 - m^2c^2 = 0$

or, $4amc = 4a^2$

$\therefore c = \frac{a}{m}$

17. 2060 Q.No. 9 b

Deduce the equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) on the parabola. [4]

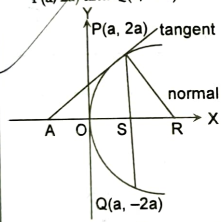
➡ Please refer to Model Set II, Q.No. 6a

18. 2061 Q.No. 9 b

Prove that the latus rectum of a parabola bisects the angle between the tangent and the normal at either extremity of the latus rectum. [4]

SOLUTION

Let the equation of parabola be $y^2 = 4ax$. Then its end points of the latus rectum PQ are P(a, 2a) and Q(a, –2a).



Slope of PQ $= \frac{-2a - 2a}{a - a} = \frac{-4a}{0} = \infty$

So, PQ is perpendicular to the axis of the parabola i.e. $\angle PSR = \angle PSA = 90°$.

Again, the equation of tangent at P(a, 2a) is

$y \cdot 2a = 2a(x + a)$

or, $y = x + a$

$\therefore$ Slope of tangent at P(a, 2a) is 1.

i.e. $\tan \angle PAR = 1 = \tan 45°$

$\therefore \angle PAR = 45°$

Then, from ΔAPS, $\angle APS = 180° - 90° - 45° = 45°$

Since tangent and normal are at right angles so $\angle RPS = 90° - 45° = 45°$.

$\therefore$ Latus rectum PQ bisects the angle between the tangent AP and normal PR at P. Similar result holds at another end Q.

19. 2062 Q.No. 9 b

Find the equation of the normal to the parabola $y^2 = 4ax$ in the slope form. [4]

SOLUTION

The equation of tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $yy_1 = 2a(x + x_1)$

So, slope of the tangent $= \frac{2a}{y_1}$

The slope of the normal $= \frac{-1}{\text{slope of tangent}} = \frac{-y_1}{2a}$

Hence the equation of the normal is

$y - y_1 = \frac{-y_1}{2a}(x - x_1)$ ...(i)

If m be the slope of the normal, then

$m = \frac{-y_1}{2a}$

$\therefore y_1 = -2am$

Since (x_1, y_1) lies in $y^2 = 4ax$,

so, $y_1^2 = 4ax_1$

Putting $y_1 = -2am$ in $y_1^2 = 4ax_1$, we get

$(-2am)^2 = 4ax_1$

$\therefore x_1 = am^2$

Putting the values of x_1 and y_1 in (i), we get

$y + 2am = m(x - am^2)$

$\therefore y = mx - 2am - am^3$ which is the required equation of the normal to the parabola $y^2 = 4ax$ in slope form.

20. 2063 Q.No. 9 b

Prove that the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ if $ln = am^2$ [4]

SOLUTION

Given parabola is $y^2 = 4ax$

Equation of given line is $lx + my + n = 0$

or, $my = -lx - n$

or, $y = \frac{-l}{m}x - \frac{n}{m}$

We know that the line $y = mx + c$ will be a tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$

So, the given line will be a tangent to the given parabola if

$\frac{-n}{m} = \frac{a}{-l/m}$

or, $\frac{ln}{m^2} = a$

$\therefore ln = am^2$

21. 2064 Q.No. 9 b

Deduce the equation of the parabola in the standard form $y^2 = 4ax$. [4]

➡ Please refer to Model Set II, Q.No. 6a or

22. 2065 Q.No 9 b

Find the equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) on the parabola. [4]

➡ Please refer to Model Set II, Q.No. 6a

23. 2066 C Q.No. 9 b

Find the equation of the parabola in the standard form $y^2 = 4ax$. [4]

➡ Please refer to Model Set II, Q.No. 6a or

24. 2067 Q.No. 9b

Show that the pair of tangents from the point (–2, 3) to the parabola $y^2 = 8x$ are at right angle. [4]

SOLUTION

Comparing the given parabola $y^2 = 8x$ with $y^2 = 4ax$, we have

$4a = 8$

$\therefore a = 2$

The equation of the tangent to the parabola is

$y = mx + \frac{a}{m}$

or $y = mx + \frac{2}{m}$ ...(i)

The tangent (i) passes through the point (–2, 3), so

$3 = -2m + \frac{2}{m}$

or, $3m = -2m^2 + 2$

or, $2m^2 + 3m - 2 = 0$

or, $2m^2 + 4m - m - 2 = 0$

or, $2m(m + 2) - 1(m + 2) = 0$

or, $(m + 2)(2m - 1) = 0$

$\therefore m = -2, \frac{1}{2}$

Product of the slopes $= (-2) \times \frac{1}{2} = -1$

Hence the tangents are at right angle.

25. 2068 Q.No. 9b

Find the coordinates of the focus, the vertex, the equation of the directrix and the length of the latus rectum of the parabola $y^2 = 6y - 12x + 45$. [4]

SOLUTION

Given equation of parabola is

$y^2 = 6y - 12x + 45$

or, $y^2 - 6y + 9 = -12x + 45 + 9$

or, $(y - 3)^2 = -12x + 54$

or, $(y - 3)^2 = -12\left(x - \frac{9}{2}\right)$ ... (i)

Comparing (i) with $(y - k)^2 = 4a(x - h)$, we get

$h = \frac{9}{2}, k = 3$

$4a = -12$

$\therefore a = -3$

Focus of the parabola = (h + a, k)

$= \left(\frac{9}{2} - 3, 3\right) = \left(\frac{3}{2}, 3\right)$

Vertex of the parabola $= (h, k) = \left(\frac{9}{2}, 3\right)$

Equation of directrix is $x = h - a$

or $x = \frac{9}{2} + 3$

$\therefore 2x - 15 = 0$

Length of latus rectum $= |4a| = |4 \times (-3)| = |-12| = 12$

26. 2069 (Set A) Q.No. 6a

Find the equation of the normal to the parabola $y^2 = 4ax$ at the point (x_1, y_1) and express this in slope form. [4]

➡ Please refer to 2062 Q.No. 9b

27. 2069 (Set A) Old Q.No. 9b

Given an equation of the parabola $y^2 = 6y - 12x + 45$, find the focus, vertex, equation of the directrix and the length of the latus rectum. [4]

➥ Please refer to 2068 Q.No. 9c

28. 2069 (Set B) Q.No. 6a

Find the condition under which the line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$. Find the equation of tangent in slope form. Also, find the point of contact. [4]

SOLUTION

First Part

Please refer to 2059 Q.No. 9b

Last Part

The equation of tangent to the parabola in the slope form is

$y = mx + \frac{a}{m}$ ...(i)

Again, the equation of tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is

$yy_1 = 2a(x + x_1)$ ...(ii)

Equation (i) and equation (ii) will represent the same line if

$\frac{1}{y_1} = \frac{m}{2a} = \frac{a/m}{2ax_1}$

From first two ratios

$\frac{1}{y_1} = \frac{m}{2a}$

$\therefore y_1 = \frac{2a}{m}$

From last two ratios

$\frac{m}{2a} = \frac{a/m}{2ax_1}$

$\therefore x_1 = \frac{a}{m^2}$

$\therefore$ The point of contact is $(x_1, y_1) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

29. 2069 Old (Set B) Q.No. 9b

Find the equation of the parabola in standard form. [4]

➥ Please refer to Model Set II, Q.No. 6a or

30. 2070 Set C Q.No. 6 a

Find the equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) [4]

➥ Please refer to Model Set II, Q.No. 6a

31. 2070 Set D Q.No. 6 a

Prove that the line $3x + 4y + 6 = 0$ is tangent to the parabola $2y^2 = 9x$. Find its point of contact. [4]

SOLUTION

Equation of line is

$3x + 4y + 6 = 0$

or, $y = \frac{-3}{4}x - \frac{3}{2}$ ...(i)

And equation of parabola is $2y^2 = 9x$ ...(ii)

From (i) and (ii), $2\left(-\frac{3}{4}x - \frac{3}{2}\right)^2 = 9x$

or, $2\left(\frac{3}{4}x + \frac{3}{2}\right)^2 = 9x$

or, $2 \times 9\left(\frac{x + 2}{4}\right)^2 = 9x$

or, $\frac{18}{6}(x^2 + 4x + 4) = 9x$

or, $x^2 + 4x + 4 = 8x$

or, $x^2 - 4x + 4 = 0$

or, $(x - 2)^2 = 0$

$\therefore x = 2, 2$

Since the line touches the parabola at two coincident points, so it is a tangent to the parabola. Putting $x = 2$ in (i)

$y = -\frac{3}{4} \times 2 - \frac{3}{2} = -3$

$\therefore$ Point of contact is $(2, -3)$

32. 2070 (Old) Q.No. 9 b

If the normal at any point $P(at_1^2, 2at_1)$ on the parabola $y^2 = 4ax$ meets the curve again in $Q(at_2^2, 2at_2)$, prove that $t_1 + \frac{2}{t_1} + t_2 = 0$.

SOLUTION

The equation of the tangent at $P(at_1^2, 2at_1)$ is

$y \cdot 2at_1 = 2a(x + at_1^2)$

or, $yt_1 = x + at_1^2$

Slope of the tangent $= \frac{1}{t_1}$

$\therefore$ Slope of the normal $= \frac{-1}{\text{slope of tangent}} = -t_1$

The equation of the normal at $P(at_1^2, 2at_1)$

$y - 2at_1 = -t_1(x - at_1^2)$

Since it passes through the point $Q(at_2^2, 2at_2)$ so

or, $2a(t_2 - t_1) = -at_1(t_2 - t_1)(t_2 + t_1)$

or, $2 = -t_1(t_2 + t_1)$

or, $\frac{2}{t_1} = -(t_2 + t_1)$

$\therefore t_1 + \frac{2}{t_1} + t_2 = 0$

33. 2071 Set C Q.No. 6 a

Find the equation of the normal to the parabola $y^2 = 4ax$ in the slope form.

➥ Please refer to 2062 Q.No. 9b

34. 2071 Set D Q.No. 6 a

Find the equation of the parabola in standard form $y^2 = 4ax$.

➥ Please refer to Model Set II, Q.No. 6a or

35. 2071 Old Q.No. 9 a

Deduce the equation of the parabola in standard from $y^2 = 4ax$.

➥ Please refer to Model Set II, Q.No. 6a or

36. 2071 Supp. Q.No. 6b

Find the area of the triangle formed by the lines joining the vertex of the parabola $y^2 = 12x$ to the ends of its latus rectum. [4]

SOLUTION

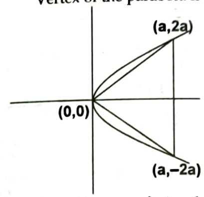
Comparing the equation of parabola $y^2 = 12x$ with $y^2 = 4ax$, we have

$4a = 12$

$\therefore a = 3$

So the end points of latus rectum are $(a, 2a)$ and $(a, -2a)$ i.e. $(3, 6)$ and $(3, -6)$.

Vertex of the parabola is $(0, 0)$



Now, the area of triangle with vertices $(x_1, y_1) = (0, 0)$, $(x_2, y_2) = (3, -6)$ and $(x_3, y_3) = (3, 6)$ is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 3 & -6 & 1 \\ 3 & 6 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \cdot 1 \begin{vmatrix} 3 & -6 \\ 3 & 6 \end{vmatrix} \quad \text{[Expanding along } R_1]$$

$= \frac{1}{2}(36) = 18$ sq. units.

37. 2072 Set C Q.No. 6a

Show that the pair of tangents from the point $(-2, 3)$ to the parabola $y^2 = 8x$ are at right angle. [4]

➥ Please refer to 2067 Q.No. 9b

38. 2072 Set D Q.No. 6a

If the tangent to the parabola $y^2 = 12x$ makes an angle 45° with the straight line $x - 2y + 3 = 0$, find its equation and the point of contact. [4]

SOLUTION

Comparing the equation of parabola $y^2 = 12x$ with $y^2 = 4ax$, we have

$4a = 12$

$\therefore a = 3$

The equation of tangent to the parabola is

$y = mx + \frac{a}{m}$

or, $y = mx + \frac{3}{m}$ ...(i)

Slope of tangent $(m_1) = m$

Slope of the line $x - 2y + 3 = 0$ is

$(m_2) = \frac{-1}{(-2)} = \frac{1}{2}$

Here, $\theta = 45°$

We have $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$

or, $\tan 45° = \pm\frac{m - 1/2}{1 + m \cdot 1/2}$

or, $1 = \pm\frac{2m - 1}{2 + m}$

Taking +ve sign

$2 + m = 2m - 1$

$\therefore m = 3$

Again, taking –ve sign

$2 + m = -2m + 1$

$\therefore m = \frac{-1}{3}$

When $m = 3$, the equation of tangent is

$y = 3x + \frac{3}{3}$

or, $3x - y + 1 = 0$

When $m = \frac{-1}{3}$, the equation of tangent is

$y = \frac{-1}{3}x + \frac{3}{(-1/3)}$

or, $3y = -x - 27$

or, $x + 3y + 27 = 0$

Hence the required equations of tangents are $3x - y + 1 = 0$ and $x + 3y + 27 = 0$

We know that the point of contact of tangent is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

When $m = 3$, the point of contact is $\left(\frac{3}{3^2}, \frac{2 \times 3}{3}\right) = \left(\frac{1}{3}, 2\right)$

When $m = \frac{-1}{3}$, the point of contact is $\left(\frac{3}{(-1/3)^2}, \frac{2 \times 3}{-1/3}\right) = (27, -18)$

Note: *The point of contact also can be obtained by solving the equation of tangent and equation of parabola.*

39. 2072 Set E Q.No. 6b

Find the condition under which the line $y = mx + c$ is tangent to the parabola $y^2 = 4ax$. Find the equation of the tangent in slope form. [4]

SOLUTION

Let $y = mx + c$ ...(i) be a straight line and

$y^2 = 4ax$ ...(ii) be a parabola.

From (i) and (ii)

$(mx + c)^2 = 4ax$

or, $m^2x^2 + 2(mc - 2a)x + c^2 = 0$ ...(iii)

The straight line will be a tangent to the parabola if the two values of x obtained from equation (iii) are equal. This will be so if discriminant of (iii) = 0

i.e. $[2(mc - 2a)]^2 - 4 \cdot m^2 \cdot c^2 = 0$

or, $4(mc-2a)^2 - 4m^2c^2 = 0$
or, $m^2c^2 - 4amc + 4a^2 - m^2c^2 = 0$
or, $4amc = 4a^2$
or, $c = \frac{a}{m}$

Thus the line $y = mx + c$ will be a tangent to the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

So the equation of tangent to the parabola in slope from is $y = mx + \frac{a}{m}$.

40. 2072 Supp Q.No. 6a

Define conic section. Find the equation of the parabola in its standard form. [4]

➥ **Conic Section:**
Please refer to Model Question Set I Q. No. 8a
Next Part:
Please refer to Model Set II, Q.No. 6a or

41. 2073 Set C Q.No. 6a

If a tangent to the parabola $y^2 = 12x$ makes an angle 45° with the straight line $x - 2y + 3 = 0$, find the equation of the tangent. [4]

SOLUTION

Comparing the equation of parabola $y^2 = 12x$ with $y^2 = 4ax$, we have
$4a = 12$
$\therefore$ $a = 3$

The equation of tangent to the parabola is

$y = mx + \frac{a}{m}$

or, $y = mx + \frac{3}{m}$ ...(i)

Slope of tangent $(m_1) = m$
Slope of the line $x - 2y + 3 = 0$ is

$(m_2) = \frac{-1}{(-2)} = \frac{1}{2}$

Here, $\theta = 45°$

We have $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1m_2}$

or, $\tan 45° = \pm\frac{m - 1/2}{1 + m\cdot 1/2}$

or, $1 = \pm\frac{2m - 1}{2 + m}$

Taking +ve sign
$2 + m = 2m - 1$
$\therefore$ $m = 3$

Again, taking –ve sign
$2 + m = -2m + 1$
$\therefore$ $m = \frac{-1}{3}$

When $m = 3$, the equation of tangent is

$y = 3x + \frac{3}{3}$

or, $3x - y + 1 = 0$

When $m = \frac{-1}{3}$, the equation of tangent is

$y = \frac{-1}{3}x + \frac{3}{(-1/3)}$

or, $3y = -x - 27$
or, $x + 3y + 27 = 0$

Hence the required equations of tangents are $3x - y + 1 = 0$ and $x + 3y + 27 = 0$

42. 2073 Set D Q.No. 6a

Find the equation of the tangent to the parab... $y^2 = 4ax$ at the point (x_1, y_1).

➥ Please refer to Model Set II, Q.No. 6a

43. 2073 Supp Q.No. 6a

Find the condition that a line $y = mx + c$ i... tangent to the parabola $y^2 = 4ax$.

➥ Please refer 2072 Set E Q.No. 6b

44. 2074 Set A Q.No. 6b

Find the condition under which the li... $y = mx + c$ is tangent to the parabola $y^2 = 4$... Also find the equation of the tangent in ... slope form.

➥ Please refer to 2072 Set E Q.No. 6b

45. 2074 Set B Q.No. 6a

Show that the pair of tangents from the po... (-2, 3) to the parabola $y^2 = 8x$ are at right ang...

➥ Pleaser refer to 2067 Q.No. 9b

46. 2074 Supp Q.No. 6a

Find the equation of the tangent to the parab... $y^2 = 4ax$ in the slope form. Also find the poin... contact.

➥ **First Part:** Please refer to 2059 Q.No. 9b
Second Part: Please refer to 2069 Set B Q.N...

47. 2075 Set A Q.No. 6a

Find the equation of the tangent to the parab... $y^2 = 4ax$ at the point (x_1, y_1).

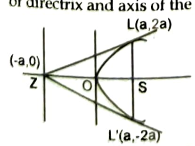
➥ Please refer to 2060 Q.No. 9b

48. 2075 Set B Q.No. 6b

Prove that the lines joining the ends of la... rectum of the parabola $y^2 = 4ax$ to the poin... intersection of the directrix and the axis ar... right angles.

SOLUTION

Let L(a, 2a) and L'(a, −2a) be the ends point... the latus rectum of the parabola $y^2 = 4$... Also, let Z (−a, 0) be the point of intersect... of directrix and axis of the parabola.



Now, Slope of LZ$(m_1) = \frac{0 - 2a}{-a - a} = 1$

Slope of L'Z$(m_2) = \frac{0 + 2a}{-a - a} = -1$

Since, $m_1 \cdot m_2 = 1 \times (-1) = -1$, LZ and L'Z are perpendicular to each other.
Hence, $\angle LZL' = 90°$. Proved

49. 2075 Set C Q.No. 6a

Find the condition of tangency of a straight line $y = mx + c$ to a parabola $y^2 = 4ax$. [4]

➥ Please refer 2059 Q.No. 9b

6 MARKS QUESTIONS

50. 2070 Supp. Q.No. 9

Prove that the tangent to a parabola $y^2 = 4ax$ at a point (x_0, y_0) on the parabola is given by the equation $yy_0 = 2a(x + x_0)$. Reduce the equation in slope form. [6]

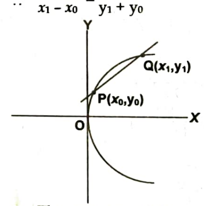
SOLUTION

Let $P(x_0, y_0)$ be the point on the parabola $y^2 = 4ax$. Let us take another point $Q(x_1, y_1)$ on the parabola which is very close to P. Then,

$y_0^2 = 4ax_0$ ...(i)
& $y_1^2 = 4ax_1$ ...(ii)

Subtracting, we have
$y_1^2 - y_0^2 = 4ax_1 - 4ax_0$
or, $(y_1 - y_0)(y_1 + y_0) = 4a(x_1 - x_0)$

$\therefore$ $\frac{y_1 - y_0}{x_1 - x_0} = \frac{4a}{y_1 + y_0}$ ... (iii)



The equation of PQ is

$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$

or, $y - y_0 = \frac{4a}{y_1 + y_0}(x - x_0)$ [using (iii)]

By definition, PQ becomes the tangent at P when $Q \to P$ i.e. when $x_1 \to x_0$ and $y_1 \to y_0$.
Hence, the equation of tangent at P is

$y - y_0 = \frac{4a}{2y_0}(x - x_0)$

or, $yy_0 - y_0^2 = 2ax - 2ax_0$
or, $yy_0 - 4ax_0 = 2ax - 2ax_0$ [using (i)]
or, $yy_0 = 2a(x + x_0)$ which is the equation of tangent to the parabola $y^2 = 4ax$ at (x_0, y_0).

The above equation can be written as

$y = \frac{2a}{y_0}x + \frac{2ax_0}{y_0}$ ...(i)

Writing $\frac{2a}{y_0} = m$ then $y_0 = \frac{2a}{m}$

Since $y_0^2 = 4ax_0$, so

$\left(\frac{2a}{m}\right)^2 = 4ax_0$

$\Rightarrow$ $x_0 = \frac{a}{m^2}$

Then from (i)

$y = mx + 2a\frac{a/m^2}{2a/m}$

$y = mx + \frac{a}{m}$ which is the equation of tangent in slope form.

B. ELLIPSE

2 MARKS QUESTIONS

51. 2069 (Set B) Q.No. 2a

Find the eccentricity and the foci of the ellipse $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$ [2]

SOLUTION

Comparing $\frac{(x+2)^2}{16} + \frac{(y-5)^2}{9} = 1$

with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we have

$h = -2, k = 5$
$a^2 = 16,\quad b^2 = 9$
$\Rightarrow$ $a = 4,\quad b = 3$

Since a > b, the major axis is along x-axis
We have,

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

Foci $= (h \pm ae, k) = \left(-2 \pm 4\cdot\frac{\sqrt{7}}{4}, 5\right) = (-2 \pm \sqrt{7}, 5)$

52. 2070 Set C Q.No. 2 a

Find the eccentricity and the foci of the ellipse: $\frac{x^2}{9} + \frac{y^2}{16} = 1$ [2]

SOLUTION

Comparing $\frac{x^2}{9} + \frac{y^2}{16} = 1$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$a^2 = 9,\quad b^2 = 16$
$\Rightarrow$ $a = 3,\quad b = 4$

Since b > a, so major axis is along y-axis.
We have

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

Foci $= (0, \pm be) = \left(0, \pm 4\cdot\frac{\sqrt{7}}{4}\right) = (0, \pm\sqrt{7})$

53. 2070 Set D Q.No. 2 a

Find the eccentricity and the foci of the ellipse $3x^2 + 4y^2 = 36$. [2]

SOLUTION

Equation of ellipse is
$3x^2 + 4y^2 = 36$

or, $\frac{x^2}{12}+\frac{y^2}{9}=1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$a^2=12$, $b^2=9$

$\Rightarrow$ $a=2\sqrt{3}$, $b=3$

Since a > b, the major axis is along x-axis.

We have,

Eccentricity (e) $=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{12}}$

$=\sqrt{\frac{3}{12}}=\frac{1}{2}$

Foci = $(\pm ae, 0)=\left(\pm 2\sqrt{3}.\frac{1}{2}, 0\right)=(\pm\sqrt{3}, 0)$

54. 2070 (Old) Q.No. 2 b

Find the equation of the ellipse in the standard form whose focus is at (-2,0) and vertex at (5,0). [2]

SOLUTION

Vertex (a, 0) = (5, 0)

$\therefore$ a = 5 ...(i)

Focus (-ae, 0) = (-2, 0)

ae = 2

5. e = 2 [using (i)]

$e=\frac{2}{5}$

We have,

$b^2=a^2(1-e^2)=5^2\left(1-\frac{4}{25}\right)=21$

Required equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore$ $\frac{x^2}{25}+\frac{y^2}{21}=1$

55. 2071 Set C Q.No. 2 a

Find the eccentricity and the foci of the ellipse $25x^2+4y^2=100$. [2]

SOLUTION

Given equation of ellipse is

$25x^2+4y^2=100$

or, $\frac{25x^2}{100}+\frac{4y^2}{100}=1$

or, $\frac{x^2}{4}+\frac{y^2}{25}=1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get

$a^2=4$, $b^2=25$

$\Rightarrow$ a = 2, b = 5

Since b > a, so the major axis is along y-axis

We have,

Eccentricity (e) $=\sqrt{1-\frac{a^2}{b^2}}$

$=\sqrt{1-\frac{4}{25}}=\sqrt{\frac{21}{25}}=\frac{\sqrt{21}}{5}$

Foci = $(0, \pm be)=\left(0, 5.\frac{\sqrt{21}}{5}\right)=(0, \pm\sqrt{21})$

56. 2072 Set E Q.No. 2a

Find the eccentricity and the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$. [2]

SOLUTION

Comparing $\frac{x^2}{9}+\frac{y^2}{25}=1$ with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$a^2=9$, $b^2=25$

$\Rightarrow$ a = 3, b = 5

Since b > a, so the major axis is along the y-axis.

We have,

Eccentricity (e) $=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{9}{25}}$

$=\sqrt{\frac{16}{25}}=\frac{4}{5}$

Foci = $(0, \pm be)=\left(0, \pm 5.\frac{4}{5}\right)=(0, \pm 4)$

57. 2072 Supp Q.No. 2a

Find the eccentricity and the coordinates of the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ [2]

➥ Please refer to 2070 Set C Q.No. 2a

58. 2074 Supp Q.No. 2a

Find the foci of the ellipse $\frac{x^2}{8}+\frac{(y-2)^2}{12}=1$. [2]

SOLUTION

Comparing $\frac{x^2}{8}+\frac{(y-2)^2}{12}=1$

with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we have

h = 0, k = 2

$a^2=8$, $b^2=12$

$\Rightarrow$ $a=2\sqrt{2}$, $b=2\sqrt{3}$

Since b > a, so the major axis is parallel to y-axis.

We have,

Eccentricity (e) $=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{8}{12}}=\frac{1}{\sqrt{3}}$

Foci = $(h, k\pm be)=\left(0, 2\pm 2\sqrt{3}.\frac{1}{\sqrt{3}}\right)$

$=(0, 2\pm 2)=(0, 0)$ and $(0, 4)$

59. 2075 Set C Q.No. 2a

Find the foci of the ellipse $\frac{(x-1)^2}{25}+\frac{y^2}{16}=1$ [2]

SOLUTION

Given equation of ellipse is $\frac{(x-1)^2}{25}+\frac{y^2}{16}=1$ ... (i)

Comparing (i) with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$,

we get, h = 1, k = 0

$a^2=25$, $b^2=16$

$\therefore$ a = 5, b = 4

Since a > b, the major axis is along x - axis.

We have,

e $=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{16}{25}}$

$=\sqrt{\frac{25-16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$

Foci = $(h\pm ae, k)$

$=\left(1\pm 5.\frac{3}{5}, 0\right)$

$=(1\pm 3, 0)$

$=(4, 0)$ and $(-2, 0)$.

4 MARKS QUESTIONS

60. 2057 Q.No. 9 b OR

Find the eccentricity, length of the latus rectum and coordinates of the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$. [4]

SOLUTION

Comparing $\frac{x^2}{16}+\frac{y^2}{4}=1$ with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we have

$a^2=16$, $b^2=4$

$\Rightarrow$ a = 4, b = 2

Since a > b, the major axis is parallel to x-axis.

We have,

Eccentricity (e) $=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{4}{16}}=\frac{\sqrt{3}}{2}$

Length of latus rectum $=\frac{2b^2}{a}=\frac{2\times 2^2}{4}=2$

Foci = $(\pm ae, 0)=\left(\pm 4.\frac{\sqrt{3}}{2}, 0\right)=(\pm 2\sqrt{3}, 0)$

61. 2058 Q.No. 9 b OR

Find the eccentricity and the foci of the ellipse. $\frac{(x+2)^2}{16}+\frac{(y-5)^2}{9}=1$ [4]

SOLUTION

Comparing $\frac{(x+2)^2}{16}+\frac{(y-5)^2}{9}=1$

with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we have

h = -2, k = 5

$a^2=16$, $b^2=9$

$\Rightarrow$ a = 4, b = 3

Since a > b, the major axis is along x-axis

We have,

Eccentricity (e) $=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$

Foci = $(h\pm ae, k)=\left(-2\pm 4.\frac{\sqrt{7}}{4}, 5\right)=(-2\pm\sqrt{7}, 5)$

62. 2059 Q.No. 9 b OR

Deduce the equation of the ellipse in the standard position if a focus is at (0, -5) and eccentricity is $\frac{1}{3}$. [4]

SOLUTION

Focus = (0, -be) = (0, -5)

$\therefore$ be = 5 ... (i)

And $e=\frac{1}{3}$

Putting the value of e in (i)

$b.\frac{1}{3}=5$

$\therefore$ b = 15

Again, we have

$e^2=1-\frac{a^2}{b^2}$

$\left(\frac{1}{3}\right)^2=1-\frac{a^2}{225}$

$\therefore$ $a^2=200$

The equation of ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\therefore$ $\frac{x^2}{200}+\frac{y^2}{225}=1$

63. 2060 Q.No. 9 b OR

Find the eccentricity and the foci of the ellipse: $\frac{x^2}{8}+\frac{(y-2)^2}{12}=1$ [4]

SOLUTION

Comparing $\frac{x^2}{8}+\frac{(y-2)^2}{12}=1$

with $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we have

h = 0, k = 2

$a^2=8$, $b^2=12$

$\Rightarrow$ $a=2\sqrt{2}$, $b=2\sqrt{3}$

Since b > a, so the major axis is parallel to y-axis.

We have,

Eccentricity (e) $=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{8}{12}}=\frac{1}{\sqrt{3}}$

Foci = $(h, k\pm be)=\left(0, 2\pm 2\sqrt{3}.\frac{1}{\sqrt{3}}\right)$

$=(0, 2\pm 2)=(0, 0)$ and $(0, 4)$

64. 2061 Q.No. 5 c

Find the equation of the ellipse in the standard position with a focus at (0, -5) and eccentricity $\frac{1}{3}$. [4]

➥ Please refer to 2059 Q.No. 9b OR

65. 2062 Q.No. 9 b OR

Find the eccentricity and the coordinates of the foci of the ellipse: $\frac{x^2}{8}+\frac{(y-2)^2}{12}=1$ [4]

➥ Please refer to 2060 Q.No. 9b OR

66. 2064 Q.No. 9 b OR

Find the eccentricity and the foci of the ellipse: $9x^2+5y^2-30y=0$ [4]

SOLUTION

Given equation of ellipse is

$9x^2 + 5y^2 - 30y = 0$

or, $9x^2 + 5(y^2 - 6y) = 0$

or, $9x^2 + 5(y^2 - 6y + 9 - 9) = 0$

or, $9x^2 + 5(y - 3)^2 - 45 = 0$

or, $9x^2 + 5(y - 3)^2 = 45$

or, $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$ ...(i)

Comparing (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

we have

$h = 0,\quad k = 3$

$a^2 = 5,\quad b^2 = 9$

$\Rightarrow a = \sqrt5,\quad b = 3$

Since b > a, so major axis is along y-axis. We have,

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Coordinates of vertices = $(h, k \pm b) = (0, 3 \pm 3)$

$= (0, 3 - 3)$ and $(0, 3 + 3)$

$= (0, 0)$ and $(0, 6)$

Foci = $(h, k \pm be) = \left(0, 3 \pm 3 \cdot \frac{2}{3}\right) = (0, 3 \pm 2)$

$= (0,1)$ and $(0, 5)$

67. 2065 Q.No 9 b OR

Show that: $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ represents the equation of an ellipse. Find its centre, vertex and focus. [4]

SOLUTION

Given equation is

$9x^2 + 4y^2 - 18x - 16y - 11 = 0$

or, $9(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = 36$

or, $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$ ...(i)

Since equation (i) is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, it represents an equation of ellipse.

Here,

$h = 1,\quad k = 2$

$a^2 = 4,\quad b^2 = 9$

$\Rightarrow a = 2,\quad b = 3$

Since b > a, so the major axis is parallel to y-axis

Centre $= (h, k) = (1, 2)$

Vertex $= (h, k \pm b)$

$= (1, 2 \pm 3) = (1, 2 - 3)$ and $(1, 2 + 3)$

$= (1, -1)$ and $(1, 5)$

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt5}{3}$

Focus = $(h, k \pm be) = \left(1, 2 \pm 3\frac{\sqrt5}{3}\right) = (1, 2 \pm \sqrt5)$

68. 2066 C Q.No. 9 b OR

Find the eccentricity and the foci of the ellipse: $9x^2 + 5y^2 - 30y = 0$ [4]

➥ Please refer to 2064 Q.No. 9b OR

69. 2066 Q.No. 9 b

Find the equation of the ellipse in the standard position whose latus rectum is equal to half its major axis and which passes through the point $(\sqrt6, 1)$. [4]

SOLUTION

Let the equation of ellipse in standard form be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (i)

By question, latus rectum $= \frac{1}{2}$ major axis

$\frac{2b^2}{a} = \frac{1}{2} \cdot (2a)$

$2b^2 = a^2$

From (i)

$\frac{x^2}{2b^2} + \frac{y^2}{b^2} = 1$ ...(ii)

Since it passes through the point $(\sqrt6, 1)$, so

$\frac{6}{2b^2} + \frac{1}{b^2} = 1$

or, $\frac{3}{b^2} + \frac{1}{b^2} = 1$

$\therefore b^2 = 4$

Putting the value of b^2 in (ii), we get

$\frac{x^2}{2 \times 4} + \frac{y^2}{4} = 1$

$\therefore x^2 + 2y^2 = 8$

70. 2067 Q.No. 9b OR

Show that $x^2 + 4y^2 - 4x + 24y + 24 = 0$ represents the equation of an ellipse. Find centre, vertex and focus. [4]

SOLUTION

Given equation is

$x^2 + 4y^2 - 4x + 24y + 24 = 0$

or, $x^2 - 4x + 4y^2 + 24y + 24 = 0$

or, $x^2 - 4x + 4 - 4 + 4(y^2 + 6y + 9 - 9) + 24 = 0$

or, $(x - 2)^2 + 4(y + 3)^2 = 16$

or, $\frac{(x-2)^2}{16} + \frac{(y+3)^2}{4} = 1$ ...(i)

Since equation (i) is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

it represents an equation of ellipse.

Here,

$h = 2,\quad k = -3$

$a^2 = 16,\quad b^2 = 4$

$\Rightarrow a = 4,\quad b = 2$

Since a > b, so major axis is along x-axis.

Centre $= (h, k) = (2, -3)$

We have,

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{16}} = \sqrt{\frac{12}{16}}$

$= \frac{2\sqrt3}{4} = \frac{\sqrt3}{2}$

Vertex = $(h \pm a, k) = (2 \pm 4, -3) = (6, -3)$ and $(-2, -3)$

Foci = $(h \pm ae, k) = \left(2 \pm 4 \cdot \frac{\sqrt3}{2}, -3\right) = (2 \pm 2\sqrt3, -3)$

71. 2068 Q.No. 9b OR

Find the coordinates of the vertices, the eccentricity and the coordinates of the foci of the ellipse $25x^2 + 4y^2 = 100$. [4]

SOLUTION

Given equation of ellipse is

$25x^2 + 4y^2 = 100$

or, $\frac{25x^2}{100} + \frac{4y^2}{100} = 1$

or, $\frac{x^2}{4} + \frac{y^2}{25} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 4,\quad b^2 = 25$

$\Rightarrow a = 2,\quad b = 5$

Since b > a, so the major axis is along y-axis

We have,

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}}$

$= \sqrt{1 - \frac{4}{25}} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$

Foci = $(0, \pm be) = \left(0, 5 \cdot \frac{\sqrt{21}}{5}\right) = (0, \pm\sqrt{21})$

72. 2069 (Set A) Q.No. 6a or

Find the eccentricity, the coordinates of the vertices and the foci of ellipse $9x^2 + 5y^2 - 30y = 0$. [4]

SOLUTION

Given equation of ellipse is

$9x^2 + 5y^2 - 30y = 0$

or, $9x^2 + 5(y^2 - 6y) = 0$

or, $9x^2 + 5(y^2 - 6y + 9 - 9) = 0$

or, $9x^2 + 5(y - 3)^2 - 45 = 0$

or, $9x^2 + 5(y - 3)^2 = 45$

or, $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$ ...(i)

Comparing (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we have

$h = 0,\quad k = 3$

$a^2 = 5,\quad b^2 = 9$

$\Rightarrow a = \sqrt5,\quad b = 3$

Since b > a, so major axis is along y-axis. We have,

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Coordinates of vertices = $(h, k \pm b)$

$= (0, 3 \pm 3)$

$= (0, 3 - 3)$ and $(0, 3 + 3)$

$= (0, 0)$ and $(0, 6)$

Foci = $(h, k \pm be) = \left(0, 3 \pm 3 \cdot \frac{2}{3}\right) = (0, 3 \pm 2)$

$= (0,1)$ and $(0, 5)$

73. 2069 (Set A) Old Q.No. 9b OR

Show that $9x^2 + 5y^2 - 30y = 0$ represents the equation of an ellipse. Find the eccentricity, the coordinates of the centre and the foci. [4]

SOLUTION

Given equation of ellipse is

$9x^2 + 5y^2 - 30y = 0$

or, $9x^2 + 5(y^2 - 6y) = 0$

or, $9x^2 + 5(y^2 - 6y + 9 - 9) = 0$

or, $9x^2 + 5(y - 3)^2 - 45 = 0$

or, $9x^2 + 5(y - 3)^2 = 45$

or, $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$ ...(i)

Since equation (i) is in the form of $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

it represents an equation of ellipse.

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Coordinates of centre $= (h, k) = (0, 3)$

Foci = $(h, k \pm be) = \left(0, 3 \pm 3 \cdot \frac{2}{3}\right) = (0, 3 \pm 2)$

$= (0,1)$ and $(0, 5)$

74. 2069 Old (Set B) Q.No. 9b OR

Find the equation of the ellipse whose latus rectum is 3 and eccentricity is $\frac{1}{\sqrt2}$. [4]

SOLUTION

Latus rectum = 3

or, $\frac{2b^2}{a} = 3$

$\therefore b^2 = \frac{3a}{2}$ ... (i)

Eccentricity (e) $= \frac{1}{\sqrt2}$

We have

$e^2 = 1 - \frac{b^2}{a^2}$

or, $\left(\frac{1}{\sqrt2}\right)^2 = 1 - \frac{3a/2}{a^2}$

or, $\frac{1}{2} = 1 - \frac{3}{2a}$

or, $\frac{3}{2a} = \frac{1}{2}$

$\therefore a = 3$

Now from (i)

$b^2 = \frac{3 \times 3}{2} = \frac{9}{2}$

The equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or, $\frac{x^2}{9} + \frac{y^2}{9/2} = 1$

or, $\frac{x^2}{9} + \frac{2y^2}{9} = 1$

$\therefore x^2 + 2y^2 = 9$

75. 2071 Set D Q.No. 6 a OR

Find the equation of the ellipse whose distance between two foci is 8 and the semi-latus rectum is 6. **[4]**

SOLUTION

Distance between two foci

$2ae = 8$

$\therefore$ $ae = 4$ ...(i)

Semi-latus rectum = 6

$\frac{1}{2}\left(\frac{2b^2}{a}\right) = 6$

or, $b^2 = 6a$ ...(ii)

We have,

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}}$

or, $a^2e^2 = a^2 - b^2$

or, $(4)^2 = a^2 - 6a$ [using (i) and (ii)]

or, $a^2 - 6a - 16 = 0$

or, $a^2 - 8a + 2a - 16 = 0$

or, $a(a - 8) + 2(a - 8) = 0$

or, $a = 8, a = -2$ (rejected)

$\therefore$ When $a = 8$, from (ii) $b^2 = 6 \times 8 = 48$

The equation of ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\therefore$ $\frac{x^2}{64} + \frac{y^2}{48} = 1$

76. 2071 Supp. Q.No. 6b OR

Find the vertices, eccentricities, foci and length of major axes of the ellipse $\frac{(x+5)^2}{9} + \frac{(y-1)^2}{4} = 1$ **[4]**

SOLUTION

Comparing $\frac{(x+5)^2}{9} + \frac{(y-1)^2}{4} = 1$ with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we have

$h = -5, k = 1$

$a^2 = 9, \quad b^2 = 4$

$\Rightarrow a = 3, \quad b = 2$

Since $a > b$, so the major axis is along x-axis.

Vertices $= (h \pm a, k) = (-5 \pm 3, 1) = (-8, 1)$ and $(-2, 1)$

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{4}{9}} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3}$

Foci $= (h \pm ae, k) = \left(-5 \pm 3.\frac{\sqrt{5}}{3}, 1\right) = (-5 \pm \sqrt{5}, 1)$

Length of major axis $= 2a = 2 \times 3 = 6$

77. 2072 Set D Q.No. 6a OR

Find the eccentricity and coordinates of the foci of the curve $\frac{(x+6)^2}{4} + \frac{y^2}{36} = 1$. **[4]**

SOLUTION

Given equation of ellipse is $\frac{(x+6)^2}{4} + \frac{y^2}{36} = 1$ ... (i)

Comparing equation (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

We have, $h = -6, k = 0$

$a^2 = 4, \quad b^2 = 36$

$\therefore$ $a = 2, \quad b = 6$

Since $b > a$, so the major axis is along y-axis.

We have,

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{36}} = \sqrt{1 - \frac{1}{9}}$

$= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

Foci $= (h, k \pm be) = \left(-6, 0 \pm 6.\frac{2\sqrt{2}}{3}\right) = (-6, \pm 4\sqrt{2})$

78. 2073 Set C Q.No. 6a OR

Find the equation of the ellipse whose major axis is twice its minor axis and passes through the point (0,1). **[4]**

SOLUTION

Let the equation of ellipse with $a > b$ be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

By given, major axis = 2 (minor axis)

i.e. $2a = 2(2b)$

$\therefore$ $a = 2b$ ...(ii)

Also, if equation (i) passes through the point (0, 1), then

$\frac{0}{a^2} + \frac{1}{b^2} = 1$

or, $b^2 = 1$

$\therefore$ $b = 1$

Then from (ii), $a = 2 \times 1 = 2$

Putting the values of a and b in (i), we get

$\frac{x^2}{4} + \frac{y^2}{1} = 1$

$\therefore$ $x^2 + 4y^2 = 4$, which is the required equation of ellipse.

79. 2073 Set D Q.No. 6a OR

Find the centre, eccentricity and foci of the ellipse $9x^2 + 5y^2 - 30y = 0$

SOLUTION

Given equation of ellipse is

$9x^2 + 5y^2 - 30y = 0$

or, $9x^2 + 5(y^2 - 6y) = 0$

or, $9x^2 + 5(y^2 - 6y + 9 - 9) = 0$

or, $9x^2 + 5(y - 3)^2 - 45 = 0$

or, $9x^2 + 5(y - 3)^2 = 45$

or, $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$ ...(i)

Comparing equation (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2}$

we get

$a^2 = 5, b^2 = 9, h = 0$ and $k = 3$

Coordinates of centre $= (h, k) = (0, 3)$

Eccentricity (e) $= \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Foci $= (h, k \pm be) = \left(0, 3 \pm 3.\frac{2}{3}\right)$

$= (0, 3 \pm 2) = (0,1)$ and $(0, 5)$

80. 2073 Supp Q.No. 6a or

Find the vertices, eccentricity and foci of the ellipse $5x^2 + 9y^2 = 45$. **[4]**

SOLUTION

Given equation of ellipse is

$5x^2 + 9y^2 = 45$

or, $\frac{5x^2}{45} + \frac{9y^2}{45} = \frac{45}{45}$

or, $\frac{x^2}{9} + \frac{y^2}{5} = 1$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 9, b^2 = 5$

$a = 3, b = \sqrt{5}$

Since $a > b$, major axis is along x – axis.

Vertices $= (\pm a, 0) = (\pm 3, 0)$

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$

Foci $= (\pm ae, 0) = (\pm 3 \times \frac{2}{3}, 0)$

$= (\pm 2, 0)$

81. 2074 Set A Q.No. 6b OR

Find the vertices, eccentricities, foci and length of major axis of the ellipse $\frac{x^2}{5} + \frac{(y+2)^2}{3} = 1$. **[4]**

SOLUTION

Given equation of ellipse is $\frac{x^2}{5} + \frac{(y+2)^2}{3} = 1$ ... (i)

Comparing equation (i) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$,

we get $h = 0, k = -2, a^2 = 5, b^2 = 3$

$\therefore$ $a = \sqrt{5}, b = \sqrt{3}$

Since $a > b$, the major axis is along x-axis

Vertices $= (h \pm a, k) = (0 \pm \sqrt{5}, -2) = (\pm\sqrt{5}, -2)$

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{3}{5}} = \sqrt{\frac{2}{5}}$

Foci $= (h \pm ae, k) = (0 \pm \sqrt{5}.\sqrt{\frac{2}{5}}, -2)$

$= (\pm\sqrt{2}, -2)$

Length of major axis $= 2a = 2 \times \sqrt{5} = 2\sqrt{5}$

82. 2074 Set B Q.No. 6a OR

Find the eccentricity and the coordinates of the foci of the ellipse: $x^2 + 4y^2 - 4x + 24y + 24 = 0$ **[4]**

➥ Pleaser refer to 2067 Q. No. 9b OR

83. 2075 Set A Q.No. 6a OR

Find the equation of the ellipse in the standard form with a vertex at (0, 8) and passing through $\left(3, \frac{32}{5}\right)$. **[4]**

SOLUTION

Let the equation of ellipse in standard form be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (1)

Given, vertex = (0, 8)

i.e. (0, b) = (0, 8)

$\therefore$ $b = 8$

Putting the value of b in (1), we get,

$\frac{x^2}{a^2} + \frac{y^2}{8^2} = 1$

or, $\frac{x^2}{a^2} + \frac{y^2}{64} = 1$ ... (2)

If equation (2) passes through the points $\left(3, \frac{32}{5}\right)$ then

$\frac{3^2}{a^2} + \frac{\left(\frac{32}{5}\right)^2}{64} = 1$

or, $\frac{9}{a^2} + \frac{(32)^2}{25 \times 64} = 1$

or, $\frac{9}{a^2} + \frac{16}{25} = 1$

or, $\frac{9}{a^2} = 1 - \frac{16}{25}$

or, $\frac{9}{a^2} = \frac{25 - 16}{25}$

or, $\frac{9}{a^2} = \frac{9}{25}$

$\therefore$ $a^2 = 25$

Substituting the value of a^2 in (2), we get,

$\frac{x^2}{25} + \frac{y^2}{64} = 1$ which is the required equation of ellipse.

84. 2075 Set B Q.No. 6b OR

Find the vertices, eccentricities, foci and length of major axis of the ellipse $\frac{(x+2)^2}{2} + y^2 = 5$.

SOLUTION

Given, equation of ellipse is $\frac{(x+2)^2}{2} + y^2 = 5$

Dividing both sides by 5, we get,

$\frac{(x+2)^2}{10} + \frac{y^2}{5} = 1$ ... (1)

Comparing (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get,

$h = -2, k = 0$

$a^2 = 10, b^2 = 5$.

$a = \sqrt{10}, b = \sqrt{5}$

Since $a > b$, the major axis is along x - axis.

Vertices $= (h \pm a, k)$
$= (-2 \pm \sqrt{10}, 0)$

Eccentricity (e) $= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{10}}$
$= \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Foci $= (h \pm ae, k)$
$= (-2 \pm \sqrt{10} \cdot \frac{1}{\sqrt{2}}, 0)$
$= (-2 \pm \sqrt{5}, 0)$

Length of major axis $= 2a = 2 \times \sqrt{10} = 2\sqrt{10}$.

C. HYPERBOLA

2 MARKS QUESTIONS

85. 2065 Q.No 5 c

Find the eccentricity and foci of the hyperbola $3x^2 - 4y^2 = 36$. [2]

SOLUTION

Given equation of the hyperbola is
$3x^2 - 4y^2 = 36$
or, $\frac{x^2}{12} - \frac{y^2}{9} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have
$a^2 = 12, \quad b^2 = 9$
$\Rightarrow a = 2\sqrt{3} \quad b = 3$

Eccentricity (e) $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{12}} = \frac{\sqrt{7}}{2}$

Foci $= (\pm ae, 0) = \left(\pm 2\sqrt{3} \cdot \frac{\sqrt{7}}{2}, 0\right) = (\pm\sqrt{21}, 0)$

86. 2066 C Q.No. 5 c

Find the equation to the hyperbola in the standard form with a focus at (–7, 0) and eccentricity $\frac{7}{4}$. [2]

SOLUTION

Focus $(-ae, 0) = (-7, 0)$
$\Rightarrow ae = 7$ ...(i)

Eccentricity (e) $= \frac{7}{4}$

Then from (i) $a \cdot \frac{7}{4} = 7$

$\therefore a = 4$

Again, we have

$e^2 = 1 + \frac{b^2}{a^2}$

$\frac{49}{16} = 1 + \frac{b^2}{16}$

$b^2 = 33$

The equation of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore \frac{x^2}{16} - \frac{y^2}{33} = 1$

87. 2067 Q.No. 5c

Find the equation of hyperbola in the standard form with a focus at (0, 5) and a vertex (0, – 3). [2]

SOLUTION

Vertex $(0, -b) = (0, -3)$
$\therefore b = 3$
Focus $(0, be) = (0, 5)$
$be = 5$
$3.e = 5 \quad (\because b = 3)$
$e = 5/3$

We have,
$e^2 = 1 + \frac{a^2}{b^2}$

or, $\left(\frac{5}{3}\right)^2 = 1 + \frac{a^2}{3^2}$

or, $25 = 9 + a^2$

$\therefore a^2 = 16$

The required equation of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\therefore \frac{x^2}{16} - \frac{y^2}{9} = -1$

88. 2069 (Set A) Q.No. 2a

Find the eccentricity and the foci of the hyperbola $3x^2 - 4y^2 = 36$. [2]

➥ Please refer to 2065 Q.No. 5c

89. 2070 Supp. Q.No. 2 a

Find the eccentricity of hyperbola $x^2 - 4y^2 - 12 = 0$. [2]

SOLUTION

Given equation of hyperbola
$x^2 - 4y^2 - 12 = 0$
or, $x^2 - 4y^2 = 12$
or, $\frac{x^2}{12} - \frac{y^2}{3} = 1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get
$a^2 = 12, \quad b^2 = 3$

Eccentricity (e) $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{3}{12}} = \frac{\sqrt{5}}{2}$

90. 2071 Set D Q.No. 2 a

Find the eccentricity and the foci of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$. [2]

SOLUTION

Comparing $\frac{x^2}{9} - \frac{y^2}{16} = 1$ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have
$a^2 = 9, \quad b^2 = 16$
$\Rightarrow a = 3, \quad b = 4$

We have,

Eccentricity (e) $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

Foci $= (\pm ae, 0) = \left(\pm 3 \cdot \frac{5}{3}, 0\right) = (\pm 5, 0)$

91. 2071 Old Q.No. 2 c

Find the coordinates of vertices and eccentricity of the hyperbola $\frac{x^2}{16} - \frac{y^2}{4} = 1$. [2]

SOLUTION

Comparing $\frac{x^2}{16} - \frac{y^2}{4} = 1$ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

we have
$a^2 = 16, \quad b^2 = 4$
$\Rightarrow a = 4, \quad b = 2$
Vertices $= (\pm a, 0) = (\pm 4, 0)$

Eccentricity (e) $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{4}{16}} = \frac{\sqrt{5}}{2}$

92. 2072 Set C Q.No. 2a

Find eccentricity and foci of the hyperbola $\frac{x^2}{36} - \frac{y^2}{64} = 1$ [2]

SOLUTION

Comparing $\frac{x^2}{36} - \frac{y^2}{64} = 1$ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have
$a^2 = 36, \quad b^2 = 64$
$\Rightarrow a = 6, \quad b = 4$

We have,

Eccentricity (e) $= \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{64}{36}} = \frac{5}{3}$

Foci $= (\pm ae, 0) = \left(\pm 6 \cdot \frac{5}{3}, 0\right) = (\pm 10, 0)$

93. 2072 Set D Q.No. 2a

Find the equation of the hyperbola with vertex (8, 0) and passing through the point $(8\sqrt{2}, 4)$. [2]

SOLUTION

Let the equation of hyperbola be
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ... (i)
Vertex $(a, 0) = (8, 0)$
$\therefore a = 8$
Then equation (i) becomes
$\frac{x^2}{64} - \frac{y^2}{b^2} = 1$ ...(ii)
Also, equation (ii) passes through the point $(8\sqrt{2}, 4)$
or, $\frac{(8\sqrt{2})^2}{64} - \frac{4^2}{b^2} = 1$
$2 - \frac{16}{b^2} = 1$
$b^2 = 16$
$\therefore$ Putting the value of b^2 in (ii)
$\frac{x^2}{64} - \frac{y^2}{16} = 1$ which is the required equation of hyperbola.

94. 2073 Set C Q.No. 2a

Find the foci and vertices of the hyperbola $9x^2 - 16y^2 = 144$. [2]

SOLUTION

Given hyperbola is
$9x^2 - 16y^2 = 144$
or, $\frac{x^2}{16} - \frac{y^2}{9} = 1$

Comparing this equation with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
we get
$a^2 = 16, b^2 = 9$
$\therefore a = 4, b = 3$

We have, $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

Coordinate of foci $= (\pm ae, 0) = \left(\pm 4 \cdot \frac{5}{4}, 0\right)$
$= (\pm 5, 0)$

Coordinate of vertices $= (\pm a, 0) = (\pm 4, 0)$

95. 2073 Set D Q.No. 2a

Find the equation of a hyperbola with a focus at (–7, 0) and eccentricity $\frac{7}{4}$. [2]

➥ Please refer to 2066 C Q.No. 5c

96. 2073 Supp Q.No. 2a

Determine the equation of the hyperbola with a focus at (–5, 0) and a vertex at (3, 0). [2]

SOLUTION

Vertex $= (3, 0)$
$(a, 0) = (3, 0)$
$\therefore a = 3$ ... (i)
Focus $= (-5, 0)$
$(-ae, 0) = (-5, 0)$
or, $ae = 5$
or, $3 \times e = 5$ [Using (i)]
$e = \frac{5}{3}$

Again, we have,
$e^2 = 1 + \frac{b^2}{a^2}$

$\left(\frac{5}{3}\right)^2 = 1 + \frac{b^2}{3^2}$

or, $\frac{25}{9} = \frac{9 + b^2}{9}$

or, $b^2 + 9 = 25$
$\therefore b^2 = 16$

Hence, the equation of hyperbola is
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
$\therefore \frac{x^2}{9} - \frac{y^2}{16} = 1$

97. 2074 Set A Q.No. 2a

Find the equation of a hyperbola in standard position such that its transverse and conjugate axes are respectively 4 and 5. [2]

SOLUTION

Given, length of transverse axis (2a) = 4
$\therefore a = 2$
And, length of conjugate axis (2b) = 5
$\therefore b = \frac{5}{2}$

The equation of hyperbola in standard position is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

or, $\frac{x^2}{2^2}-\frac{y^2}{(5/2)^2}=1$

or, $\frac{x^2}{4}-\frac{y^2}{(25/4)}=1$

∴ $\frac{x^2}{4}-\frac{4y^2}{25}=1$ which is the required equation of hyperbola.

98. 2074 Set B Q.No. 2a

Find the equation of hyperbola with a focus at (7, 0) and a vertex at (5, 0). [2]

SOLUTION

Given, vertex = (5, 0)

or, (a, o) = (5, 0)

∴ $a = 5$

Also, focus = (7, 0)

or, (ae, o) = (7, 0)

$ae = 7$

$5.e = 7$

∴ $e = \frac{7}{5}$

We have $e^2 = 1 + \frac{b^2}{a^2}$

or, $\left(\frac{7}{5}\right)^2 = 1 + \frac{b^2}{5^2}$

or, $\frac{35}{25} = \frac{25+b^2}{25}$

or, $25 + b^2 = 35$

∴ $b^2 = 10$

The equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

∴ $\frac{x^2}{25}-\frac{y}{10}=1$

99. 2075 Set A Q.No. 2a

Find the foci of the hyperbola $\frac{x^2}{9}-\frac{y^2}{16}=1$. [2]

➥ Please refer to 2071 Set D Q.No. 2a

4 MARKS QUESTIONS

100. 2061 Q.No. 9 b OR

Find the eccentricity and the foci of the hyperbola: $3x^2 - 4y^2 = 36$ [4]

SOLUTION

Given equation of the hyperbola is

$3x^2 - 4y^2 = 36$

or, $\frac{x^2}{12}-\frac{y^2}{9}=1$ ... (i)

Comparing (i) with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have

$a^2 = 12$, $b^2 = 9$

⇒ $a = 2\sqrt{3}$ $b = 3$

Eccentricity (e) = $\sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{9}{12}} = \frac{\sqrt{7}}{2}$

Foci = $(\pm ae, 0) = \left(\pm 2\sqrt{3}\,.\,\frac{\sqrt{7}}{2}, 0\right) = (\pm\sqrt{21}, 0)$

101. 2063 Q.No. 9 b OR

Find the eccentricity and the coordinates of the foci of the hyperbola $\frac{x^2}{16}-\frac{y^2}{4}=1$ [4]

SOLUTION

Comparing $\frac{x^2}{16}-\frac{y^2}{4}=1$ with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we have

$a^2 = 16$, $b^2 = 4$

⇒ $a = 4$, $b = 2$

We have,

Eccentricity (e) = $\sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{16}{4}} = \frac{\sqrt{5}}{2}$

Foci = $(\pm ae, 0) = \left(\pm 4\,.\,\frac{\sqrt{5}}{2}, 0\right) = (\pm 2\sqrt{5})$

102. 2066 Q.No. 9 b OR

Determine the equation of the hyperbola in the standard position with focus at (−7, 0) and eccentricity $\frac{7}{4}$. [4]

➥ Please refer to 2066 C Q.No. 5c

103. 2069 (Set B) Q.No. 6a Or

Find the equation of the hyperbola with focus at (−5, 0) and vertex at (2, 0). [4]

SOLUTION

Vertex (a, 0) = (2, 0)

∴ $a = 2$

Focus (−ae, 0) = (−5, 0)

⇒ $ae = 5$

⇒ $2\cdot e = 5$ (∵ a = 2)

⇒ $e = \frac{5}{2}$

We have,

$e^2 = 1 + \frac{b^2}{a^2}$

or, $\left(\frac{5}{2}\right)^2 = 1 + \frac{b^2}{4}$

or, $\frac{25}{4} = \frac{4+b^2}{4}$

∴ $b^2 = 21$

Hence, the required equation of hyperbola is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

or, $\frac{x^2}{4}-\frac{y^2}{21}=1$

∴ $21x^2 - 4y^2 = 84$

104. 2070 Set C Q.No. 6 a or

Find the coordinates of the vertices, the eccentricity and the coordinates of the foci of the hyperbola $5x^2 - 20y^2 - 20x = 0$. [4]

SOLUTION

Given equation of hyperbola is

$5x^2 - 20y^2 - 20x = 0$

or, $5x^2 - 20x - 20y^2 = 0$

or, $5(x^2 - 4x) - 20y^2 = 0$

or, $5(x^2 - 4x + 4 - 4) - 20y^2 = 0$

or, $5(x-2)^2 - 20 - 20y^2 = 0$

or, $5(x-2)^2 - 20y^2 = 20$

or, $\frac{(x-2)^2}{4}-\frac{y^2}{1}=1$ ... (i)

Comparing (i) with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$,

We have, $h = 2$, $k = 0$

$a^2 = 4$, $b^2 = 1$

⇒ $a = 2$, $b = 1$

Vertices= $(h \pm a, k) = (2 \pm 2, 0) = (0, 0)$ and $(4, 0)$

Eccentricity (e) = $\sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

Foci = $(h \pm ae, k) = \left(2 \pm 2\,.\,\frac{\sqrt{5}}{2}, 0\right) = (2 \pm \sqrt{5}, 0)$

105. 2070 Set D Q.No. 6 a Or

Deduce the equation of a hyperbola with a focus at (6, 0) and a vertex at (4, 0). [4]

SOLUTION

Vertex (a, 0) = (4, 0)

⇒ $a = 4$

Focus (ae, 0) = (6, 0)

⇒ $ae = 6$

⇒ $4\,.\,e = 6$ (∵ a = 4)

⇒ $e = \frac{3}{2}$

We have

$e^2 = 1 + \frac{b^2}{a^2}$

$\frac{9}{4} = 1 + \frac{b^2}{16}$

⇒ $b^2 = 20$

The equation of hyperbola is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

∴ $\frac{x^2}{16}-\frac{y^2}{20}=1$

106. 2070 (Old) Q.No. 9 b Or

Find the equation to the hyperbola in standard form whose focus is at (0, 5) and vertex at (0, −3). [4]

➥ Please refer to 2067 Q.No. 5c

107. 2071 Set C Q.No. 6 a OR

Find the equation of the hyperbola with a focus at (0,5) and a vertex at (0,−3). [4]

➥ Please refer to 2067 Q.No. 5c

108. 2072 Set E Q.No. 6b OR

Find the equation of the hyperbola with vertex at (0, 8) and passing through the point $(4, 8\sqrt{2})$. [4]

SOLUTION

Vertex (0, b) = (0, 8)

∴ $b = 8$

The equation of hyperbola is

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

or, $\frac{x^2}{a^2}-\frac{y^2}{64}=-1$ ... (i)

If equation (i) passes through the point $(4, 8\sqrt{2})$ then

$\frac{4^2}{a^2}-\frac{(8\sqrt{2})^2}{64}=-1$

or, $a^2 = 16$

Putting the value of a^2 in equation (i), we get

$\frac{x^2}{16}-\frac{y^2}{64}=-1$ which is the required equation of hyperbola.

109. 2072 Supp Q.No. 6a OR

Find the eccentricity, coordinates of the vertices and the foci of the hyperbola $5x^2 - 20y^2 - 20x = 0$. [4]

➥ Please refer to 2070 Set C Q.No. 6a OR

110. 2074 Supp Q.No. 6a OR

Obtain the equation to the hyperbola in the standard form with a focus at (−7,0) and eccentricity $\frac{7}{4}$. [4]

➥ Please refer to 2066 Q.N. 9b OR

111. 2075 Set C Q.No. 6a OR

Find the vertices, centre, eccentricity and foci of the hyperbola $9(x-1)^2 - 16(y+2)^2 = 144$. [4]

SOLUTION

Given hyperbola is

$9(x-1)^2 - 16(y+2)^2 = 144$

Dividing both sides by 144 we get,

$\frac{9(x-1)^2}{144}-\frac{16(y+2)^2}{144}=\frac{144}{144}$

or, $\frac{(x-1)^2}{16}-\frac{(y+2)^2}{9}=1$ ... (1)

Comparing (1) with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, we get,

$h = 1$, $k = -2$

$a^2 = 16$, $b^2 = 9$

∴ $a = 4$, $b = 3$

Vertices $= (h \pm a, k)$

$= (1 \pm 4, -2)$

$= (5, -2)$ and $(-3, -2)$

Centre $= (h, k) = (1, -2)$

Eccentricity (e) = $\sqrt{1+\frac{b^2}{a^2}}$

$= \sqrt{1+\frac{9}{16}} = \sqrt{\frac{16+9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

Foci $= (h \pm ae, k)$

$= \left(1 \pm 4.\frac{5}{4}, -2\right)$

$= (1 \pm 5, -2)$

$= (1 + 5, -2)$ and $(1 - 5, -2)$

$= (6, -2)$ and $(-4, -2)$

UNIT 5

CO-ORDINATES IN SPACE

A. CO-ORDINATE IN SPACE

2 MARKS QUESTIONS

1. 2056 Q.No. 5b

Show that the line joining the points (1, 2, 3) and (4, 5, 7) is parallel to the line joining the points (-4, 3, -6) and (2, 9, 2). [2]

SOLUTION

The direction ratios of the line joining the points (1, 2, 3) and (4, 5, 7) are

$a_1 = 4 - 1 = 3,$

$b_1 = 5 - 2 = 3,$

$c_1 = 7 - 3 = 4$

The direction ratios of the line joining the points (-4, 3, -6) and (2, 9, 2) are

$a_2 = 2 + 4 = 6, \quad b_2 = 9 - 3 = 6, \quad c_2 = 2 + 6 = 8$

Here,

$\frac{a_1}{a_2} = \frac{3}{6} = \frac{1}{2}$

$\frac{b_1}{b_2} = \frac{3}{6} = \frac{1}{2}$

$\frac{c_1}{c_2} = \frac{4}{8} = \frac{1}{2}$

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence, the given lines are parallel.

2. 2057 Q.No. 5b

What are direction cosines of a line? Prove: $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$ [2]

SOLUTION

If a line makes angle α, β and γ with positive x-axis, y-axis and z-axis respectively then cos α, cos β and cos γ are called the d.c's of the line.

Now,

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

$= 1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma$

$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$

$= 3 - 1 \quad (\because \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1)$

$= 2$

3. 2058 Q.No. 5b

Find direction cosines of a line joining the points (-1, 2, 5) and (-2, 4, 3). [2]

SOLUTION

Given points are A(-1, 2, 5) and B(-2, 4, 3).

$AB = \sqrt{(-2+1)^2 + (4-2)^2 + (3-5)^2}$

$= \sqrt{1+4+4} = 3.$

If l, m, n be the d.c's of AB, then

$l = \frac{x_2 - x_1}{AB} = \frac{-2+1}{3} = \frac{-1}{3}$

$m = \frac{y_2 - y_1}{AB} = \frac{4-2}{3} = \frac{2}{3}$

$n = \frac{z_2 - z_1}{AB} = \frac{3-5}{3} = \frac{-2}{3}$

$\therefore$ Required d.c's are $\frac{-1}{3}, \frac{2}{3}, \frac{-2}{3}$.

4. 2059 Q.No. 5b

Find the angle between the lines wh... direction cosines are proportional to 1, 2, 4 a... -2, -9, 5.

SOLUTION

Given, $a_1 = 1, \quad b_1 = 2, \quad c_1 = 4$

and $a_2 = -2, \quad b_2 = -9, \quad c_2 = 5$

If θ be the angle between the lines, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{1\cdot(-2) + 2\cdot(-9) + 4\cdot 5}{\sqrt{1^2+2^2+4^2}\sqrt{(-2)^2+(-9)^2+5^2}}$$

$$= \frac{-2-18+20}{\sqrt{21}\sqrt{110}} = 0 = \cos 90°$$

$\therefore \theta = 90°$

5. 2060 Q.No. 5b

Find the co-ordinates of the point which divi... the line joining (2, -4, 3) and (5, 5, -6) in the r... 2:1.

SOLUTION

Here, $(x_1, y_1, z_1) = (2, -4, 3)$

$(x_2, y_2, z_2) = (5, 5, -6)$

$m_1 : m_2 = 2 : 1$

$(x, y, z) = ?$

From section formula, we have

$$(x, y, z) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, \frac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$$

$$= \left(\frac{2\times5 + 1\times2}{2+1}, \frac{2\times5 + 1\times(-4)}{2+1}, \frac{2\times(-6) + 1\times 3}{2+1}\right)$$

$$= \left(\frac{12}{3}, \frac{6}{3}, \frac{-9}{3}\right) = (4, 2, -3)$$

6. 2061 Q.No. 5b

If the section of two points P(2, -4, 3) ... Q(x, y, z) in the ratio 2:1 is (-2, 2, -3), then fin...

SOLUTION

Given relations are

$l + m + n = 0 \quad \ldots$ (i)

and $2lm + 2ln - mn = 0 \quad \ldots$ (ii)

Eliminating n from (i) and (ii), we have

$2lm + 2l(-l - m) - m(-l - m) = 0$

or, $2lm - 2l^2 - 2lm + lm + m^2 = 0$

or, $2l^2 - lm - m^2 = 0$

or, $(l - m)(2l + m) = 0$

Either, $l - m = 0 \quad \ldots$(iii)

or, $2l + m = 0 \quad \ldots$(iv)

Solving (i) and (iii) by cross multiplication

$$\frac{l}{1\cdot 0 - 1\cdot(-1)} = \frac{m}{1\cdot1 - 1\cdot0} = \frac{n}{1\cdot(-1) - 1\cdot1}$$

or, $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2} = \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}} = \frac{1}{\sqrt6}$

$\therefore l = \frac{1}{\sqrt6}, m = \frac{1}{\sqrt6}, n = \frac{-2}{\sqrt6}$

which are the d.c's of first line.

Again, from (i) and (iv) we have

$l + m + n = 0$

$2l + m + 0\cdot n = 0$

By the rule of cross multiplication, we have

$$\frac{l}{1\cdot0 - 1\cdot1} = \frac{m}{1\cdot2 - 0\cdot1} = \frac{n}{1\cdot1 - 1\cdot2}$$

or, $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

or, $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = \frac{1}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{1}{\sqrt6}$

$\therefore l = \frac{1}{\sqrt6}, m = \frac{-2}{\sqrt6}, n = \frac{1}{\sqrt6}$

If θ be the angle between two lines, then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$= \frac{1}{\sqrt6}\cdot\frac{1}{\sqrt6} + \frac{1}{\sqrt6}\cdot\left(\frac{-2}{\sqrt6}\right) + \left(\frac{-2}{\sqrt6}\right)\cdot\frac{1}{\sqrt6}$

$= \frac{1-2-2}{6} = \frac{-1}{2} = \cos 120°$

$\therefore \theta = 120°$

7. 2062 Q.No. 5b

Show that P (1, 2, 3), Q(4, 0, 4) and R(-2, 4, 2) are collinear. [2]

SOLUTION

Given points are A(1, 2, 3), B(4, 0, 4) and C(-2, 4, 2).

$AB = \sqrt{(4-1)^2 + (0-2)^2 + (4-3)^2} = \sqrt{14}$

$BC = \sqrt{(-2-4)^2 + (4-0)^2 + (2-4)^2} = \sqrt{56}$

$= 2\sqrt{14}$

$AC = \sqrt{(-2-1)^2 + (4-2)^2 + (2-3)^2} = \sqrt{14}$

Here $AB + AC = \sqrt{14} + \sqrt{14} = 2\sqrt{14} = BC$

$\therefore$ The points A, B and C are collinear.

8. 2063 Q.No. 6a

Find the direction cosines of a line whose direction ratios are 1, 2, 2. [2]

SOLUTION

Given direction ratios are

$a = 1, b = 2, c = 2$

If l, m, n be the d.c's of the line, then

$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{1^2+2^2+2^2}} = \frac{1}{3}$

$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{1^2+2^2+2^2}} = \frac{2}{3}$

$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{1^2+2^2+2^2}} = \frac{2}{3}$

Hence, the required d.c's are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$.

9. 2064 Q.No. 5 a

Prove that the points (-4, 9, 6), (0, 7, 10) and (-1, 6, 6) are the vertices of a right angled isosceles triangle. [2]

SOLUTION

Let A(-4, 9, 6), B(0, 7, 10) and C(-1, 6, 6) be the given points. Now,

$AB = \sqrt{(0+4)^2 + (7-9)^2 + (10-6)^2}$

$= \sqrt{16+4+16} = 6$

$BC = \sqrt{(-1-0)^2 + (6-7)^2 + (6-10)^2} = \sqrt{18}$

$AC = \sqrt{(-1+4)^2 + (6-9)^2 + (6-6)^2} = \sqrt{18}$

Since BC = AC, so ΔABC is an isosceles triangle.

Also, $BC^2 + AC^2 = 18 + 18 = 36 = (6)^2 = (AB)^2$

$\therefore \angle C = 90°$

Hence, ΔABC is a right angled isosceles triangle.

10. 2065 Q. No. 5 b

Show that the points A (1, 2, 3), B (4, 0, 4) and C (-2, 4, 2) are collinear. [2]

SOLUTION

Please refer to 2062 Q.No. 5b

11. 2066 Q.No. 5b

Find the ratio in which the line joining the points (2, 4, 5) and (3, 5, -4) is divided by xy-plane. [2]

SOLUTION

Here, $(x_1, y_1, z_1) = (2, 4, 5)$

$(x_2, y_2, z_2) = (3, 5, -4)$

In xy-plane, we have z = 0.

Now, from section formula, we have

$z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$

or, $0 = \frac{m_1\cdot(-4) + m_2\cdot(5)}{m_1 + m_2}$

or, $4m_1 = 5m_2$

$\therefore \frac{m_1}{m_2} = \frac{5}{4}$

Required ratio is 5 : 4.

12. 2067 Q.No. 5 b

The section of two points P(2, -4, 3) and Q(x, y, z) in the ratio 2 : 1 is (-2, 2, -3). Find Q. [2]

SOLUTION

Please refer to 2061 Q.No. 5b

13. 2068 Q.No. 5c

Show that the points (3, 0, 1), (2, 2, 2), (-1, 3, 3) and (0, 1, 2) are the vertices of a parallelogram. [2]

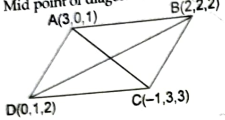
SOLUTION

Let A(3, 0, 1), B(2, 2, 2), C(-1, 3, 3) and D(0, 1, 2) be given points.

Mid point of diagonal AC

$= \left(\frac{3-1}{2}, \frac{0+3}{2}, \frac{1+3}{2}\right) = \left(1, \frac{3}{2}, 2\right)$

Mid point of diagonal BD



$= \left(\frac{2+0}{2}, \frac{2+1}{2}, \frac{2+2}{2}\right) = \left(1, \frac{3}{2}, 2\right)$

Since the coordinates of the midpoints of the diagonals AC and BD are the same, so diagonals AC and BD bisects each other.

∴ ABCD is a parallelogram.

14. 2069 (Set B) Q.No. 2b

Find the direction cosines of a line which are equally inclined to the axes. [2]

SOLUTION

Given that the line is equally inclined to the axes. So $\alpha = \beta = \gamma$. Hence, the d.c's are

$l = \cos\alpha$, $m = \cos\alpha$ and $n = \cos\alpha$

We have,

$l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

or, $3\cos^2\alpha = 1$

∴ $\cos\alpha = \pm\frac{1}{\sqrt{3}}$

∴ Required d.c's are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.

15. 2070 Set C Q.No. 2 b

Show that the direction cosines of a line equally inclined to the axes are: $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$. [2]

SOLUTION

Please refer to 2069 (Set B) Q.No. 2b

16. 2070 Supp. Q.No. 2 b

Find the locus of a point which is equidistant from the points (1, 2, 3) and (3, 2, -1). [2]

SOLUTION

Please see Model Set I, Q.No. 2b

17. 2071 Set D Q.No. 2 b

Find the direction cosines of a line passing through the points M(-2, 4, 3) and N(-1, 2, 5). [2]

SOLUTION

Given points are M(-2, 4, 3) and N(-1, 2, 5)

Now,

$MN = \sqrt{(-1+2)^2 + (2-4)^2 + (5-3)^2} = 3$

If l, m, n be the direction cosines of line then

$l = \frac{x_2 - x_1}{MN} = \frac{-1+2}{3} = \frac{1}{3}$

$m = \frac{y_2 - y_1}{MN} = \frac{2-4}{3} = \frac{-2}{3}$

$n = \frac{z_2 - z_1}{MN} = \frac{5-3}{3} = \frac{2}{3}$

∴ Required direction cosines of MN are $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$

18. 2072 Set C Q.No. 2b

Find the ratio in which the line joining points P(-2, 4, 7) and Q(3, -5, -1) is divided by the yz- plane.

SOLUTION

Given points are $(x_1, y_1, z_1) = (-2, 4, 7)$ and $(x_2, y_2, z_2) = (3, -5, -1)$

The x-coordinate of any point on the yz plane is 0. Now, from section formula, we have

$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

or, $0 = \frac{m_1 . 3 + m_2 . (-2)}{m_1 + m_2}$

or, $3m_1 = 2m_2$

or, $\frac{m_1}{m_2} = \frac{2}{3}$

∴ $m_1 : m_2 = 2 : 3$

19. 2072 Set D Q.No. 2b

If P and Q denote the coordinates (2, 6, 2) and (4, 5, 0) respectively, find the direction cosines of the line PQ.

SOLUTION

Given points are P(2, 6, 2) and Q(4, 5, 0)

$PQ = \sqrt{(4-2)^2 + (5-6)^2 + (0-2)^2} = 3$

If l, m, n be the direction of PQ, then

$l = \frac{x_2 - x_1}{PQ} = \frac{4-2}{3} = \frac{2}{3}$

$m = \frac{y_2 - y_1}{PQ} = \frac{5-6}{3} = -\frac{1}{3}$

$n = \frac{z_2 - z_1}{PQ} = \frac{0-2}{3} = \frac{-2}{3}$

∴ Required direction cosines of PQ are $\frac{2}{3}, -\frac{1}{3}, \ldots$

20. 2072 Set E Q.No. 2b

Find the angle between the two lines whose direction ratios are 2, 3, 4 and 1, -2, 1.

SOLUTION

Here $a_1 = 2$, $b_1 = 3$, $c_1 = 4$

and $a_2 = 1$, $b_2 = -2$, $c_2 = 1$

If θ be the angle between the two lines

$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$= \frac{2 . 1 + 3 . (-2) + 4 . 1}{\sqrt{2^2 + 3^2 + 4^2}\sqrt{1^2 + (-2)^2 + 1^2}} = 0 = \cos\frac{\pi}{2}$

∴ $\theta = \frac{\pi}{2}$

21. 2072 Supp Q.No. 2b

Find the direction cosines of a line passing through the points P(2, 3, 4) and Q(1, 4, 6). [2]

SOLUTION

Here, $x_1 = 2, y_1 = 3, z_1 = 4$

$x_2 = 1, y_2 = 4, z_2 = 6$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(1-2)^2 + (4-3)^2 + (6-4)^2} = \sqrt{6}$

Direction cosines of PQ are

$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$

i.e. $\frac{1-2}{\sqrt{3}}, \frac{4-3}{\sqrt{6}}, \frac{6-4}{\sqrt{6}}$

i.e. $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

22. 2073 Set C Q.No. 2b

If O is the origin, P(2, 3, 4) and Q(1, -2, 1) be any two points, show that OP is perpendicular to OQ. [2]

SOLUTION

Direction ratios of OP are

$a_1 = x_2 - x_1 = 2 - 0 = 2$, $b_1 = y_2 - y_1 = 3 - 0 = 3$

$c_1 = z_2 - z_1 = 4 - 0 = 4$

and direction of ratios of OQ are

$a_2 = x_2 - x_1 = 1 - 0 = 1$, $b_2 = y_2 - y_1 = -2 - 0 = -2$,

$c_2 = z_2 - z_1 = 1 - 0 = 1$

Now,

$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 3 \times (-2) + 4 \times 1$

$= 2 - 6 + 4 = 0$

Hence, OP is perpendicular to OQ.

23. 2073 Set D Q.No. 2b

If α, β and γ are the direction cosines of a line, prove that $\cos2\alpha + \cos2\beta + \cos2\gamma + 1 = 0$ [2]

SOLUTION

L.H.S. $= \cos2\alpha + \cos2\beta + \cos2\gamma + 1$

$= 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 + 1$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 2$

$= 2 \times 1 - 2$ [∵ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$]

= 0 = R.H.S.

24. 2073 Supp Q.No. 2b

Find the direction cosines of a line joining the points (1,2,3) and (4,5,7). [2]

SOLUTION

Let given points (1, 2, 3) and (4, 5, 7) be denoted by A and B respectively. Then,

$AB = \sqrt{(4-1)^2 + (5-2)^2 + (7-3)^2}$

$= \sqrt{9 + 9 + 16} = \sqrt{34}$

If l, m, n be the d.c's of AB then

$l = \frac{x_2 - x_1}{AB} = \frac{4-1}{\sqrt{34}} = \frac{3}{\sqrt{34}}$

$m = \frac{y_2 - y_1}{AB} = \frac{5-2}{\sqrt{34}} = \frac{3}{\sqrt{34}}$

$n = \frac{z_2 - z_1}{AB} = \frac{7-3}{\sqrt{34}} = \frac{4}{\sqrt{34}}$

Required d. c's of AB are $\frac{3}{\sqrt{34}}, \frac{3}{\sqrt{34}}$ and $\frac{4}{\sqrt{34}}$.

25. 2074 Set B Q.No. 2b

Find the direction cosines of the line PQ passing through P(2, 3, 4) and Q(5, 9, 13) [2]

SOLUTION

Given points are

$(x_1, y_1, z_1) = (2, 3, 4)$

& $(x_2, y_2, z_2) = (5, 9, 13)$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(5-2)^2 + (9-3)^2 + (13-4)^2}$

$= \sqrt{9 + 36 + 81} = \sqrt{126} = 3\sqrt{14}$

If the direction cosines of PQ be l, m, n then

$l = \frac{x_2 - x_1}{PQ} = \frac{5-2}{3\sqrt{14}} = \frac{3}{3\sqrt{14}} = \frac{1}{\sqrt{14}}$

$m = \frac{y_2 - y_1}{PQ} = \frac{9-3}{3\sqrt{14}} = \frac{6}{3\sqrt{14}} = \frac{2}{\sqrt{14}}$

$n = \frac{z_2 - z_1}{PQ} = \frac{13-4}{3\sqrt{14}} = \frac{9}{3\sqrt{14}} = \frac{3}{\sqrt{14}}$

26. 2074 Supp Q.No. 2b

Find the direction cosines of a line which is equally inclined to the axes. [2]

➥ Please refer to 2069 Set B Q.N. 2b

27. 2075 Set C Q.No. 2b

Find the direction cosines of the line passing through the points A(-1, 2, 5) and B(-2, 4, 3). [2]

SOLUTION

Given points A and B are

$(x_1, y_1, z_1) = (-1, 2, 5)$

and $(x_2, y_2, z_2) = (-2, 4, 3)$

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(-2+1)^2 + (4-2)^2 + (3-5)^2}$

$= \sqrt{1 + 4 + 4}$

= 3

The direction cosines of AB are

$l = \frac{x_2 - x_1}{AB} = \frac{-2+1}{3} = \frac{-1}{3}$

$m = \frac{y_2 - y_1}{AB} = \frac{4-2}{3} = \frac{2}{3}$

$n = \frac{z_2 - z_1}{AB} = \frac{3-5}{3} = \frac{-2}{3}$

4 MARKS QUESTIONS

28. 2056 Q.No. 14a

Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to 3, -1, 1 and -3, 2, 4. [4]

SOLUTION

Let l, m, n be the d.c's of the line which is perpendicular to the lines with d.c's proportional to 3, -1, 1 and -3, 2, 4. Then, using the condition of perpendicular, we have

$3l - m + n = 0$

and $-3l + 2m + 4n = 0$

By the rule of cross multiplication, we have

$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3}$$

or, $\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3}$

or, $\frac{l}{2} = \frac{m}{5} = \frac{n}{-1} = \frac{1}{\sqrt{2^2+5^2+(-1)^2}} = \frac{1}{\sqrt{30}}$

$\therefore$ $l = \frac{2}{\sqrt{30}}$, $m = \frac{5}{\sqrt{30}}$, $n = \frac{-1}{\sqrt{30}}$

$\therefore$ Required d.c's are $\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}$.

29. 2056 Q.No. 14a OR

Find the ratio in which the yz-plane divides the line joining (4, 6, 7) and (-1, 2, 5). Also find the coordinates of the point on the yz plane. **[4]**

SOLUTION

Here, $(x_1, y_1, z_1) = (4, 6, 7)$

$(x_2, y_2, z_2) = (-1, 2, 5)$

$m_1 : m_2 = ?$

In yz-plane, we have $x = 0$

i.e. any point in yz-plane is of the form (0, y, z).

From section formula, we have

$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

or, $0 = \frac{m_1 \cdot (-1) + m_2 \cdot 4}{m_1 + m_2}$

or, $m_1 = 4m_2$

or, $\frac{m_1}{m_2} = \frac{4}{1}$

$\therefore$ $m_1 : m_2 = 4 : 1$

Again from section formula, we have

$$(0, y, z) = \left(0, \frac{m_1y_2 + m_2y_1}{m_1+m_2}, \frac{m_1z_2 + m_2z_1}{m_1+m_2}\right)$$

$$= \left(0, \frac{4\cdot 2 + 1\cdot 6}{4+1}, \frac{4\cdot 5 + 1\cdot 7}{4+1}\right) = \left(0, \frac{14}{5}, \frac{27}{5}\right)$$

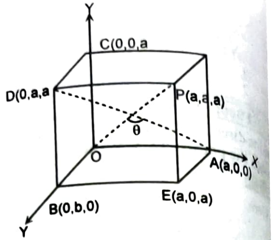
$\therefore$ Required point in yz-plane is $\left(0, \frac{14}{5}, \frac{27}{5}\right)$.

30. 2057 Q.No. 14a

Show that the angle between two diagonals of a cube is $\cos^{-1}\frac{1}{3}$. **[4]**

SOLUTION

Let us suppose that one vertex of a cube of length 'a' be at origin O(0,0,0). Consider two diagonals OP and AD where the coordinates of O, P, A and D be (0, 0, 0), (a, a, a), (a, 0, 0) and (0, a, a) respectively as shown in the figure.



Here,

$OP = \sqrt{(a-0)^2 + (a-0)^2 + (a-0)^2} = \sqrt{3a^2}$

$= a\sqrt{3}$

$AD = \sqrt{(0-a)^2 + (a-0)^2 + (a-0)^2} = \sqrt{3a^2}$

$= a\sqrt{3}$

Now, d.c's of OP are

$\frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}$

i.e. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

And d.c's of AD are

$\frac{0-a}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}, \frac{a-0}{a\sqrt{3}}$

i.e. $-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

If θ be the angle between OP and AD then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$= \frac{1}{\sqrt{3}}\left(\frac{-1}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{3}$

$\therefore$ $\theta = \cos^{-1}\left(\frac{1}{3}\right)$.

31. 2057 Q.No. 14a OR

Find the angles between the two lines whose direction cosines are (l_1, m_1, n_1) (l_2, m_2, n_2).

SOLUTION

Let θ be the angle between two lines AB and CD whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Let the coordinates of A and B be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then,

$l_1 = \frac{x_2 - x_1}{AB}$

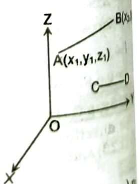
$\therefore$ $x_2 - x_1 = ABl_1$

Similarly,

$y_2 - y_1 = ABm_1$

$z_2 - z_1 = AB\,n_1$

The projection of AB on CD is given by



$AB\cos\theta = (x_2 - x_1)\,l_2 + (y_2 - y_1)m_2 + (z_2 - z_1)n_2$

or, $AB\cos\theta = AB\,l_1l_2 + AB\,m_1m_2 + ABn_1n_2$

$\therefore$ $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

32. 2058 Q.No. 14a

Find the angle between the lines whose direction cosines are given by (l_1, m_1, n_1) and (l_2, m_2, n_2).

➥ Please refer to 2057 Q.No. 14a

33. 2058 Q.No. 14a OR

Show that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$. **[4]**

➥ Please refer to 2057 Q.No. 14a

34. 2059 Q.No. 14a

Find the angle between the lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) **[4]**

➥ Please refer to 2057 Q.No. 14a OR

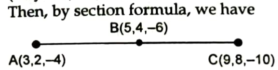
35. 2059 Q.No. 14a OR

Given three collinear points A(3, 2, -4), B(5, 4, -6) and C(9, 8, -10). Find the ratio in which B divides AC. **[4]**

SOLUTION

Let (x, y, z) = (5, 4, –6) divide the line joining the points (x_1, y_1, z_1) = (3, 2, –4) and (x_2, y_2, z_2) = (9, 8, –10) in the ratio $m_1 : m_2$.

Then, by section formula, we have



$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$

or, $5 = \frac{m_1 \cdot 9 + m_2 \cdot 3}{m_1 + m_2}$

or, $5m_1 + 5m_2 = 9m_1 + 3m_2$

or, $4m_1 = 2m_2$

or, $\frac{m_1}{m_2} = \frac{2}{4}$

$\therefore$ $m_1 : m_2 = 1 : 2$

36. 2060 Q.No. 14a

The projection of a line on the axes are 6, 2, 3. Find the length of the line and its direction cosines. **[4]**

SOLUTION

Let l, m, n be the d.c's of a line and r be its length. Then,

$l = \frac{6}{r}$, $m = \frac{2}{r}$, $n = \frac{3}{r}$

$\therefore$ $lr = 6$, $mr = 2$ and $nr = 3$

Squaring and adding,

$r^2(l^2 + m^2 + n^2) = 36 + 4 + 9 = 49$

or, $r^2 = 49$ $(\because l^2 + m^2 + n^2 = 1)$

$\therefore$ $r = 7$

So, the length of the line = 7

Now, $l = \frac{6}{7}$, $m = \frac{2}{7}$ and $n = \frac{3}{7}$

Hence, d.c's of the line are $\frac{6}{7}, \frac{2}{7}$ and $\frac{3}{7}$.

37. 2060 Q.No. 14a OR

Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to (1, -2, -2) and (0, 2, 1). **[4]**

SOLUTION

Let l, m, n be the d.c's of a line which is perpendicular to the lines with d.c's proportional to 1, –2, –2 and 0, 2, 1. Then, by the condition of perpendicularity, we have,

$l \cdot 1 + m.(-2) + n.(-2) = 0$

or, $l - 2m - 2n = 0$ ...(i)

and $l.0 + m.2 + n.1 = 0$

$0 + 2m + n = 0$ ... (ii)

By the rule of cross multiplication, we have

$$\frac{l}{(-2).1 - (-2).2} = \frac{m}{-2.0 - 1.1} = \frac{n}{1.2 - (-2).0}$$

or, $\frac{l}{-2+4} = \frac{m}{-1} = \frac{n}{2}$

or, $\frac{l}{2} = \frac{m}{-1} = \frac{n}{2} = \frac{1}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{1}{3}$

$\therefore$ $l = \frac{2}{3}$, $m = \frac{-1}{3}$, $n = \frac{2}{3}$.

Required d.c's of the line are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

38. 2061 Q.No. 14a

Prove that the lines whose direction cosines are given by the relation $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$. **[4]**

SOLUTION

Given relations are

$al + bm + cn = 0$ ... (i)

and $fmn + gnl + hlm = 0$ ... (ii)

Eliminating n between (i) and (ii), we have

$$fm\left(\frac{-4l + bm}{c}\right) + gl\left(\frac{-4l + bm}{c}\right) + hlm = 0$$

or, $ag\,l^2 + (af + bg - ch)\,lm + vfm^2 = 0$

or, $ag\left(\frac{l}{m}\right)^2 + (ch - af + bg)\left(\frac{l}{m}\right) + bf = 0$

which is quadratic in $\frac{l}{m}$.

Let the two roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

Then, $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bf}{ag}$ $\left[\because \text{Product of the roots} = \frac{C}{A}\right]$

or, $\frac{l_1l_2}{b} = \frac{m_1m_2}{ag}$

or, $\frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b}$ ... (iii)

Similarly, if we eliminate l between (i) and (ii), we have

$\frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$ ... (iv)

From (iii) and (iv),

$\frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c} = k$ (suppose)

$\therefore\ l_1l_2 = \frac{kf}{a},\ m_1m_2 = \frac{kg}{b},\ n_1n_2 = \frac{kh}{c}$

The two lines will be perpendicular if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

or, $\frac{kf}{a} + \frac{kg}{b} + \frac{kh}{c} = 0$

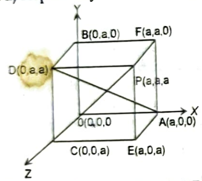
$\therefore\ \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

39. 2061 Q.No. 14b OR

Prove that a line which makes angle x, y, z, t with four diagonals of a cube is $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2 t = \frac{4}{3}$. [4]

SOLUTION

Suppose that one of the vertex of the cube of length a is at origin O(0, 0, 0) and OA, OB and OC are along the coordinate axes. Then the coordinates of O, A, B, C, D, E, F and P are (0,0,0), (a, 0, 0), (0, a, 0), (0, 0, a), (0, a, a), (a, 0, a), (a, a, 0) and (a, a, a) respectively as shown in figure.



The four diagonals of cube are OP, AD, BE and CF.

Now,

The direction ratios of OP are a − 0, a − 0, a − 0 i.e. a, a, a.

The direction ratios of AD are 0 − a, a − 0, a − 0 i.e. −a, a, a

The direction ratios of BE are a − 0, 0 − a, a − 0 i.e. a, −a, a

The direction ratios of CF are a − 0, a − 0, 0 − a i.e. a, a, −a

So, the d.c's of OP are

$\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}$

i.e. $\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}$

Similarly, the d.c's of AD, BE and CF are $\left(\frac{-1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right), \left(\frac{1}{\sqrt3}, \frac{-1}{\sqrt3}, \frac{1}{\sqrt3}\right)$ and $\left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{-1}{\sqrt3}\right)$ respectively.

Let l, m, n be the d.c's of the line making angles α, β, γ and t with OP, AD, BE and CF. Then

$\cos\alpha = l \cdot \frac{1}{\sqrt3} + m \cdot \frac{1}{\sqrt3} + n \cdot \frac{1}{\sqrt3} = \frac{1}{\sqrt3}(l + m + n)$

Similarly,

$\cos\beta = \frac{1}{\sqrt3}(-l + m + n)$

$\cos\gamma = \frac{1}{\sqrt3}(l - m + n)$

$\cos t = \frac{1}{\sqrt3}(l + m - n)$

Now, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2 t$

$= \frac{1}{3}(l+m+n)^2 + \frac{1}{3}(-l+m+n)^2 + \frac{1}{3}(l-m+n)^2 + \frac{1}{3}(l+m-n)^2$

$= \frac{1}{3}(4l^2 + 4m^2 + 4n^2) = \frac{4}{3}(l^2 + m^2 + n^2) = \frac{4}{3}$

$(\because l^2 + m^2 + n^2 = 1)$

40. 2062 Q.No. 14a

Find the direction cosines l, m, n of two lines which are connected by the relation $4l + 3m - 2n = 0$ and $lm - mn + nl = 0$.

SOLUTION

Given relations are

$4l + 3m - 2n = 0$ … (i)

and $lm - mn + nl = 0$ … (ii)

Eliminating n from (i) and (ii), we have

$lm - m\left(\frac{4l+3m}{2}\right) + \left(\frac{4l+3m}{2}\right) \cdot l = 0$

or, $2lm - 4ml - 3m^2 + 4l^2 + 3lm = 0$

or, $4l^2 + lm - 3m^2 = 0$

or, $4l^2 + 4lm - 3lm - 3m^2 = 0$

or, $4l(l + m) - 3m(l + m) = 0$

or, $(l + m)(4l - 3m) = 0$

Either $l + m = 0$ …(iii)

and $4l - 3m = 0$ …(iv)

From (i) and (iii)

$4l + 3m - 2n = 0$

$l + m + 0.n = 0$

By the rule of cross multiplication, we have

$\frac{l}{3\cdot0-(-2)\cdot1} = \frac{m}{(-2)\cdot1 - 0\cdot4} = \frac{n}{4\cdot1-3\cdot1}$

or, $\frac{l}{2} = \frac{m}{-2} = \frac{n}{1} = \frac{1}{\sqrt{2^2+(-2)^2+1^2}} = \frac{1}{3}$

$\therefore l = \frac{2}{3}, m = \frac{-2}{3}, n = \frac{1}{3}$ which are the d.c's of first line

Again, from (i) and (iv)

$4l + 3m - 2n = 0$

$4l - 3m + 0n = 0$

By the rule of cross multiplication, we have

$\frac{l}{3\cdot0-(-2)\cdot(-3)} = \frac{m}{(-2)\cdot4-0\cdot4} = \frac{n}{4\cdot(-3)-4\cdot3}$

or, $\frac{l}{-6} = \frac{m}{-8} = \frac{n}{-24}$

or, $\frac{l}{3} = \frac{m}{4} = \frac{n}{12} = \frac{1}{\sqrt{3^2+4^2+12^2}} = \frac{1}{13}$

$\therefore l = \frac{3}{13}, m = \frac{4}{13}, n = \frac{12}{13}$

which are the d.c's of second line.

Hence, the required d.c's are $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$ and $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$

41. 2062 Q.No. 14a OR

Find the ratio in which the line joining the points (−2, 4, 7) and (3, −5, −8) is divided by xy-plane. [4]

SOLUTION

Here, $(x_1, y_1, z_1) = (-2, 4, 7)$

$(x_2, y_2, z_2) = (3, -5, -8)$

We have, in xy-plane, z = 0

Now, from section formula, we have

$z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$

or, $0 = \frac{m_1\cdot(-8) + m_2\cdot(7)}{m_1 + m_2}$

or, $8m_1 = 7m_2$

or, $\frac{m_1}{m_2} = \frac{7}{8}$

$\therefore m_1 : m_2 = 7 : 8$

42. 2064 Q.No. 14 a

Find the direction cosines l, m, n of two lines which are connected by the relations: $l + m + n = 0$ and $mn - 2nl - 2lm = 0$ [4]

SOLUTION

Given relations are

$l + m + n = 0$ … (i)

and $2lm + 2ln - mn = 0$ … (ii)

Eliminating n from (i) and (ii), we have

$2lm + 2l(-l - m) - m(-l - m) = 0$

or, $2lm - 2l^2 - 2lm + lm + m^2 = 0$

or, $2l^2 - lm - m^2 = 0$

or, $(l - m)(2l + m) = 0$

Either, $l - m = 0$ …(iii)

or, $2l + m = 0$ …(iv)

Solving (i) and (iii) by cross multiplication

$\frac{l}{1\cdot0-1\cdot(-1)} = \frac{m}{1\cdot1-1\cdot0} = \frac{n}{1\cdot(-1)-1\cdot1}$

or, $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2} = \frac{1}{\sqrt{1^2+1^2+(-2)^2}} = \frac{1}{\sqrt6}$

$\therefore l = \frac{1}{\sqrt6}, m = \frac{1}{\sqrt6}, n = \frac{-2}{\sqrt6}$

which are the d.c's of first line.

Again, from (i) and (iv) we have

$l + m + n = 0$

$2l + m + 0\cdot n = 0$

By the rule of cross multiplication, we have

$\frac{l}{1\cdot0-1\cdot1} = \frac{m}{1\cdot2-0\cdot1} = \frac{n}{1\cdot1-1\cdot2}$

or, $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

or, $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = \frac{1}{\sqrt{1^2+(-2)^2+1^2}} = \frac{1}{\sqrt6}$

$\therefore l = \frac{1}{\sqrt6}, m = \frac{-2}{\sqrt6}, n = \frac{1}{\sqrt6}$

which are the d.c's of second line

If θ be the angle between two lines, then

$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

$= \frac{1}{\sqrt6}\cdot\frac{1}{\sqrt6} + \frac{1}{\sqrt6}\cdot\left(\frac{-2}{\sqrt6}\right) + \left(\frac{-2}{\sqrt6}\right)\cdot\frac{1}{\sqrt6}$

$= \frac{1-2-2}{6} = \frac{-1}{2} = \cos 120°$

$\therefore\ \theta = 120°$

43. 2064 Q.No. 14 a OR

Find the ratio in which the line joining the points (− 3, 4, − 8) and (5, − 6, 4) is divided by the xy plane. Find also the coordinates of the point of intersection of the line with the place. [4]

SOLUTION

Here, $(x_1, y_1, z_1) = (-3, 4, -8)$

$(x_2, y_2, z_2) = (5, -6, 4)$

In xy-plane, we have z = 0

i.e. any point in xy-plane is of the form (x, y, 0)

Now, from section formula, we have

$z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}$

or, $0 = \frac{m_1\cdot4 + m_2(-8)}{m_1 + m_2}$

or, $4m_1 = 8m_2$

or, $\frac{m_1}{m_2} = \frac{2}{1}$

$\therefore\ m_1 : m_2 = 2 : 1$

The required point is

$(x, y, 0) = \left(\frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \frac{m_1y_2 + m_2y_1}{m_1 + m_2}, 0\right)$

$= \left(\frac{2\cdot5 + 1\cdot(-3)}{2+1}, \frac{2\cdot(-6) + 1\cdot4}{2+1}, 0\right) = \left(\frac{7}{3}, \frac{-8}{3}, 0\right)$

44. 2065 Q.No. 14 a

Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to 3, −1, 1 and −3, 2, 4. [4]

SOLUTION

Please refer to 2056 Q.No. 14a

45. 2065 Q. No. 14 a OR

The projection of a line on the axis are 6, 2, 3. Find the length of the line and its direction cosines. [4]

SOLUTION

Please refer to 2060 Q.No. 14a

46. 2066 Q.No. 14 a

Prove that the straight lines whose dc's are given by ul + vm + wn = 0 and fmn + gnl + hlm = 0 are perpendicular if $\frac{f}{u} + \frac{g}{v} + \frac{h}{w} = 0$ [4]

SOLUTION

Given relations are

ul + vm + wn = 0 … (i)

and fmn + gnl + hlm = 0 … (ii)

Eliminating n between (i) and (ii), we have

$fm\left(\frac{-ul + vm}{c}\right) + gl\left(\frac{-ul + vm}{c}\right) + hlm = 0$

or, $ug\,l^2 + (uf + vg - wh)\,lm + vfm^2 = 0$

or, $ug\left(\frac{l}{m}\right)^2 + (wh - uf + vg)\left(\frac{l}{m}\right) + vf = 0$

which is quadratic in $\frac{l}{m}$. So it has two roots.

Let the two roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

or, $\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{vf}{ug}$

or, $\frac{l_1 l_2}{vf} = \frac{m_1 m_2}{ug}$

or, $\frac{l_1 l_2}{f/u} = \frac{m_1 m_2}{g/v}$ ... (iii)

Similarly, if we eliminate l between (i) and (ii), we have

$\frac{m_1 m_2}{g/v} = \frac{n_1 n_2}{h/w}$ ... (iv)

From (iii) and (iv),

$\frac{l_1 l_2}{f/u} = \frac{m_1 m_2}{g/v} = \frac{n_1 n_2}{h/w} = k$ (suppose)

$\therefore\ l_1 l_2 = \frac{kf}{u},\ m_1 m_2 = \frac{kg}{v},\ n_1 n_2 = \frac{kh}{w}$

The two lines will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

or, $\frac{kf}{u} + \frac{kg}{v} + \frac{kh}{w} = 0$

$\therefore\ \frac{f}{u} + \frac{g}{v} + \frac{h}{w} = 0$

47. 2066 Q.No. 14 a OR

A (2, 3, –1), B (5, 2, 3), C (4, 3, –5), D (–2, 1, –3) are four points in space. Find the projection of AB on CD. **[4]**

SOLUTION

Given points are A (2, 3, –1), B (5, 2, 3), C (4, 3, –5) and D(–2, 1, –3).

Now,

$CD = \sqrt{(-2-4)^2 + (1-3)^2 + (-3+5)^2}$

$= \sqrt{36 + 4 + 4} = \sqrt{44} = 2\sqrt{11}$

If the d.c's of CD are l, m, n then

$l = \frac{x_2 - x_1}{CD} = \frac{-2-4}{2\sqrt{11}} = \frac{-6}{2\sqrt{11}} = \frac{-3}{\sqrt{11}}$

$m = \frac{y_2 - y_1}{CD} = \frac{1-3}{2\sqrt{11}} = \frac{-2}{2\sqrt{11}} = \frac{-1}{\sqrt{11}}$

$n = \frac{z_2 - z_1}{CD} = \frac{-3+5}{2\sqrt{11}} = \frac{2}{2\sqrt{11}} = \frac{1}{\sqrt{11}}$

Now, the projection of AB on CD is

$(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$

$= (5-2)\left(\frac{-3}{\sqrt{11}}\right) + (2-3)\left(\frac{-1}{\sqrt{11}}\right) + (3+1)\cdot\frac{1}{\sqrt{11}}$

$= \frac{-9}{\sqrt{11}} + \frac{1}{\sqrt{11}} + \frac{4}{\sqrt{11}} = \frac{-4}{\sqrt{11}}$

48. 2067 Q.No. 14 a

Find the angle between the lines whose direction cosines are given by l + m + n = 0 and 2 lm + 2ln – mn = 0 **[4]**

➥ Please refer to 2064 Q.No. 14a

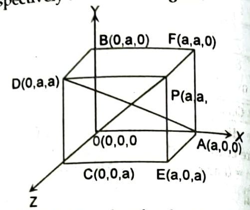
49. 2067 Q.No. 14 a OR

Prove that line which makes angle, x, y, z, δ with four diagonals of a cube is

$\cos^2 x + \cos^2 y + \cos^2 z + \cos^2\delta = \frac{4}{3}$.

SOLUTION

Suppose that one of the vertex of the cube of length a is at origin O(0, 0, 0) and OA, OB and OC are along the coordinate axes. Then the coordinates of O, A, B, C, D, E, F and P are (0, 0, 0), (a, 0, 0), (0, a, 0), (0, 0, a), (0, a, a), (a, 0, a), (a, a, 0) and (a, a, a) respectively as shown in figure.



The four diagonals of cube are OP, AD, BE and CF.

Now, the direction ratios of OP are a – 0, a – 0, a – 0

i.e. a, a, a.

The direction ratios of AD are 0 – a, a – 0, a – 0

i.e. –a, a, a

The direction ratios of BE are a –0, 0 – a, a – 0

i.e. a, –a, a

The direction ratios of CF are a – 0, a – 0, 0 – a

i.e. a, a, –a

So, the d.c's of OP are

$\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{a}{\sqrt{a^2+a^2+a^2}}$

i.e. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Similarly, the d.c's of AD, BE and CF are

$\left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

respectively.

Let l, m, n be the d.c's of the line making angles α, β, γ and δ with OP, AD, BE and CF. Then

$\cos\alpha = l\cdot\frac{1}{\sqrt{3}} + m\cdot\frac{1}{\sqrt{3}} + n\cdot\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}(l + m + n)$

Similarly, $\cos\beta = \frac{1}{\sqrt{3}}(-l + m + n)$

$\cos\gamma = \frac{1}{\sqrt{3}}(l - m + n)$

$\cos\delta = \frac{1}{\sqrt{3}}(l + m - n)$

Now,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= \frac{1}{3}(l + m + n)^2 + \frac{1}{3}(-l + m + n)^2 + \frac{1}{3}(l - m + n)^2 + \frac{1}{3}(l + m - n)^2$

$= \frac{1}{3}(4l^2 + 4m^2 + 4n^2) = \frac{4}{3}(l^2 + m^2 + n^2) = \frac{4}{3}$

$(\because l^2 + m^2 + n^2 = 1)$

50. 2068 Q.No. 14 a

Find the angle between the two straight lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2. Also, find the condition for the two lines to be perpendicular to each other. **[4]**

SOLUTION

First part:

Please refer to 2057 Q.No. 14a OR

Second part:

Two line will be perpendicular to each other if $\theta = 90°$

i.e. $\cos 90° = l_1 l_2 + m_1 m_2 + n_1 n_2$

$\therefore\ l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

51. 2068 Q.No. 14 a Or

Show that the line joining the points (1, 2, 3) and (–1, –2, –3) is parallel to the line joining the points (2, 3, 4) and (5, 9, 13). **[4]**

SOLUTION

Let the given points be A(1, 2, 3), B(–1, –2, –3), C(2, 3, 4), D(5, 9, 13). We have to show that AB is parallel to CD.

For AB:

If a_1, b_1, c_1 are the direction ratios of AB then

$a_1 = x_2 - x_1 = -1 - 1 = -2$

$b_1 = y_2 - y_1 = -2 - 2 = -4$

$c_1 = z_2 - z_1 = -3 - 3 = -6$

For CD:

If a_2, b_2, c_2 are the direction cosines of CD, then

$a_2 = 5 - 2 = 3$

$b_2 = 9 - 3 = 6$

$c_2 = 13 - 4 = 9$

Here,

$\frac{a_1}{a_2} = \frac{-2}{3}$

$\frac{b_1}{b_2} = \frac{-4}{6} = \frac{-2}{3}$

$\frac{c_1}{c_2} = \frac{-6}{9} = \frac{-2}{3}$

$\therefore\ \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Hence, AB and CD are parallel lines.

52. 2069 (Set A) Q.No. 6b

Find the angle between the two lines whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2. Also, find the condition under which the two lines are perpendicular. **[4]**

SOLUTION

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of two lines whose corresponding d.c's are l_1, m_1, n_1 and l_2, m_2, n_2. Then,

$l_1 = \frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$,

$m_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$,

$n_1 = \frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}$

$l_2 = \frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$,

$m_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$,

$n_2 = \frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$

If θ be the angle between two lines, then

$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$= \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

The two lies will be perpendicular if θ = 90°.

i.e. $\cos 90° = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

or, $0 = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$\therefore\ a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$

53. 2070 Set D Q.No. 6 b

Find the angle between two straight lines whose direction cosiness are l_1, m_1, n_1 and l_2, m_2, n_2. **[4]**

SOLUTION

Please refer to 2057 Q.No. 14a OR

54. 2071 Set C Q.No. 6 b

Show that the line AB is perpendicular to CD if A, B, C, D are the points (2, 3, 4), (5, 4, -1), (3, 6, 2) and (1, 2, 0) respectively. **[4]**

SOLUTION

For AB:

If a_1, b_1, c_1 be the direction ratios of AB, then

$a_1 = 5 - 2 = 3,\quad b_1 = 4 - 3 = 1,\quad c_1 = -1 - 4 = -5$

For CD:

Again, if a_2, b_2, c_2 be the direction cosines of CD, then

$a_2 = 1 - 3 = -2,\quad b_2 = 2 - 6 = -4,\quad c_2 = 0 - 2 = -2$

Now,

$a_1 a_2 + b_1 b_2 + c_1 c_2 = 3(-2) + 1(-4) + (-5)(-2) = 0$

Hence AB is perpendicular to CD.

55. 2075 Set A Q.No. 6b

Show that the angle between the two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$. **[4]**

➥ Please refer to 2057 Q.No. 14a

6 MARKS QUESTIONS

56. 2071 Supp. Q.No. 9

Show that the straight lines whose direction cosines are given by the equations $al+bm+cn=0$ and $ul^2+vm^2+wn^2=0$ are perpendicular if $a^2(v+w)+b^2(u+w)+c^2(u+v)=0$ and parallel if $\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$. [6]

SOLUTION

Given relations are

$al+bm+cn=0$ ... (i)

& $ul^2+vm^2+wn^2=0$ ... (ii)

Eliminating n between (i) and (ii), we have

$$ul^2+vm^2+w\left(\frac{-al-bm}{c}\right)^2=0$$

or, $ul^2+vm^2+w\left(\frac{a^2l^2}{c^2}+\frac{2ablm}{c^2}+\frac{b^2m^2}{c^2}\right)=0$

or, $ul^2c^2+vm^2c^2+a^2l^2w+2ablmw+b^2m^2w=0$

or, $(c^2u+a^2w)\,l^2+(2abw)\,lm+(c^2v+b^2w)\,m^2=0$

or, $(c^2u+a^2w)\left(\frac{l}{m}\right)^2+2abw\left(\frac{l}{m}\right)+(c^2v+b^2w)=0$ ...(iii)

which is quadratic in $\frac{l}{m}$. Let the two roots be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$

Now, product of the roots $=\frac{C}{A}$

or, $\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\frac{c^2v+b^2w}{c^2u+a^2w}$

or, $\frac{l_1l_2}{c^2v+b^2w}=\frac{m_1m_2}{c^2u+a^2w}$ ...(iv)

Similarly, if we eliminate l between (i) and (ii), we have

$\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u}$ ...(v)

From (iv) and (v)

$\frac{l_1l_2}{c^2v+b^2w}=\frac{m_1m_2}{c^2u+a^2w}=\frac{n_1n_2}{a^2v+b^2u}=k$ (suppose)

∴ $l_1l_2=k(c^2v+b^2w)$

$m_1m_2=k(c^2u+a^2w)$

$n_1n_2=k(a^2v+b^2u)$

The two lines will be perpendicular if

$l_1l_2+m_1m_2+n_1n_2=0$

or, $c^2v+b^2w+c^2u+a^2w+a^2v+b^2u=0$

∴ $a^2(v+w)+b^2(u+w)+c^2(u+v)=0$

Again, the lines are parallel if

$\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$

Taking first two ratios, we have

$\frac{l_1}{l_2}=\frac{m_1}{m_2}$

i.e. $\frac{l_1}{m_1}=\frac{l_2}{m_2}$

We know that the quadratic equation (iii) have equal roots if its discriminate is zero.

i.e. $(2abw)^2-4(c^2u+a^2w)(c^2v+b^2w)=0$

or, $a^2b^2w^2-c^4uv-b^2c^2uw-a^2c^2wv-a^2b^2w^2=0$

or, $c^4uv+b^2c^2uw+a^2c^2vw=0$

or, $c^2uv+b^2uv+a^2vw=0$

Dividing both sides by uvw, we get

$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0$

57. 2074 Set A Q.No. 9

Prove that the lines whose direction cosines given by the relations $al+bm+cn=0$, $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$.

SOLUTION

Please refer to 2066 Q.No. 14a

[Put u = a, v = b, w = c]

58. 2075 Set B Q.No. 9

Prove that the straight lines whose direction cosines are given by the relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel if $\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$.

SOLUTION

First Part: Please refer to 2061 Q.No. 14a

Second Part:

The lines are parallel if $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$.

Taking first two ratios, we have

$\frac{l_1}{l_2}=\frac{m_1}{m_2}$

i.e. $\frac{l_1}{m_1}=\frac{l_2}{m_2}$

We know that the quadratic equation (i) have two equal roots if the discriminate is zero.

i.e. $(af+bg-ch)^2-4.ag.bf=0$ [∵ $B^2-4AC=0$]

or, $(af+bg-ch)^2=(\pm2\sqrt{ag.bf})^2$

or, $af+bg-ch=\pm2\sqrt{ag.bf}$

or, $af+bg\pm2\sqrt{ag.bf}=ch$

or, $(\sqrt{af}\pm\sqrt{bg})^2=(\pm\sqrt{ch})^2$

or, $\sqrt{af}\pm\sqrt{bg}=\pm\sqrt{ch}$

or, $\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$

Hence the given lines are parallel if

$\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0$

B. PLANE

2 MARKS QUESTIONS

59. 2069 (Set A) Q.No. 2b

Find the equation of the plane through the point (3, -4, 5) and parallel to the plane $3x-4y+5z=0$.

SOLUTION

The equation of plane through the point (3, -4, 5) is

$a(x-3)+b(y+4)+c(z-5)=0$ ... (i)

If equation (i) is parallel to $3x-4y+5z=7$, then $\frac{a}{3}=\frac{b}{-4}=\frac{c}{5}=k$ (Suppose)

∴ $a=3k$, $b=-4k$, $c=5k$

Putting the values of a, b, c in (i), we have

$3k(x-3)-4k(y+4)+5k(z-5)=0$

or, $3x-9-4y-16+5z-25=0$

∴ $3x-4y+5z=50$ is the required equation of plane.

60. 2070 Set D Q.No. 2 b

Find the equation of the plane which makes equal intercepts on the axes and passes through the point (2, 3, 4). [2]

SOLUTION

The equation of plane in intercept form is

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Since the plane makes equal intercepts on the axes, so a = b = c. So the equation of plane becomes

$\frac{x}{a}+\frac{y}{a}+\frac{z}{a}=1$

or, $x+y+z=a$ ... (i)

Since equation (i) passes through the point (2, 3, 4), so

$2+3+4=a$

∴ $a=9$

Substituting the value of a in equation (i), we have

$x+y+z=9$ is the required equation of plane.

61. 2071 Set C Q.No. 2 b

Find the equation of the plane through (1, 2, 3) and parallel to the plane $3x-4y+5z=0$. [2]

SOLUTION

The equation of plane through the point (1, 2, 3) is

$a(x-1)+b(y-2)+c(z-3)=0$ ...(i)

Since, plane (i) is parallel to the plane $3x-4y+5z=0$, so

$\frac{a}{3}=\frac{b}{-4}=\frac{c}{5}=k$ (suppose)

∴ $a=3k$, $b=-4k$, $c=5k$

Substituting the value of a, b and c in (i)

$3k(x-1)-4k(y-2)+5k(z-3)=0$

or, $3x-3-4y+8+5z-15=0$

∴ $3x-4y+5z=10$ is the required equation of plane.

62. 2071 Supp. Q.No. 2b

Find k so that the planes $x-2y+kz=0$ and $2x+5y-z=0$ are at right angles. [2]

SOLUTION

Two planes $x-2y+kz=0$ and $2x+5y-z=0$ are at right angles to each other if

$1\cdot2+(-2)\cdot5+k\cdot(-1)=0$ [∵ $a_1a_2+b_1b_2+c_1c_2=0$]

or, $2-10-k=0$

∴ $k=-8$

63. 2074 Set A Q.No. 2b

Find the equation of the plane whose intercepts on the axes are 2, 3 and 4 respectively. [2]

SOLUTION

Given, a = 2, b = 3, c = 4

The equation of plane in intercept form is

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

or, $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

or, $\frac{6x+4y+3z}{12}=1$

∴ $6x+4y+3z=12$ which is the required equation of plane.

64. 2075 Set A Q.No. 2b

Find the equation of the plane which makes equal intercepts on the axes and passes through the point (2, 3, 4). [2]

Please refer to 2070 Set D Q.No. 2b

65. 2075 Set B Q.No. 2b

Find the intercepts made by the plane $2x+3y+4z=24$ on the coordinate axes. [2]

SOLUTION

Given, equation of plane is

$2x+3y+4z=24$ ... (i)

Dividing both sides of (i) by 24, we get,

$\frac{2x}{24}+\frac{3y}{24}+\frac{4z}{24}=\frac{24}{24}$

or, $\frac{x}{12}+\frac{y}{8}+\frac{z}{6}=1$ ... (ii)

Comparing (ii) with $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, we get,

x - intercept (a) = 12

y - intercept (b) = 8

z - intercept (c) = 6

4 MARKS QUESTIONS

66. 2069 (Set B) Q.No. 6b

Find the equation of the plane through the points (1, 1, 0) (-2, 2, -1) and (1, 2, 1). [4]

SOLUTION

The equation of plane through the point (1, 1, 0) is

$a(x-1)+b(y-1)+c(z-0)=0$ ... (i)

If equation (i) passes through the points (-2, 2, -1) and (1, 2, 1), then

$a(-2-1)+b(2-1)+c(-1-0)=0$

or, $-3a+b-c=0$ ...(ii)

and $a(1-1)+b(2-1)+c(1-0)=0$

or, $0\cdot a+b+c=0$ ...(iii)

Solving (ii) and (iii) by the method of cross multiplication, we have

-3	1	-1	-3	1
0	1	1	0	1

$\frac{a}{1\cdot1-(-1)\cdot1}=\frac{b}{(-1)\cdot0-(-3)\cdot1}=\frac{c}{(-3)\cdot1-0\cdot1}$

or, $\frac{a}{2} = \frac{b}{3} = \frac{c}{-3} = k$ (suppose)

∴ $a = 2k,\ b = 3k,\ c = -3k$

Substituting the value of a, b and c in (i), we have

$2k(x-1) + 3k(y-1) + (-3k)(z-0) = 0$

or, $2x - 2 + 3y - 3 - 3z = 0$

∴ $2x + 3y - 3z = 5$ is the required equation of plane.

67. 2070 Set C Q.No. 6 b

Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6), and normal to the plane $2x + 6y + 6z = 9$. [4]

SOLUTION

The equation of plane through (2, 2, 1) is

$a(x-2) + b(y-2) + c(z-1) = 0$ ...(i)

If equation (i) passes through the point (9, 3, 6), then

$a(9-2) + b(3-2) + c(6-1) = 0$

or $7a + b + 5c = 0$ ...(ii)

Since equation (i) is normal to the plane $2x + 6y + 6z = 9$, so

$2 \cdot a + 6 \cdot b + 6 \cdot c = 0$ ...(iii)

Solving (ii) and (iii),

7 1 5 7 1

2 6 6 2 6

$\frac{a}{1 \cdot 6 - 5 \cdot 6} = \frac{b}{5 \cdot 2 - 6 \cdot 7} = \frac{c}{7 \cdot 6 - 1 \cdot 2}$

or, $\frac{a}{6-30} = \frac{b}{10-42} = \frac{c}{42-2}$

or, $\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$

or, $\frac{a}{3} = \frac{b}{4} = \frac{c}{-5} = k$ (Suppose)

∴ $a = 3k$, $b = 4k$ and $c = -5k$

Substituting the values of a, b, c in (i), we have

$3k(x-2) + 4k(y-2) - 5k(z-1) = 0$

or, $3x - 6 + 4y - 8 - 5z + 5 = 0$

∴ $3x + 4y - 5z = 9$ is the required equation of plane.

68. 2071 Set D Q.No. 6 b

Find the equation of the plane through the points (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane $x + 2y + 2z = 5$. [4]

SOLUTION

The equation of plane through the point (–1, 1, 1) is

$a(x+1) + b(y-1) + c(z-1) = 0$ ...(i)

Since equation (i) passes through the point (1, –1, 1), so $a(1+1) + b(-1-1) + c(1-1) = 0$

or, $2a - 2b + 0 \cdot c = 0$ ...(ii)

Since equation (i) is perpendicular to the plane $x + 2y + 2z = 5$, so

$a \cdot 1 + b \cdot 2 + c \cdot 2 = 0$ ...(iii)

Solving (ii) and (iii), we have

2 –2 0 2 –2

1 2 2 1 2

$\frac{a}{(-2) \cdot 2 - 2 \cdot 0} = \frac{b}{0 \cdot 1 - 2 \cdot 2} = \frac{c}{2 \cdot 2 - (-2) \cdot 1}$

or, $\frac{a}{-4} = \frac{b}{-4} = \frac{c}{6}$

or, $\frac{a}{2} = \frac{b}{2} = \frac{c}{-3} = k$ (suppose)

∴ $a = 2k,\ b = 2k,\ c = -3k$

Substituting the value of a, b and c in equation (i), we have

$2k(x+1) + 2k(y-1) - 3k(z-1) = 0$

or, $2x + 2 + 2y - 2 - 3z + 3 = 0$

∴ $2x + 2y - 3z + 3 = 0$ is the required equation of plane.

69. 2072 Set C Q.No. 6b

Find the equation of the plane through the intersection of the planes $2x + 3y + 10z = 8$ and $2x - 3y + 7z = 2$, and perpendicular to the plane $3x - 2y + 4z = 5$. [4]

SOLUTION

The equation of plane through the intersection of the planes $2x + 3y + 10z = 8$ and $2x - 3y + 7z = 2$ is

$2x + 3y + 10z - 8 + k(2x - 3y + 7z - 2) = 0$

or, $(2 + 2k)x + (3 - 3k)y + (10 + 7k)z - 8 - 2k = 0$ ...(i)

Since (i) is perpendicular to $3x - 2y + 4z = 5$,

$(2 + 2k) \cdot 3 + (3 - 3k) \cdot (-2) + (10 + 7k) \cdot 4 = 0$

or, $6 + 6k - 6 + 6k + 40 + 28k = 0$

or, $40k + 40 = 0$

∴ $k = -1$

Putting the value of k in equation (i), we have

$2x + 3y + 10z - 8 - 1(2x - 3y + 7z - 2) = 0$

or, $2x + 3y + 10z - 8 - 2x + 3y - 7z + 2 = 0$

or, $6y + 3z - 6 = 0$

∴ $2y + z = 2$

which is the required equation of plane.

70. 2072 Set D Q.No. 6b

Show that the plane $2x + 3y - 4z = 3$ is parallel to the plane $10x + 15y - 20z = 12$ and perpendicular to the plane $3x + 2y + 3z = 5$. [4]

SOLUTION

Given planes are

$2x + 3y - 4z = 3$ ...(i)

$10x + 15y - 20z = 12$ ...(ii)

$3x + 2y + 3z = 5$ ...(iii)

We know that two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

So, plane (i) and (ii) are parallel if

$\frac{2}{10} = \frac{3}{15} = \frac{-4}{-20}$

i.e. $\frac{1}{5} = \frac{1}{5} = \frac{1}{5}$ (true)

Hence the planes (i) and (ii) are parallel.

Again, we know that two planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

So, planes (i) and (iii) are perpendicular if

$2 \cdot 3 + 3 \cdot 2 + (-4) \cdot 3 = 0$

or, $12 - 12 = 0$

i.e. $0 = 0$ (true)

Hence, the planes (i) and (iii) are perpendicular.

71. 2072 Set E Q.No. 6a

Find the equation of the plane passing through the points (1, 1, 0), (–2, 2, –1) and (1, 2, 1). [4]

SOLUTION

Please refer to 2069 (Set B) Q.No. 6b

72. 2072 Supp Q.No. 6b

Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and normal to the plane $2x + 6y + 6z = 9$. [4]

SOLUTION

Please refer to 2070 Set C Q.No. 6b

73. 2073 Set C Q.No. 6b

Find the equation of the plane through the point (2,–3, 1) and perpendicular to the line joining the two points (3, 4, –1) and (2, –1, 5). [4]

SOLUTION

Direction ratios of the line joining the points (3, 4, –1) and (2, –1, 5) are

$a = 2-3 = -1$, $b = -1-4 = -5$, $c = 5-(-1) = 6$

Now, the equation of plane through the point (2, –3, 1) is,

$a(x-2) + b(y+3) + c(z-1) = 0$ ...(i)

Since equation (i) is perpendicular to the line joining the points (3, 4, –1) and (2, –1, 5), so

$\frac{a}{-1} = \frac{b}{-5} = \frac{c}{6} = k$ (say)

∴ $a = -k,\ b = -5k,\ c = 6k$

Putting the values of a, b, c in (i), we get

$-k(x-2) - 5k(y+3) + 6k(z-1) = 0$

or, $-(x-2) - 5(y+3) + 6(z-1) = 0$

or, $-x + 2 - 5y - 15 + 6z - 6 = 0$

∴ $x + 5y - 6z + 19 = 0$

74. 2073 Set D Q.No. 6b

Find the equation of the plane passing through the points (1,1,0), (–2, 2,–1) and (1, 2, 1). [4]

SOLUTION

Please refer to 2069 (Set B) Q.No. 6b

75. 2073 Supp Q.No. 6b

Find the equation of the plane through (–2, 3, 4) and perpendicular to the planes $2x + 3y + 4z = 6$ and $3x + 2y + 2z = 8$. [4]

SOLUTION

Equation of plane through the point (–2, 3, 4) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

or, $a(x+2) + b(y-3) + c(z-4) = 0$ ... (i)

If plane (i) is perpendicular to the planes $2x + 3y + 4z = 6$ and $3x + 2y + 2z = 8$, then

$2a + 3b + 4c = 0$ [∵ $a a_2 + b_1 b_2 + c_1 c_2 = 0$]

and $3a + 2b + 2c = 0$

Solving these equations by cross multiplication method

$\frac{a}{3 \times 2 - 4 \times 2} = \frac{b}{4 \times 3 - 2 \times 2} = \frac{c}{2 \times 2 - 3 \times 3}$

or, $\frac{a}{-2} = \frac{b}{8} = \frac{c}{-5}$

or, $\frac{a}{2} = \frac{b}{-8} = \frac{c}{5} = k$ (Suppose)

∴ $a = 2k,\ b = -8k,\ c = 5k$

Putting the value of a, b, c in (i)

$2k(x+2) - 8k(y-3) + 5k(z-4) = 0$

or, $2x + 4 - 8y + 24 + 5z - 20 = 0$

∴ $2x - 8y + 5z + 8 = 0$

76. 2074 Set B Q.No. 6b

Find the equation of the plane through the intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$. [4]

SOLUTION

The equation of the plane through the intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ is

$x + y + z - 6 + k(2x + 3y + 4z + 5) = 0$ ...(i)

or, $(1 + 2k)x + (1 + 3k)y + (1 + 4k)z + 5k - 6 = 0$

If this plane is perpendicular to the plane $4x + 5y - 3z = 8$ then,

$4.(1 + 2k) + 5.(1 + 3k) + (-3)(1 + 4k) = 0$

or, $4 + 8k + 5 + 15k - 3 - 12k = 0$

or, $6 + 11k = 0$

∴ $k = -\frac{6}{11}$

Substituting the value of k in (i)

$x + y + z - 6 - \frac{6}{11}(2x + 3y + 4z + 5) = 0$

or, $11x + 11y + 11z - 66 - 12x - 18y - 24z - 30 = 0$

or, $-x - 7y - 13z - 96 = 0$

∴ $x + 7y + 13z + 96 = 0$ which is required equation of plane.

77. 2074 Supp Q.No. 6b

Show that the plane $2x + 3y - 4z = 3$ is parallel to the plane $10x + 15y - 20z = 12$ and perpendicular to the plane $3x + 2y + 3z = 5$. [4]

➥ Please refer to 2072 Set D Q.N. 6b

78. 2075 Set C Q.No. 6b

Find the equation of the plane through (–2, 3, 4) and perpendicular to the planes $3x + 2y + 2z - 8 = 0$ and $2x + 3y + 4z - 6 = 0$. [4]

➥ Please refer to 2073 Supp Q.No. 6b

6 MARKS QUESTIONS

79. 2070 Supp. Q.No. 10

Prove that a plane through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

Also, find the equation of the plane passing through (2, 2, –1), (3, 4, 2) and (7, 0, 6). **[6]**

SOLUTION

The equation of plane through the point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ ... (i)

Since (i) passes through the points (x_2, y_2, z_2) and (x_3, y_3, z_3), so

$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$ ...(ii)

& $a(x_3 - x_1) + b(y_3 - y_1) + c(z_3 - z_1) = 0$ ... (iii)

Eliminating a, b and c from (i), (ii) and (iii), we have

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$ is the required equation of plane.

The equation of plane through the po[int] $(x_1, y_1, z_1) = (2, 2, -1)$, $(x_2, y_2, z_2) = (3,$ [...] and $(x_3, y_3, z_3) = (7, 0, 6)$ is

$$\begin{vmatrix} x - 2 & y - 2 & z + 1 \\ 3 - 2 & 4 - 2 & 2 + 1 \\ 7 - 2 & 0 - 2 & 6 + 1 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x - 2 & y - 2 & z + 1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$$

or, $(x - 2)\begin{vmatrix} 2 & 3 \\ -2 & 7 \end{vmatrix} - (y - 2)\begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} + (z + 1)\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix} = 0$

or, $(x - 2)(14 + 6) - (y - 2)(7 - 15) + (z + 1)(-2 - 10) = 0$

or, $20x - 40 + 8y - 16 - 12z - 12 = 0$

or, $20x + 8y - 12z - 68 = 0$

$\therefore$ $5x + 2y - 3z = 17$ is the required equation of plane.

UNIT 6

VECTORS AND ITS APPLICATIONS

A. ELEMENTS OF VECTORS AND ITS APPLICATIONS

2 MARKS QUESTIONS

1. 2057 Q.No. 4 a

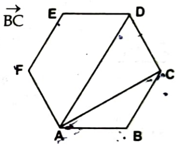
ABCDEF is a regular hexagon. Express $\vec{AC}$ and $\vec{AD}$ in terms of $\vec{AB}$ and $\vec{BC}$. **[2]**

SOLUTION

Let ABCDEF be a regular hexagon.

From ΔABC, $\vec{AC} = \vec{AB} + \vec{BC}$

Also, $\vec{AD} = 2\vec{BC}$



2. 2058 Q.No. 4 a

If $\vec{a} = (3, -1, -4)$, $\vec{b} = (-2, 4, -3)$ and $\vec{c} = (-5, 7, -1)$ find $|\vec{a} - 2\vec{b} + \vec{c}|$. **[2]**

SOLUTION

Here, $\vec{a} = (3, -1, -4)$, $\vec{b} = (-2, 4, -3)$ and $\vec{c} = (-5, 7, -1)$

$\vec{a} - 2\vec{b} + \vec{c} = (3, -1, -4) - 2(-2, 4, -3) + (-5, 7, -1)$
$= (3 + 4 - 5, -1 - 8 + 7, -4 + 6 - 1)$
$= (2, -2, 1)$

$|\vec{a} - 2\vec{b} + \vec{c}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$

3. 2059 Q.No. 3 b

If $\vec{a} + \vec{b} = (5, 6)$ and $\vec{a} - \vec{b} = (3, 2)$, find $\vec{a}$ and $\vec{b}$. **[2]**

SOLUTION

Here, $\vec{a} + \vec{b} = (5, 6)$ ... (i)

$\vec{a} - \vec{b} = (3, 2)$ ... (ii)

Adding (i) and (ii)

$2\vec{a} = (8, 8)$

$\therefore$ $\vec{a} = (4, 4)$

Putting the value of $\vec{a}$ in (i)

$(4, 4) + \vec{b} = (5, 6)$

or, $\vec{b} = (5, 6) - (4, 4) = (1, 2)$

$\therefore$ $\vec{b} = (1, 2)$

4. 2060 Q.No. 4 a

If $\vec{a} = (2, -3)$ and $\vec{b} = (4, -2)$. Find the unit vector along $4\vec{a} - 3\vec{b}$. **[2]**

SOLUTION

Here, $\vec{a} = (2, -3)$ and $\vec{b} = (4, -2)$

$4\vec{a} - 3\vec{b} = 4(2, -3) - 3(4, -2)$
$= (8, -12) - (12, -6)$
$= (8 - 12, -12 + 6) = (-4, -6)$

$|4\vec{a} - 3\vec{b}| = \sqrt{(-4)^2 + (-6)^2} = \sqrt{16 + 36}$
$= \sqrt{52} = 2\sqrt{13}$

Unit vector along $4\vec{a} - 3\vec{b}$ is $\dfrac{4\vec{a} - 3\vec{b}}{|4\vec{a} - 3\vec{b}|}$

$= \dfrac{1}{2\sqrt{3}}(-4, -6) = \left(\dfrac{-4}{2\sqrt{13}}, \dfrac{-6}{2\sqrt{13}}\right)$

$= \left(\dfrac{-2}{\sqrt{13}}, \dfrac{-3}{\sqrt{13}}\right)$

5. 2062 Q.No. 4 a

If $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} + 2\vec{j} - 4\vec{k}$ find $\vec{PQ}$ and determine its direction cosines. **[2]**

SOLUTION

Here, $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{PQ} = \vec{OQ} - \vec{OP}$
$= (5\vec{i} - 2\vec{j} + 4\vec{k}) - (\vec{i} + 3\vec{j} - 7\vec{k})$
$= 4\vec{i} - 5\vec{j} + 11\vec{k}$

$|\vec{PQ}| = \sqrt{4^2 + (-5)^2 + 11^2} = 9\sqrt{2}$

Unit vector along the direction of $\vec{PQ} = \frac{\vec{PQ}}{|\vec{PQ}|}$

$$= \frac{4\vec{i} - 5\vec{j} + 11\vec{k}}{9\sqrt{2}} = \frac{4}{9\sqrt{2}}\vec{i} - \frac{5}{9\sqrt{2}}\vec{j} + \frac{11}{9\sqrt{2}}\vec{k}$$

6. 2063 Q.No. 3 b

If $\vec{a} = (2, -3)$ and $\vec{b} = (4, -2)$, find unit vector along $4\vec{a} - 3\vec{b}$. [2]

➥ Please refer to 2060 Q.No. 4a

7. 2064 Q.No. 4 a

If $\vec{a} = (3, 4)$ and $3\vec{a} + 2\vec{b} = (5, 6)$, find $\vec{b}$. [2]

SOLUTION

Here,

$\vec{a} = (3, 4)$ ...(i)

$3\vec{a} + 2\vec{b} = (5, 6)$ ...(ii)

From (i) and (ii)

$3(3, 4) + 2\vec{b} = (5, 6)$

or, $2\vec{b} = (5, 6) - (9, 12)$

or, $2\vec{b} = (5 - 9, 6 - 12)$

or, $\vec{b} = \frac{1}{2}(-4, -6)$

$\therefore \vec{b} = (-2, -3)$

8. 2065 Q.No 3 b

If $3\vec{i} + \vec{j} - \vec{k}$ and $\lambda\vec{i} - 4\vec{j} + 4\vec{k}$ are collinear vector. Find λ. [2]

SOLUTION

Since the vectors $3\vec{i} + \vec{j} - \vec{k}$ and $\lambda\vec{i} - 4\vec{j} + 4\vec{k}$ are collinear, so

$A(3\vec{i} + \vec{j} - \vec{k}) = \lambda\vec{i} - 4\vec{j} + 4\vec{k}$ for some scalar A.

or $3A\vec{i} + A\vec{j} - A\vec{k} = \lambda\vec{i} - 4\vec{j} + 4\vec{k}$

Equating the coefficient of $\vec{i}$, $\vec{j}$ and $\vec{k}$, we get

$\lambda = 3A$ ... (i)

$A = -4$ ... (ii)

From (i) and (ii)

$\lambda = 3 \times (-4) = -12$

9. 2066 Q.No. 4 a

Prove that the points A, B, C are collinear, if $\vec{OA} = \vec{i} + 2\vec{j} + 4\vec{k}$, $\vec{OB} = 2\vec{i} + 5\vec{j} - \vec{k}$ and $\vec{OC} = 3\vec{i} + 8\vec{j} - 6\vec{k}$ [2]

SOLUTION

Let A, B and C be three points with position vectors $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$ respectively. Let O be the origin. Then

$\vec{OA} = \vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{OB} = 2\vec{i} + 5\vec{j} - \vec{k}$

$\vec{OC} = 3\vec{i} + 8\vec{j} - 6\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + 3\vec{j} - 5\vec{k}$

$\vec{AC} = \vec{OC} - \vec{CA} = 2\vec{i} + 6\vec{j} - 10\vec{k}$

$= 2(\vec{i} + 3\vec{j} - 5\vec{k}) = 2\vec{AB}$

This shows that AB and AC are paral they start from the same point A. So A, B and C are collinear.

10. 2066 C Q.No. 3 b

Find a unit vector parallel to the sum of vectors $2\vec{i} + 4\vec{j} - 5\vec{k}$ and $\vec{i} + 2\vec{j} + \vec{k}$

SOLUTION

Let $\vec{a} = 2\vec{i} + 4\vec{j} - 5\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} + \vec{k}$

Now,

$\vec{a} + \vec{b} = 3\vec{i} + 6\vec{j} - 4\vec{k}$

$|\vec{a} + \vec{b}| = \sqrt{3^2 + 6^2 + (-4)^2} = \sqrt{61}$

A unit vector parallel to the sum of $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|}$

$$= \frac{3\vec{i} + 6\vec{j} - 4\vec{k}}{\sqrt{61}} = \frac{3}{\sqrt{61}}\vec{i} + \frac{6}{\sqrt{61}}\vec{j} - \frac{4}{\sqrt{61}}\vec{k}$$

11. 2067 Q.No. 4a

Determine the unit vector of $2\vec{a} - 3\vec{b}$, $\vec{a} = 4\vec{i} + 3\vec{j}$ and $\vec{b} = -2\vec{i} - 3\vec{j}$.

SOLUTION

Here, $\vec{a} = 4\vec{i} + 3\vec{j}$ and $\vec{b} = -2\vec{i} - 3\vec{j}$

$2\vec{a} - 3\vec{b} = 2(4\vec{i} + 3\vec{j}) - 3(-2\vec{i} - 3\vec{j})$

$= 8\vec{i} + 6\vec{j} + 6\vec{i} + 9\vec{j} = 14\vec{i} + 15\vec{j}$

$|2\vec{a} - 3\vec{b}| = \sqrt{14^2 + 15^2} = \sqrt{421}$

Unit vector in the direction of $2\vec{a} - 3\vec{b}$

$$\frac{2\vec{a} - 3\vec{b}}{|2\vec{a} - 3\vec{b}|} = \frac{14}{\sqrt{421}}\vec{i} + \frac{15}{\sqrt{421}}\vec{j}$$

12. 2068 Q.No. 4a

ABCD is a parallelogram. G is the point of intersection of its diagonals and if O is any point, show that: $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 4\vec{OG}$

SOLUTION

Given, ABCD is a parallelogram. Also G is the point of intersection of its diagonals and O is any point.

Now,

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}$

$= (\vec{OG} + \vec{GA}) + (\vec{OG} + \vec{GB}) + (\vec{OG} + \vec{GC}) + (\vec{OG} + \vec{GD})$

$= 4\vec{OG} + (\vec{GA} + \vec{GC}) + (\vec{GD} + \vec{GB})$

$= 4\vec{OG} + (\vec{GA} - \vec{GA}) + (\vec{GD} - \vec{GD})$

(∵ The diagonals of parallelogram bisects each other)

$= 4\vec{OG}$

13. 2069 (Set A) Old Q.No. 4a

If $\vec{a} = (3, 4)$ and $3\vec{a} + 2\vec{b} = (5, 6)$ find $\vec{b}$. [2]

➥ Please refer to 2064 Q.No. 4a

14. 2069 (Set B) Q.No. 2c

If $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$, find $\vec{PQ}$ and a unit vector along the direction of $\vec{PQ}$. [2]

➥ Please refer to 2062 Q.No. 4a

15. 2069 (Set A) Q.No. 2c

Show that the three points with position vectors $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} - 6\vec{k}$ are collinear. [2]

➥ Please refer to 2066 Q.No. 4a

16. 2070 (Old) Q.No. 3 b

Find the direction cosines of the vector $\vec{MN}$ where position vectors of M is $-\vec{i} + 6\vec{j} + 6\vec{k}$ and N is $-4\vec{i} + 9\vec{j} + 6\vec{k}$. [2]

SOLUTION

Let M and N be two points with position vectors $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively. Let O be the origin. Then,

$\vec{OM} = -\vec{i} + 6\vec{j} + 6\vec{k}$

$\vec{ON} = -4\vec{i} + 9\vec{j} + 6\vec{k}$

$\vec{MN} = \vec{ON} - \vec{OM} = -3\vec{i} + 3\vec{j}$

$|\vec{MN}| = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$

Unit vector in the direction of $\vec{MN}$ is $\frac{\vec{MN}}{|\vec{MN}|}$

$$= \left(\frac{-3\vec{i} + 3\vec{j}}{3\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

∴ Required d.c's of the vector $\vec{MN}$ are $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$

17. 2070 Set C Q.No. 2 c

ABCD is a parallelogram. G is the point of intersection of its diagonals and if O is any point show that: $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 4\vec{OG}$. [2]

➥ Please refer to 2068 Q.No. 4a

18. 2070 Set D Q.No. 2 c

The vertices A, B, C of a triangle are (2, –1, –3), (4, 2, 3) and (6, 3, 4) respectively. Show that $\vec{AB} = (2, 3, 6)$ and AC = 9. [2]

SOLUTION

Let A (2, –1, –3), B(4, 2, 3) and C(6, 3, 4) be the given vertices of ΔABC. Let O be the origin. Then,

$\vec{OA} = (2, -1, -3)$

$\vec{OB} = (4, 2, 3)$

$\vec{OC} = (6, 3, 4)$

Now,

$\vec{AB} = \vec{OB} - \vec{OA} = (4, 2, 3) - (2, -1, -3)$

$= (4-2, 2+1, 3+3) = (2, 3, 6)$

$\vec{AC} = \vec{OC} - \vec{CA} = (6, 3, 4) - (2, -1, -3)$

$= (4, 4, 7)$

$AC = |\vec{AC}| = \sqrt{4^2 + 4^2 + 7^2} = \sqrt{81} = 9$

19. 2071 Old Q.No. 3 b

If D is the middle point of BC of the triangle ABC show that $\vec{AB} + \vec{AC} = 2\vec{AD}$. [2]

SOLUTION

Let D be the middle point of BC of the triangle ABC.

From ΔABD, $\vec{AB} + \vec{BD} = \vec{AD}$ ...(i)

From ΔACD, $\vec{AC} + \vec{CD} = \vec{AD}$ ... (ii)

Adding (i) and (ii)

$\vec{AB} + \vec{AC} + \vec{BD} + \vec{CD} = 2\vec{AD}$

or $\vec{AB} + \vec{AC} + \vec{BD} - \vec{BD} = 2\vec{AD}$

(∵ D is the midpoint of BC)

∴ $\vec{AB} + \vec{AC} = 2\vec{AD}$

20. 2071 Set C Q.No. 2 c

If $3\vec{i} + \vec{j} - \vec{k}$ and $\lambda\vec{i} - 4\vec{j} + 4\vec{k}$ are collinear vectors. Find the value of λ. [2]

➥ Please refer to 2065 Q.No. 3b

21. 2071 Set D Q.No. 2 c

Show that the three points whose position vectors are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ form an isosceles triangle. [2]

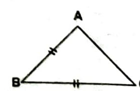
SOLUTION

Let A, B and C be the three points with position vectors $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively. Let O be the origin. Then,

$\vec{OA} = 7\vec{j} + 10\vec{k}$

$\vec{OB} = -\vec{i} + 6\vec{j} + 6\vec{k}$

$\vec{OC} = -4\vec{i} + 9\vec{j} + 6\vec{k}$



Now,

$\vec{AB} = \vec{OB} - \vec{OA} = -\vec{i} - \vec{j} - 4\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = -3\vec{i} + 3\vec{j}$

$\vec{AC} = \vec{OC} - \vec{OA} = -4\vec{i} + 2\vec{j} - 4\vec{k}$

$AB = |\vec{AB}| = \sqrt{(-1)^2 + (-1)^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2}$

$BC = |\vec{BC}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 9\sqrt{2}$

$AC = |\vec{AC}| = \sqrt{(-4)^2 + 2^2 + (-4)^2} = 6$

Here, AB = BC

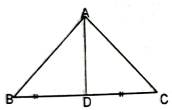
So, A, B and C are vertices of an isosceles triangle.

22. 2071 Supp. Q.No. 2c

Find the value of λ if the points with position vectors $\vec{i} + 2\vec{j} + \vec{k}$, $2\vec{i} - \vec{j} + 3\vec{k}$ and $5\vec{i} - 10\vec{j} + \lambda\vec{k}$ are collinear. [2]

SOLUTION

Let A, B and C be three points with position vectors $\vec{i} + 2\vec{j} + \vec{k}$, $2\vec{i} - \vec{j} + 3\vec{k}$ and $5\vec{i} - 10\vec{j} + \lambda\vec{k}$ respectively. Let O be the origin. Then,



$\vec{OA} = \vec{i} + 2\vec{j} + \vec{k}$

$\vec{OB} = 2\vec{i} - \vec{j} + 3\vec{k}$

$\vec{OC} = 5\vec{i} - 10\vec{j} + \lambda\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA}$

$= \vec{i} - 3\vec{j} + 2\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = 4\vec{i} - 12\vec{j} + (\lambda-1)\vec{k}$

Since A, B and C are collinear, so $\vec{AB}$ is parallel $\vec{AC}$. Then,

$\alpha(\vec{i} - 3\vec{j} + 2\vec{k}) = 4\vec{i} - 12\vec{j} + (\lambda-1)\vec{k}$ for some scalar α.

or $\alpha\vec{i} - 3\alpha\vec{j} + 2\alpha\vec{k} = 4\vec{i} - 12\vec{j} + (\lambda-1)\vec{k}$

Equating,

$\alpha = 4$ ... (i)

& $2\alpha = \lambda - 1$ ... (ii)

From (i) and (ii)

$2 \times 4 = \lambda - 1$

$\therefore \lambda = 9$

23. 2072 Set C Q.No. 2c

Prove that the vectors $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $-7\vec{j} + 10\vec{k}$ are collinear. [2]

SOLUTION

Let A, B and C be three points with position vectors $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $-7\vec{j} + 10\vec{k}$ respectively. Let O be the origin. Then,

$\vec{OA} = \vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{OB} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$\vec{OC} = -7\vec{j} + 10\vec{k}$

Now, $\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + 5\vec{j} - 7\vec{k}$

and $\vec{AC} = \vec{OC} - \vec{OA} = -\vec{i} - 5\vec{j} + 7\vec{k}$

$= -(\vec{i} + 5\vec{j} - 7\vec{k})$

$= -\vec{AB}$

Since AC and AB are parallel and both start from the same point A; so A, B and C are collinear.

24. 2072 Set D Q.No. 2c

If $\vec{a} = (3, -1, -4)$, $\vec{b} = (-2, 4, -3)$ find unit vector along $\vec{a} - 2\vec{b}$.

SOLUTION

Here, $\vec{a} = (3, -1, -4)$

$\vec{b} = (-2, 4, -3)$

$\vec{a} - 2\vec{b} = (3, -1, -4) - 2(-2, 4, -3)$

$= (3, -1, -4) - (-4, 8, -6) = (3+4, -1-8, -4+6) = (7, -9, 2)$

$|\vec{a} - 2\vec{b}| = \sqrt{7^2 + (-9)^2 + 2^2}$

$= \sqrt{134}$

Unit vector along $\vec{a} - 2\vec{b}$ is $\frac{\vec{a} - 2\vec{b}}{|\vec{a} - 2\vec{b}|}$

$= \frac{1}{\sqrt{134}}(7, -9, 2) = \left(\frac{7}{\sqrt{134}}, \frac{-9}{\sqrt{134}}, \frac{2}{\sqrt{134}}\right)$

25. 2072 Set E Q.No. 2c

If $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$, find $\vec{PQ}$ and its direction cosines.

SOLUTION

Here, $\vec{OP} = \vec{i} + 3\vec{j} - 7\vec{k}$ and $\vec{OQ} = 5\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{PQ} = \vec{OQ} - \vec{OP} = (5\vec{i} - 2\vec{j} + 4\vec{k}) - (\vec{i} + 3\vec{j} - 7\vec{k})$

$= 4\vec{i} - 5\vec{j} + 11\vec{k}$

$|\vec{PQ}| = \sqrt{4^2 + (-5)^2 + 11^2} = 9\sqrt{2}$

Unit vector in the direction of $\vec{PQ}$ is $\frac{\vec{PQ}}{|\vec{PQ}|}$

$= \frac{4\vec{i} - 5\vec{j} + 11\vec{k}}{9\sqrt{2}}$

$\therefore$ Required d.c's are $\frac{4}{9\sqrt{2}}, \frac{-5}{9\sqrt{2}}, \frac{11}{9\sqrt{2}}$

26. 2072 Supp Q.No. 2c

If $3\vec{i} + \vec{j} - \vec{k}$ and $\lambda\vec{i} - 4\vec{j} + 4\vec{k}$ are collinear vectors, find the value of λ. [2]

➥ Please refer to 2065 Q.No. 3b

27. 2073 Set C Q.No. 2c

If $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$ and $\vec{b} = -\vec{i} + 2\vec{j} - 2\vec{k}$, find a unit vector along the direction of $2\vec{a} + 3\vec{b}$. [2]

SOLUTION

Here, $\vec{a} = 2\vec{i} - 3\vec{j} + 4\vec{k}$

$\vec{b} = -\vec{i} + 2\vec{j} - 2\vec{k}$

$2\vec{a} + 3\vec{b} = 2(2\vec{i} - 3\vec{j} + 4\vec{k}) + 3(-\vec{i} + 2\vec{j} - 2\vec{k})$

$= 4\vec{i} - 6\vec{j} + 8\vec{k} - 3\vec{i} + 6\vec{j} - 6\vec{k}$

$= \vec{i} + 2\vec{k}$

$|2\vec{a} + 3\vec{b}| = \sqrt{1^2 + 2^2} = \sqrt{5}$

Unit vector along the direction of $2\vec{a} + 3\vec{b}$ is

$\frac{2\vec{a} + 3\vec{b}}{|2\vec{a} + 3\vec{b}|} = \frac{\vec{i} + 2\vec{k}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\vec{i} + \frac{2}{\sqrt{5}}\vec{k}$

28. 2073 Set D Q.No. 2c

ABCD is a parallelogram. G is the point of intersection of its diagonals and if O is any point show that: $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 4\vec{OG}$ [2]

➥ Please refer to 2068 Q.No. 4a

29. 2073 Supp Q.No. 2c

Show that the vectors $\vec{i} + 2\vec{j} + 4\vec{k}$, $2\vec{i} + 5\vec{j} - \vec{k}$ and $3\vec{i} + 8\vec{j} + 6\vec{k}$ are collinear. [2]

➥ Please refer 2066 Q.No. 4a

30. 2074 Set B Q.No. 2c

If $\vec{a} = (3, -1, -4)$, $\vec{b} = (-2, 4, -3)$ and $\vec{c} = (-5, 7, -1)$, find the unit vector along $\vec{a} - 2\vec{b} + \vec{c}$. [2]

SOLUTION

$\vec{a} - 2\vec{b} + \vec{c} = (3, -1, -4) - 2(-2, 4, -3) + (-5, 7, -1)$

$= (3, -1, -4) - (-4, 8, -6) + (-5, 7, -1)$

$= (3 + 4 - 5, -1 - 8 + 7, -4 + 6 - 1)$

$= (2, -2, 1)$

$|\vec{a} - 2\vec{b} + \vec{c}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$

Unit vector along $\vec{a} - 2\vec{b} + \vec{c}$

$= \frac{\vec{a} - 2\vec{b} + \vec{c}}{|\vec{a} - 2\vec{b} + \vec{c}|} = \left(\frac{2, -2, 1}{3}\right) = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$

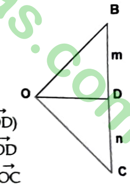
31. 2074 Supp Q.No. 2c

OB and OC are two straight lines and D is a point on BC such that BD : DC = m : n, show that $\vec{OD} = \frac{n\vec{OB} + m\vec{OC}}{m + n}$. [2]

SOLUTION

Suppose OB and OC are two straight lines and D is a point on BC such that BD:DC = m : n.

So, $\frac{BD}{DC} = \frac{m}{n}$



or, $\frac{\vec{BD}}{\vec{DC}} = \frac{m}{n}$

or, $n\vec{BD} = m\vec{DC}$

or, $n(\vec{OD} - \vec{OB}) = m(\vec{OC} - \vec{OD})$

or, $n\vec{OD} - n\vec{OB} = m\vec{OC} - m\vec{OD}$

or, $n\vec{OD} + m\vec{OD} = n\vec{OB} + m\vec{OC}$

$\therefore \vec{OD} = \frac{n\vec{OB} + m\vec{OC}}{m + n}$

32. 2075 Set A Q.No. 2c

Express $\vec{r}$ = (4,7) as the linear combination of $\vec{a}$ = (5, –4) and $\vec{b}$ = (–2, 5). [2]

SOLUTION

Let,

$\vec{r} = x\vec{a} + y\vec{b}$ ... (1) where x and y are scalars to be determined.

or, (4, 7) = x(5, –4) + y(–2, 5)

or, (4, 7) = (5x – 2y, – 4x + 5y)

Then,

$5x - 2y = 4$ ... (2)

and,

$-4x + 5y = 7$ ... (3)

Solving (2) and (3), we get,

x = 2, y = 3

Substituting the values of x and y in (1), we get,

$\vec{r} = 2\vec{a} + 3\vec{b}$

33. 2075 Set B Q.No. 2c

Show that the points $2\vec{i} + \vec{j} - \vec{k}$, $3\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{i} + 4\vec{j} - 3\vec{k}$ are collinear. [2]

SOLUTION

Let O be the origin. Let A, B and C be there points with position vectors $2\vec{i} + \vec{j} - \vec{k}$, $3\vec{i} - 2\vec{j} + \vec{k}$ and $\vec{i} + 4\vec{j} - 3\vec{k}$, respectively. Then,

$\vec{OA} = 2\vec{i} + \vec{j} - \vec{k}$

$\vec{OB} = 3\vec{i} - 2\vec{j} + \vec{k}$

$\vec{OC} = \vec{i} + 4\vec{j} - 3\vec{k}$

Now,

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (3\vec{i} - 2\vec{j} + \vec{k}) - (2\vec{i} + \vec{j} - \vec{k})$

$= \vec{i} - 3\vec{j} + 2\vec{k}$

And,

$\vec{AC} = \vec{OC} - \vec{OA}$

$= \vec{i} + 4\vec{j} - 3\vec{k} - (2\vec{i} + \vec{j} - \vec{k})$

$= -\vec{i} + 3\vec{j} - 2\vec{k}$

$= -(\vec{i} - 3\vec{j} + 2\vec{k})$

$= -\vec{AB}$

$\therefore$ $\vec{AC} = -\vec{AB}$ which shows that $\vec{AC}$ and $\vec{AB}$ are parallel. But they start from the same point A. So, A, B & C are collinear.

4 MARKS QUESTIONS

34. 2057 Q.No. 10 a

If the position vector of M and N are $3\vec{i} + \vec{j} - 3\vec{k}$ and $4\vec{i} - 2\vec{j} + \vec{k}$ respectively, find $\vec{MN}$ and determine its direction cosines. [4]

SOLUTION

Let M and N be two points with position vectors $3\vec{i} + \vec{j} - 3\vec{k}$ and $4\vec{i} - 2\vec{j} + \vec{k}$ respectively. Let O be the origin. Then,

$\vec{OM} = 3\vec{i} + \vec{j} - 3\vec{k}$

$\vec{ON} = 4\vec{i} - 2\vec{j} + \vec{k}$

$\vec{MN} = \vec{ON} - \vec{OM} = \vec{i} - 3\vec{j} + 4\vec{k}$

$|\vec{MN}| = \sqrt{1^2 + (-3)^2 + 4^2} = \sqrt{26}$

Unit vector in the direction of $\vec{MN}$ is

$$\frac{\vec{MN}}{|\vec{MN}|} = \frac{\vec{i} - 3\vec{j} + 4\vec{k}}{\sqrt{26}}$$

$$= \frac{1}{\sqrt{26}}\vec{i} - \frac{3}{\sqrt{26}}\vec{j} + \frac{4}{\sqrt{26}}\vec{k}$$

$\therefore$ Required d.c's are $\frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{4}{\sqrt{26}}$

35. 2058 Q.No. 10 a

Show that the points A, B and C with position vectors $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$, $-7\vec{j} + 10\vec{k}$ respectively are collinear. [4]

SOLUTION

Let A, B and C be three points with position vectors $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $-7\vec{j} + 10\vec{k}$ respectively. Let O be the origin. Then,

$\vec{OA} = \vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{OB} = 2\vec{i} + 3\vec{j} - 4\vec{k}$

$\vec{OC} = -7\vec{j} + 10\vec{k}$

Now, $\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + 5\vec{j} - 7\vec{k}$

and $\vec{AC} = \vec{OC} - \vec{OA} = -\vec{i} - 5\vec{j} + 7\vec{k}$

$= -(\vec{i} + 5\vec{j} - 7\vec{k})$

$= -\vec{AB}$

Since AC and AB are parallel and both from the same point A; so A, B and C collinear.

36. 2059 Q.No. 10 a

Prove that the vectors $-\vec{a} + 4\vec{b} + 3\vec{c}$; $2\vec{a} - 3\vec{b}$... and $2\vec{a} + 7\vec{b} - 3\vec{c}$ are coplanars, where $\vec{a}, \vec{b}$, ... are any vectors.

SOLUTION

Let, $\vec{r_1} = -\vec{a} + 4\vec{b} + 3\vec{c}$

$\vec{r_2} = 2\vec{a} - 3\vec{b} - 5\vec{c}$

$\vec{r_3} = 2\vec{a} + 7\vec{b} - 3\vec{c}$

If three vectors are coplanar than one ... can be written as the sum of the ... multiple of other two say

$\vec{r_3} = x\vec{r_1} + y\vec{r_2}$ ...(i)

$2\vec{a} + 7\vec{b} - 3\vec{c} = x(-\vec{a} + 4\vec{b} + 3\vec{c}) + y(2\vec{a} - 3\vec{b}$ ...

or, $2\vec{a} + 7\vec{b} - 3\vec{c} = (-x + 2y)\vec{a} + (4x - 3y)\vec{b} + (3x$ ...

Equating, we get,

$-x + 2y = 2$ ... (ii)

$4x - 3y = 7$ ...(iii)

$3x - 5y = -3$ ... (iv)

Solving (ii) and (iii), $x = 4$, $y = 3$

Putting the values of x and y in (iv), we g...

$3 \times 4 - 5 \times 3 = -3$ (true)

$\therefore$ From (i), $\vec{r_3} = 4\vec{r_1} + 3\vec{r_2}$

Hence the given vectors are coplanar.

37. 2060 Q.No. 10 a

Show that the three points whose pos... vectors are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and ... $9\vec{j} + 6\vec{k}$ form an isosceles right angled tri...

SOLUTION

Let A, B and C be three points with p... vectors $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6$... $-4\vec{i} + 9\vec{j} + 6\vec{k}$ respectively. Let O be the ... Then,

$\vec{OA} = 7\vec{j} + 10\vec{k}$

$\vec{OB} = -\vec{i} + 6\vec{j} + 6\vec{k}$

$\vec{OC} = -4\vec{i} + 9\vec{j} + 6\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = -\vec{i} - \vec{j} - 4\vec{k}$

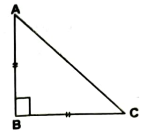
$\vec{BC} = \vec{OC} - \vec{OB} = -3\vec{i} + 3\vec{j}$

$\vec{CA} = \vec{OA} - \vec{OC} = 4\vec{i} - 2\vec{j} + 4\vec{k}$

$AB = |\vec{AB}| = \sqrt{(-1)^2 + (-1)^2 + (-4)^2} = \sqrt{18} = 3\sqrt{2}$

$BC = |\vec{BC}| = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$

$CA = |\vec{CA}| = \sqrt{4^2 + (-2)^2 + 4^2} = 6$



Here, AB = BC

Also,

$AB^2 + BC^2 = (3\sqrt{2})^2 + (3\sqrt{2})^2 = 18 + 18 = 36 = 6^2 = (CA)^2$

$\therefore$ $\angle B = 90°$

Hence, A, B and C are vertices of isosceles right angled triangle.

38. 2061 Q.No. 10 a

Prove that the following vectors are coplanar; $\vec{a} - 3\vec{b} + 5\vec{c}$, $\vec{a} - 2\vec{b} + 3\vec{c}$, $-2\vec{a} + 3\vec{b} - 4\vec{c}$ [4]

SOLUTION

Let, $\vec{r_1} = \vec{a} - 3\vec{b} + 5\vec{c}$

$\vec{r_2} = \vec{a} - 2\vec{b} + 3\vec{c}$

$\vec{r_3} = -2\vec{a} + 3\vec{b} - 4\vec{c}$

If three vectors are coplanar then one vector can be written as the sum of the scalars multiples of other two say

$\vec{r_3} = x\vec{r_1} + y\vec{r_2}$ ...(i)

where x and y are scalars.

or, $-2\vec{a} + 3\vec{b} - 4\vec{c} = x(\vec{a} - 3\vec{b} + 5\vec{c}) + y(\vec{a} - 2\vec{b} + 3\vec{c})$

or, $-2\vec{a} + 3\vec{b} - 4\vec{c} = (x + y)\vec{a} + (-3x - 2y)\vec{b} + (5x + 3y)\vec{c}$

Equating, we have

$x + y = -2$ ...(ii)

$-3x - 2y = 3$ ...(iii)

$5x + 3y = -4$ ... (iv)

Solving (ii) and (iii), $x = 1$, $y = -3$

Putting the values of x and y in (iv), we get

$5 \times 1 + 3 \times (-3) = -4$ (true)

Hence from (i) $\vec{r_3} = \vec{r_1} - 3\vec{r_2}$

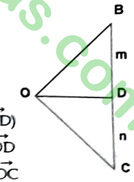
So, the given vectors are coplanar.

39. 2062 Q.No. 10 a

OB and OC are two straight lines and D is a point on BC such that BD:DC = m:n, show that $\vec{OD} = \frac{n\vec{OB} + m\vec{OC}}{m + n}$ [4]

SOLUTION

Suppose OB and OC are two straight lines and D is a point on BC such that BD:DC = m : n.



So, $\frac{BD}{DC} = \frac{m}{n}$

or, $\frac{\vec{BD}}{\vec{DC}} = \frac{m}{n}$

or, $n\vec{BD} = m\vec{DC}$

or, $n(\vec{OD} - \vec{OB}) = m(\vec{OC} - \vec{OD})$

or, $n\vec{OD} - n\vec{OB} = m\vec{OC} - m\vec{OD}$

or, $n\vec{OD} + m\vec{OD} = n\vec{OB} + m\vec{OC}$

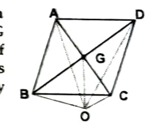
$\therefore$ $\vec{OD} = \frac{n\vec{OB} + m\vec{OC}}{m + n}$

40. 2063 Q.No. 10 a

ABCD is a parallelogram G is the point of intersection of the diagonals and if O is any point, show that: $\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 4\vec{OG}$ [4]

SOLUTION

Given, ABCD is a parallelogram. Also G is the point of intersection of its diagonals and O is any point.



Now,

$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD}$

$= (\vec{OG} + \vec{GA}) + (\vec{OG} + \vec{GB}) + (\vec{OG} + \vec{GC}) + (\vec{OG} + \vec{GD})$

$= 4\vec{OG} + (\vec{GA} + \vec{GC}) + (\vec{GD} + \vec{GB})$

$= 4\vec{OG} + (\vec{GA} - \vec{GA}) + (\vec{GD} - \vec{GD})$

($\because$ The diagonals of parallelogram bisects each other)

$= 4\vec{OG}$

41. 2064 Q.No. 10 a

Show that the three points whose position vectors are $7\vec{j} + 10\vec{k}$, $-\vec{i} + 6\vec{j} + 6\vec{k}$ and $-4\vec{i} + 9\vec{j} + 6\vec{k}$ form an isosceles right angled triangle. [4]

➡ Please refer to 2060 Q.No. 10a

42. 2065 Q.No 10 a

Show that the following vectors are linearly dependent: $5\vec{i} + 6\vec{j} + 7\vec{k}$, $7\vec{i} - 8\vec{j} + 9\vec{k}$ and $3\vec{i} + 20\vec{j} + 5\vec{k}$ [4]

SOLUTION

Given vectors are $5\vec{i} + 6\vec{j} + 7\vec{k}$, $7\vec{i} - 8\vec{j} + 9\vec{k}$ and $3\vec{i} + 20\vec{j} + 5\vec{k}$

Now, $\begin{vmatrix} 5 & 6 & 7 \\ 7 & -8 & 9 \\ 3 & 20 & 5 \end{vmatrix}$

$= 5\begin{vmatrix} -8 & 9 \\ 20 & 5 \end{vmatrix} - 6\begin{vmatrix} 7 & 9 \\ 3 & 5 \end{vmatrix} + 7\begin{vmatrix} 7 & -8 \\ 3 & 20 \end{vmatrix}$

$= 5(-40-180) - 6(35-27) + 7(140+24)$

$= -1100 - 48 + 1148 = 0$

So the given vectors are linearly dependent.

43. 2066 Q.No. 10 a

Prove that the vectors $5\vec{a}+6\vec{b}+7\vec{c}$, $7\vec{a}-8\vec{b}+9\vec{c}$ and $3\vec{a}+20\vec{b}+5\vec{c}$ are coplanar. **[4]**

SOLUTION

Let $\vec{r_1} = 5\vec{a}+6\vec{b}+7\vec{c}$

$\vec{r_2} = 7\vec{a}-8\vec{b}+9\vec{c}$

$\vec{r_3} = 3\vec{a}+20\vec{b}+5\vec{c}$

If three vectors are coplanar then one vector can be written as the sum of the scalar multiples of other two.

Let $\vec{r_3} = x\vec{r_1} + y\vec{r_2}$ ... (i)

or $3\vec{a}+20\vec{b}+5\vec{c} = x(5\vec{a}+6\vec{b}+7\vec{c}) + y(7\vec{a}-8\vec{b}+9\vec{c})$

or, $3\vec{a}+20\vec{b}+5\vec{c} = (5x+7y)\vec{a} + (6x-8y)\vec{b} + (7x+9y)\vec{c}$

Equating, we have

$5x + 7y = 3$ ... (ii)

$6x - 8y = 20$ ... (iii)

$7x + 9y = 5$ ... (iv)

Solving (ii) and (iii), we get $x = 2$, $y = -1$

Putting the values of x & y in (iv)

$7 \times 2 + 9 \times (-1) = 5$ (true)

$\therefore$ From (i) $\vec{r_3} = 2\vec{r_1} - \vec{r_2}$

Hence given vectors are coplanar.

44. 2066 C Q.No. 10 a

Show that the points A, B and C with position vectors $\vec{i}-2\vec{j}+3\vec{k}$, $2\vec{i}+3\vec{j}-4\vec{k}$, $-7\vec{j}+10\vec{k}$ respectively are collinear. **[4]**

➥ Please refer to 2058 Q.No. 10a

45. 2067 Q.No. 10a

Show that the three points whose position vectors are $2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}-3\vec{j}-5\vec{k}$ and $3\vec{i}-4\vec{j}-4\vec{k}$ form the sides of a right angled triangle. **[4]**

SOLUTION

Let A, B and C be three points with position vectors $2\vec{i}-\vec{j}+\vec{k}$, $\vec{i}-3\vec{j}-5\vec{k}$ and $3\vec{i}-4\vec{j}-4\vec{k}$ respectively. Let O be the origin. Then,

$\vec{OA} = 2\vec{i}-\vec{j}+\vec{k}$

$\vec{OB} = \vec{i}-3\vec{j}-5\vec{k}$

$\vec{OC} = 3\vec{i}-4\vec{j}-4\vec{k}$

$\vec{AB} = \vec{OB}-\vec{OA} = -\vec{i}-2\vec{j}-6\vec{k}$

$\vec{BC} = \vec{OC}-\vec{OB} = 2\vec{i}-\vec{j}+\vec{k}$

$\vec{CA} = \vec{OA}-\vec{OC} = -\vec{i}+3\vec{j}+5\vec{k}$

$AB = |\vec{AB}| = \sqrt{(-1)^2+(-2)^2+(-6)^2} = \sqrt{41}$

$BC = |\vec{BC}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$

$CA = |\vec{CA}| = \sqrt{(-1)^2+3^2+5^2} = \sqrt{35}$

Here,

$BC^2 + CA^2 = (\sqrt{6})^2 + (\sqrt{35})^2 = 41 = (\sqrt{41})^2 = (AB)^2$

$\therefore \angle C = 90°$

Hence A, B and C are the vertices of a right angled triangle.

46. 2068 Q.No. 10a

Prove that the three vectors $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-4\vec{c}$ and $-\vec{b}+2\vec{c}$ are coplanar.

SOLUTION

Let $\vec{r_1} = \vec{a}-2\vec{b}+3\vec{c}$

$\vec{r_2} = -2\vec{a}+3\vec{b}-4\vec{c}$

$\vec{r_3} = -\vec{b}+2\vec{c}$

If three vectors are coplanar then one vector can be written as the sum of the scalar multiples of other two vectors. So, let

$\vec{r_3} = x\vec{r_1} + y\vec{r_2}$ ... (i)

where x and y are scalars

or, $-\vec{b}+2\vec{c} = x(\vec{a}-2\vec{b}+3\vec{c}) + y(-2\vec{a}+3\vec{b}-4\vec{c})$

or, $-\vec{b}+2\vec{c} = (x-2y)\vec{a} + (-2x+3y)\vec{b} + (3x-4y)\vec{c}$

Equating, we have

$x - 2y = 0$ ... (ii)

$-2x + 3y = -1$ ... (iii)

$3x - 4y = 2$ ... (iv)

Solving (ii) and (iii), we get $x = 2$, $y = 1$

Substituting the value of x and y in (iv)

$3 \times 2 - 4 \times 1 = 2$ (true)

$\therefore$ From (i)

$\vec{r_3} = 2\vec{r_1} + \vec{r_2}$

Hence the given vectors are coplanar.

47. 2069 (Set A) Old Q.No. 10a

OB and OC are two straight lines and D is a point on BC such that BD:DC = m:n, show that

$$\vec{OD} = \frac{n\vec{OB} + m\vec{OC}}{m+n}$$

➥ Please refer to 2062 Q.No. 10a

48. 2070 (Old) Q.No. 10 a

Prove that the vectors $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-4\vec{c}$ and $-\vec{b}+2\vec{c}$ are coplanar.

➥ Please refer to 2068 Q.No. 10a

B. PRODUCT OF VECTORS

2 MARKS QUESTIONS

49. 2057 Q.No. 3 b

Show that the vectors $2\vec{i}+3\vec{j}-8\vec{k}$ and $2\vec{i}+4\vec{j}+2\vec{k}$ are orthogonal. **[2]**

SOLUTION

Let $\vec{a} = 2\vec{i}+3\vec{j}-8\vec{k}$

$\vec{b} = 2\vec{i}+4\vec{j}+2\vec{k}$

Now,

$\vec{a}\cdot\vec{b} = (2\vec{i}+3\vec{j}-8\vec{k}).(2\vec{i}+4\vec{j}+2\vec{k})$

$= 4 + 12 - 16 = 0$

This shows that $\vec{a}$ and $\vec{b}$ are orthogonal.

50. 2058 Q.No. 3 b

Show that the area of the parallelogram determined by $\vec{i}+\vec{j}-3\vec{k}$ and $-\vec{i}-2\vec{j}-3\vec{k}$ is $\sqrt{118}$ sq. units **[2]**

SOLUTION

Let $\vec{a} = \vec{i}+\vec{j}-3\vec{k}$

$\vec{b} = -\vec{i}-2\vec{j}-3\vec{k}$

Now,

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -3 \\ -1 & -2 & -3 \end{vmatrix}$

$= \begin{vmatrix} 1 & -3 \\ -2 & -3 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & -3 \\ -1 & -3 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix}\vec{k}$

$= -9\vec{i} + 6\vec{j} - \vec{k}$

Area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$

$= \sqrt{(-9)^2+6^2+(-1)^2} = \sqrt{198}$ sq. units.

51. 2059 Q.No. 4 a

Find the angle between two vectors: $\vec{a} = \vec{i}+\vec{j}-2\vec{k}$ and $\vec{b} = 2\vec{i}-\vec{j}-\vec{k}$. **[2]**

SOLUTION

Here, $\vec{a} = \vec{i}+\vec{j}-2\vec{k}$

$\vec{b} = 2\vec{i}-\vec{j}-\vec{k}$

$|\vec{a}| = \sqrt{1^2+1^2+(-2)^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{2^2+(-1)^2+(-1)^2} = \sqrt{6}$

$\vec{a}\cdot\vec{b} = (\vec{i}+\vec{j}-2\vec{k}).(2\vec{i}-\vec{j}-\vec{k})$

$= 2 - 1 + 2 = 3$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3}{(\sqrt{6})(\sqrt{6})} = \frac{3}{6} = \frac{1}{2} = \cos 60°$

$\therefore \theta = 60°$

52. 2060 Q.No. 3 b

Find the cosine of the angle between the vectors $2\vec{i}+\vec{j}+\vec{k}$ and $4\vec{i}+3\vec{j}+5\vec{k}$. **[2]**

SOLUTION

Let $\vec{a} = 2\vec{i}+\vec{j}+\vec{k}$

$\vec{b} = 4\vec{i}+3\vec{j}+5\vec{k}$

Now,

$|\vec{a}| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{4^2+3^2+5^2} = 5\sqrt{2}$

$\vec{a}\cdot\vec{b} = (2\vec{i}+\vec{j}+\vec{k}).(4\vec{i}+3\vec{j}+5\vec{k})$

$= 8 + 3 + 5 = 16$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{16}{\sqrt{6}(5\sqrt{2})} = \frac{8}{5\sqrt{3}}$

53. 2061 Q.No. 3 b

Find a unit vector perpendicular to $2\vec{i}+3\vec{j}-\vec{k}$ and $\vec{i}+\vec{j}-2\vec{k}$. **[2]**

SOLUTION

Let $\vec{a} = 2\vec{i}+3\vec{j}-\vec{k}$

$\vec{b} = \vec{i}+\vec{j}-2\vec{k}$

Now,

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}\vec{k}$

$= -5\vec{i} + 3\vec{j} - \vec{k}$

$|\vec{a}\times\vec{b}| = \sqrt{(-5)^2+3^2+(-1)^2} = \sqrt{35}$

Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$= \frac{1}{\sqrt{35}}(-5\vec{i}+3\vec{j}-\vec{k}) = \frac{-5}{\sqrt{35}}\vec{i} + \frac{3}{\sqrt{35}}\vec{j} - \frac{1}{\sqrt{35}}\vec{k}$

54. 2061 Q.No. 4 a

If $\vec{a}$ and $\vec{b}$ are two vectors of unit length and θ is the angle between them. Show that $\frac{1}{2}|\vec{a}-\vec{b}| = \sin\frac{\theta}{2}$. **[2]**

SOLUTION

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})^2$

$= a^2 - 2\vec{a}\cdot\vec{b} + b^2$

$= 1 - 2\vec{a}\cdot\vec{b} + 1 \;(\because |\vec{a}| = |\vec{b}| = 1)$

$= 2 - 2|\vec{a}||\vec{b}|\cos\theta$

$(\because \vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta)$

$= 2 - 2\cos\theta = 2(1 - \cos\theta) = 2\left(2\sin^2\theta/2\right)$

or, $|\vec{a} - \vec{b}|^2 = (2\sin\theta/2)^2$

or, $|\vec{a} - \vec{b}| = 2\sin\dfrac{\theta}{2}$

$\therefore \dfrac{1}{2}|\vec{a} - \vec{b}| = \sin\dfrac{\theta}{2}$

55. 2062 Q.No. 3 b

Find the area of the parallelogram determined by the vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$ [2]

SOLUTION

Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{b} = -3\vec{i} - 2\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 2 \\ -3 & -2 \end{vmatrix}\vec{k}$$

$= 8\vec{i} - 10\vec{j} + 4\vec{k}$

Area of the parallelogram determined by $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}| = \sqrt{8^2 + (-10)^2 + 4^2} = 6\sqrt{5}$ sq. units

56. 2063 Q.No. 4 a

If $\vec{i}, \vec{j}, \vec{k}$ are three mutually perpendicular unit vectors and $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} - 3\vec{j} - \vec{k}$, find the cosine of the angle between the two vectors. [2]

SOLUTION

Here, $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = 2\vec{i} - 3\vec{j} - \vec{k}$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{2^2 + (-3)^2 + (-1)^2} = \sqrt{14}$

$\vec{a} \cdot \vec{b} = (\vec{i} - 2\vec{j} + \vec{k}) \cdot (2\vec{i} - 3\vec{j} - \vec{k})$
$= 2 + 6 - 1 = 7$

If θ be the angle between $\vec{a}$ and $\vec{b}$ then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{7}{\sqrt{6}\sqrt{14}} = \sqrt{\frac{7}{12}}$$

57. 2064 Q.No. 3 b

Find the area of the triangle determined by the vectors $3\vec{i} + 4\vec{j}$ and $-5\vec{i} + 7\vec{j}$. [2]

SOLUTION

Let $\vec{a} = 3\vec{i} + 4\vec{j}$ $\vec{b} = -5\vec{i} + 7\vec{j}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ -5 & 7 & 0 \end{vmatrix}$$

$= 0\vec{i} + 0\vec{j} + 41\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 0^2 + 41^2} = 41$

Area of triangle determined by $\vec{a}$ and $\vec{b} = \dfrac{1}{2}|\vec{a} \times \vec{b}| = \dfrac{41}{2} = 20.5$ sq. units

58. 2065 Q.No 4 a

Find the value of r if the vectors $3\vec{i} - \vec{j} - 2\vec{k}$ and $2\vec{i} - 2\vec{j} + r\vec{k}$ are orthogonal. [2]

SOLUTION

Let $\vec{a} = 3\vec{i} - \vec{j} - 2\vec{k}$

$\vec{b} = 2\vec{i} - 2\vec{j} + r\vec{k}$

The vectors $\vec{a}$ and $\vec{b}$ will be orthogonal if $\vec{a} \cdot \vec{b} = 0$

i.e. $(3\vec{i} - \vec{j} - 2\vec{k}) \cdot (2\vec{i} - 2\vec{j} + r\vec{k}) = 0$

or, $6 + 2 - 2r = 0$

$\therefore r = 4$

59. 2066 C Q.No. 4 a

If $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, find the projection of $\vec{a}$ on $\vec{b}$.

SOLUTION

Here, $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$

$\vec{b} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$\vec{a} \cdot \vec{b} = (\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (2\vec{i} + 3\vec{j} + 4\vec{k})$
$= 2 + 6 + 12 = 20$

$|\vec{b}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{20}{\sqrt{29}}$.

60. 2066 Q.No. 3 b

For what value of m are the vectors $\vec{i} - 2\vec{j} + 4\vec{k}$ and $2\vec{i} + 7\vec{j} + m\vec{k}$ orthogonal?

SOLUTION

Let $\vec{a} = \vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{b} = 2\vec{i} + 7\vec{j} + m\vec{k}$

The vectors $\vec{a}$ and $\vec{b}$ will be orthogonal if $\vec{a} \cdot \vec{b} = 0$

or, $(\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (2\vec{i} + 7\vec{j} + m\vec{k}) = 0$

or, $2 - 14 + 4m = 0$

$\therefore m = 3$

61. 2067 Q.No. 3b

Given $\vec{a} = (3, 1, 2)$ and $\vec{b} = (2, -2, 4)$, find the projection of $\vec{a}$ on $\vec{b}$.

SOLUTION

Here, $\vec{a} = (3, 1, 2)$

$\vec{b} = (2, -2, 4)$

Now, $\vec{a} \cdot \vec{b} = (3, 1, 2) \cdot (2, -2, 4) = 6 - 2 + 8$

$|\vec{b}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}$

Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{12}{\sqrt{24}} = \sqrt{6}$

62. 2068 Q.No. 3b

Find the area of the triangle determined by the vectors $3\vec{i} + 4\vec{j}$ and $-5\vec{i} + 7\vec{j}$. [2]

➥ Please refer to 2064 Q.No. 3b

63. 2069 (Set A) Old Q.No. 3b

If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that $\vec{a}$ is perpendicular to $\vec{b}$. [2]

SOLUTION

Here,

$|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

or, $|\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

or, $(\vec{a} + \vec{b})^2 = (\vec{a} - \vec{b})^2$

or, $a^2 + 2\vec{a} \cdot \vec{b} + b^2 = a^2 - 2\vec{a} \cdot \vec{b} + b^2$

or, $4\vec{a} \cdot \vec{b} = 0$

or, $\vec{a} \cdot \vec{b} = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{b}$.

64. 2069 (Set B) Q.No. 3c

If $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$ are any two vectors, find the cosine of the angle between the two vectors. [2]

SOLUTION

Here,

$\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$

$\vec{b} = 2\vec{i} - \vec{j} - \vec{k}$

$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{2^2 + (-1)^2 + (-1)^2} = \sqrt{6}$

$\vec{a} \cdot \vec{b} = (\vec{i} + \vec{j} - 2\vec{k}) \cdot (2\vec{i} - \vec{j} - \vec{k})$
$= 2 - 1 + 2 = 3$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

65. 2069 Old (Set B) Q.No. 3b

For what value of x is the pair of vectors $x\vec{i} - 2\vec{j} + 4\vec{k}$ and $2\vec{i} + 7\vec{j} + \vec{k}$ orthogonal? [2]

SOLUTION

Let $\vec{a} = x\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{b} = 2\vec{i} + 7\vec{j} + \vec{k}$

The vectors $\vec{a}$ and $\vec{b}$ will be orthogonal if $\vec{a} \cdot \vec{b} = 0$

i.e. $(x\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (2\vec{i} + 7\vec{j} + \vec{k}) = 0$

or, $2x - 14 + 4 = 0$

$\therefore x = 5$

66. 2069 Old (Set B) Q.No. 4a

Find the area of the triangle determine by the vectors: $3\vec{i} + 4\vec{j} + \vec{k}$ and $-5\vec{i} + 7\vec{j}$. [2]

SOLUTION

Let $\vec{a} = 3\vec{i} + 4\vec{j} + \vec{k}$

$\vec{b} = -5\vec{i} + 7\vec{j}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 1 \\ -5 & 7 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 \\ 7 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 1 \\ -5 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 4 \\ -5 & 7 \end{vmatrix}\vec{k}$$

$= -7\vec{i} + 5\vec{j} + 41\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-7)^2 + 5^2 + 41^2} = \sqrt{1755}$

Area of triangle determined by $\vec{a}$ and $\vec{b}$ is

$\dfrac{1}{2}|\vec{a} \times \vec{b}| = \dfrac{1}{2}\sqrt{1755} = 20.95$ sq. units.

67. 2069 (Set A) Q.No. 3c

Find the area of the parallelogram determined by the vectors $\vec{i} + 2\vec{j} + 3\vec{k}$ and $-3\vec{i} - 2\vec{j} + \vec{k}$. [2]

➥ Please refer to 2062 Q.No. 3b

68. 2070 Set C Q.No. 3 c

If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, prove that $\vec{a}$ is perpendicular to $\vec{b}$. [2]

➥ Please refer to 2069 Set A (Old) Q.No. 3b

69. 2070 Set D Q.No. 3 c

Find the sine of the angle between the two vectors $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$ [2]

SOLUTION

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 1 \\ 4 & -1 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}\vec{k}$$

$= -3\vec{i} + 5\vec{j} + 11\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + 5^2 + 11^2} = \sqrt{155}$

If θ be the angle between $\vec{a}$ and $\vec{b}$, then

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}\sqrt{26}} = \sqrt{\frac{155}{156}}$$

70. 2070 (Old) Q.No. 4 b

Find the unit vector perpendicular to the vectors $4\vec i - 2\vec j + 3\vec k$ and $5\vec i + \vec j - 4\vec k$ [2]

SOLUTION

Let $\vec a = 4\vec i - 2\vec j + 3\vec k$

$\vec b = 5\vec i + \vec j - 4\vec k$

$$\vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 4 & -2 & 3 \\ 5 & 1 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 3 \\ 1 & -4 \end{vmatrix}\vec i - \begin{vmatrix} 4 & 3 \\ 5 & -4 \end{vmatrix}\vec j + \begin{vmatrix} 4 & -2 \\ 5 & 1 \end{vmatrix}\vec k$$

$= 5\vec i - 31\vec j + 14\vec k$

$|\vec a \times \vec b| = \sqrt{5^2 + (-31)^2 + 14^2} = \sqrt{1182}$

Unit vector perpendicular to $\vec a$ and $\vec b$ is

$$\frac{\vec a \times \vec b}{|\vec a \times \vec b|}$$

$$= \frac{1}{\sqrt{1182}}(5\vec i - 31\vec j + 14\vec k) = \frac{5\vec i - 31\vec j + 14\vec k}{\sqrt{1182}}$$

71. 2070 Supp. Q.No. 2 c

If θ is the angle between two unit vectors $\vec a$ and $\vec b$, show that $\frac{1}{2}|\vec a - \vec b| = \sin\frac{\theta}{2}$. [2]

➥ Please refer to 2061 Q.No. 4a

72. 2071 Set C Q.No. 3 c

For what value of m is the pair of vectors $\vec i - 2\vec j + 4\vec k$ and $2\vec i + 7\vec j + m\vec k$ orthogonal? [2]

SOLUTION

Let $\vec a = \vec i - 2\vec j + 4\vec k$

$\vec b = 2\vec i + 7\vec j + m\vec k$

The vectors $\vec a$ and $\vec b$ will be orthogonal

if $\vec a \cdot \vec b = 0$

i.e. $(\vec i - 2\vec j + 4\vec k) \cdot (2\vec i + 7\vec j + m\vec k) = 0$

or, $2 - 14 + 4m = 0$

∴ $m = -3$

73. 2071 Set D Q.No. 3 c

Find a unit vector perpendicular to each of the vectors $3\vec i + \vec j + 2\vec k$ and $2\vec i - 2\vec j + 4\vec k$ [2]

SOLUTION

Let $\vec a = 3\vec i + \vec j + 2\vec k$

$\vec b = 2\vec i - 2\vec j + 4\vec k$

Unit vector perpendicular to $\vec a$ and $\vec b$ is

$$\frac{\vec a \times \vec b}{|\vec a \times \vec b|}$$

Here, $\vec a \times \vec b = \begin{vmatrix} \vec i & \vec j & \vec k \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}\vec i - \begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix}\vec j + \begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}\vec k$$

$= 8\vec i - 8\vec j - 8\vec k$

$|\vec a \times \vec b| = \sqrt{8^2 + (-8)^2 + (-8)^2} = 8\sqrt3$

Unit vector perpendicular to $\vec a \times \vec b$

$$\frac{\vec a \times \vec b}{|\vec a \times \vec b|}$$

$$= \frac{1}{8\sqrt3}(8\vec i - 8\vec j - 8\vec k) = \frac{1}{\sqrt3}(\vec i - \vec j - \vec k)$$

74. 2071 Old Q.No. 3 c

Find the vector perpendicular to each of vectors (1, 3, – 4) and (2, 1, –1).

SOLUTION

Let $\vec a = (1, 3, -4)$

$\vec b = (2, 1, -1)$

Now,

$$\begin{array}{ccccc} 1 & 3 & -4 & 1 & 3 \\ 2 & 1 & -1 & 2 & 1 \end{array}$$

$\vec a \times \vec b = (-3 + 4, -8 + 1, 1 - 6) = (1, -7, -5)$

∴ The vector perpendicular to $\vec a$ and $\vec b$ is a
$= (1, -7, -5)$

75. 2072 Set C Q.No. 3c

Find the angle between the vectors $2\vec i - \vec j + \vec k$ and $\vec i - 3\vec j - 5\vec k$.

SOLUTION

Let $\vec a = 2\vec i - \vec j + \vec k$

$\vec b = \vec i - 3\vec j - 5\vec k$

$|\vec a| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt6$

$|\vec b| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{35}$

$\vec a \cdot \vec b = (2\vec i - \vec j + \vec k) \cdot (\vec i - 3\vec j - 5\vec k)$

$= 2 + 3 - 5 = 0$

If θ be the angle between $\vec a$ and $\vec b$ then

$$\cos\theta = \frac{\vec a \cdot \vec b}{|\vec a|\,|\vec b|} = \frac{0}{\sqrt6\sqrt{35}} = 0 = \cos 90°$$

∴ θ = 90°

76. 2072 Set D Q.No. 3c

If $(\vec a + \vec b) \cdot (\vec a - \vec b) = 0$, prove that $|\vec a| = |\vec b|$

SOLUTION

Here, $(\vec a + \vec b) \cdot (\vec a - \vec b) = 0$

or, $\vec a \cdot \vec a - \vec a \cdot \vec b + \vec b \cdot \vec a - \vec b \cdot \vec b = 0$

or, $|\vec a|^2 - \vec a \cdot \vec b + \vec a \cdot \vec b - |\vec b|^2 = 0$

$(\because \vec a \cdot \vec b = \vec b \cdot \vec a)$

or, $|\vec a|^2 = |\vec b|^2$

∴ $|\vec a| = |\vec b|$

77. 2072 Set E Q.No. 3c

Find the area of the triangle determined by the vectors $3\vec i + 4\vec j$ and $-5\vec i + 7\vec j$. [2]

➥ Please refer to 2064 Q.No. 3b

78. 2072 Supp Q.No. 3c

If $|\vec a + \vec b| = |\vec a - \vec b|$ prove that $\vec a$ is perpendicular to $\vec b$. [2]

➥ Please refer to 2069 (Set A) Old Q.No. 3b

79. 2073 Set C Q.No. 3c

Show that vector product $\vec a \times \vec b$ is perpendicular to both vectors $\vec a$ and $\vec b$. [2]

SOLUTION

Let $\vec a = (a_1, a_2, a_3)$ and $\vec b = (b_1, b_2, b_3)$

Then,

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \end{array}$$

$\vec a \times \vec b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

Now, $(\vec a \times \vec b) \cdot \vec a$

$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (a_1, a_2, a_3)$

$= a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1)$

$= a_1a_2b_3 - a_1b_2a_3 + b_1a_2a_3 - a_1a_2b_3 + a_1b_2a_3 - b_1a_2a_3$

$= 0$

∴ $\vec a \times \vec b$ is perpendicular to $\vec a$.

Similarly, $(\vec a \times \vec b) \cdot \vec b$

$= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot (b_1, b_2, b_3)$

$= b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1)$

$= 0$

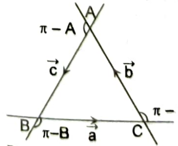
∴ $\vec a \times \vec b$ is perpendicular to $\vec b$.

80. 2073 Set D Q.No. 3c

If $\vec a + \vec b + \vec c = 0$, prove that: $\vec a \times \vec b = \vec b \times \vec c = \vec c \times \vec a$. [2]

SOLUTION

In ΔABC, let $\vec{BC} = \vec a$, $\vec{CA} = \vec b$, and $\vec{AB} = \vec c$



By definition of vector addition,

$\vec{AB} = \vec{AC} + \vec{CB}$

or, $\vec c = -\vec b - \vec a$

or, $\vec a + \vec b + \vec c = 0$ … (i)

From (i)

$\vec a \times \vec a + \vec a \times \vec b + \vec a \times \vec c = 0$

or, $0 + \vec a \times \vec b = -\vec a \times \vec c$ $(\because \vec a \times \vec a = 0)$

or, $\vec a \times \vec b = \vec c \times \vec a$ …(ii)

Again, from (i)

$\vec a \times \vec b + \vec b \times \vec b + \vec c \times \vec b = 0$

or, $\vec a \times \vec b + 0 = -\vec c \times \vec b$

or, $\vec a \times \vec b = \vec b \times \vec c$ …(iii)

From (ii) and (iii),

$\vec a \times \vec b = \vec b \times \vec c = \vec c \times \vec a$

81. 2073 Supp Q.No. 3c

If $\vec a = \vec i + 2\vec j - \vec k$ and $\vec b = \vec i - \vec j + \vec k$ find the projection of $\vec a$ on $\vec b$. [2]

SOLUTION

Here,

$\vec a = \vec i + 2\vec j - \vec k$

$\vec b = \vec i - \vec j + \vec k$

Now, $\vec a \cdot \vec b = (\vec i + 2\vec j - \vec k) \cdot (\vec i - \vec j + \vec k)$

$= 1 - 2 - 1$

$= -2$

$|\vec b| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt3$

Projection of $\vec a$ on $\vec b = \frac{\vec a \cdot \vec b}{|\vec b|} = \frac{-2}{\sqrt3}$

82. 2074 Set A Q.No. 2c

Prove that $(\vec a - \vec b) \times (\vec a + \vec b) = 2\vec a \times \vec b$. [2]

SOLUTION

L.H.S $= (\vec a - \vec b) \times (\vec a + \vec b)$

$= \vec a \times \vec a + \vec a \times \vec b - \vec b \times \vec a - \vec b \times \vec b$

$= \vec 0 + \vec a \times \vec b - \vec b \times \vec a - \vec 0$

$[\because \vec a \times \vec a = \vec 0 \ \& \ \vec b \times \vec b = \vec 0]$

$= \vec a \times \vec b + \vec a \times \vec b$ $[\because \vec a \times \vec b = -(\vec b \times \vec a)]$

$= 2\vec a \times \vec b$ = R.H.S.

83. 2074 Set B Q.No. 3c

Find the area of the parallelogram determined by the vectors $\vec i + 2\vec j + 3\vec k$ and $-3\vec i - 2\vec j + \vec k$ [2]

➥ Please refer to 2069 Set A Q. No. 3c

84. 2074 Supp Q.No. 3c

If $\vec a = 6\vec i + 3\vec j - 5\vec k$ and $\vec b = \vec i - 4\vec j + 2\vec k$, show that $\vec a \times \vec b$ is perpendicular to $\vec a$. [2]

SOLUTION

Here,

$\vec{a} = 6\vec{i} + 3\vec{j} - 5\vec{k}$

$\vec{b} = \vec{i} - 4\vec{j} + 2\vec{k}$

Now,

$\vec{a} \times \vec{b}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 3 & -5 \\ 1 & -4 & 2 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 3 & -5 \\ -4 & 2 \end{vmatrix} - \vec{j}\begin{vmatrix} 6 & -5 \\ 1 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 6 & 3 \\ 1 & -4 \end{vmatrix}$$

$= \vec{i}(6 - 20) - \vec{j}(12 + 5) + \vec{k}(-24 - 3)$

$= -14\vec{i} - 17\vec{j} - 27\vec{k}$

To show $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$, we have to prove $\vec{a} \times \vec{b} \cdot \vec{a} = 0$.

Now,

$(\vec{a} \times \vec{b}) \cdot \vec{a}$

$= (-14\vec{i} - 17\vec{j} - 27\vec{k}) \cdot (6\vec{i} + 3\vec{j} - 5\vec{k})$

$= -84 - 51 + 135 = 0$

This shows that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

85. 2075 Set A Q.No. 3c

Find the cosine of the angle between the two vectors $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j} = 2\vec{k}$. [2]

SOLUTION

Given,

$\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$

$\vec{b} = \vec{i} + 3\vec{j} + 2\vec{k}$

$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$

$\vec{a} \cdot \vec{b} = (\vec{i} - 2\vec{j} + 3\vec{k}) \cdot (\vec{i} + 3\vec{j} + 2\vec{k})$
$= 1 - 6 + 6 = 1$

If θ be the angle between $\vec{a}$ and $\vec{b}$ then,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{\sqrt{14}\sqrt{14}} = \frac{1}{14}$$

86. 2075 Set C Q.No. 2c

If $\vec{a} = (1, 2)$ and $\vec{b} = (-3, 1)$, find the projection of $\vec{a}$ on $\vec{b}$. [2]

SOLUTION

Given,

$\vec{a} = (1, 2)$, $\vec{b} = (-3, 1)$

Now,

$\vec{a} \cdot \vec{b} = (1, 2) \cdot (-3, 1) = -3 + 2 = -1$

And,

$|\vec{b}| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = -\frac{1}{\sqrt{10}}$$

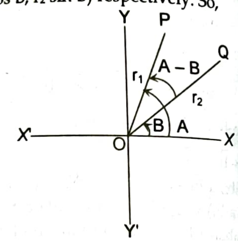
4 MARKS QUESTIONS

87. 2057 Q.No. 11 a

Prove by vector method:
cos (A – B) = cosA cosB + sinA sinB

SOLUTION

Let XOX' and YOY' be two mutual perpendicular straight lines representing axis and y-axis respectively. Let ∠XOQ = and ∠XOP = A so that ∠QOP = A – B. Again let OP = r_1 and OQ = r_2. Then the coordinates of P and Q are (r_1 cos A, r_1 sin A) and (r_2 cos B, r_2 sin B) respectively. So,



$\vec{OP}$ = (r_1 cos A, r_1 sin A)

$\vec{OQ}$ = (r_2 cos B, r_2 sin B)

Now,

$\vec{OP} \cdot \vec{OQ}$ = (r_1 cos A, r_1 sin A) . (r_2 cos B, r_2 sin B)
= $r_1 r_2$ cos A cos B + $r_1 r_2$ sin A sin B
= $r_1 r_2$ (cos A cos B + sin A sin B)

Since (A – B) is the angle between $\vec{OQ}$ and $\vec{OP}$,

$$\cos(A-B) = \frac{\vec{OQ} \cdot \vec{OP}}{|\vec{OQ}|\,|\vec{OP}|}$$

$$= \frac{r_1 r_2(\cos A \cos B + \sin A \sin B)}{r_1 r_2}$$

= cos A cos B + sin A sin B

88. 2058 Q.No. 11 a

Prove by vector method.
cos(A – B) = cosA cosB + sin A sin B

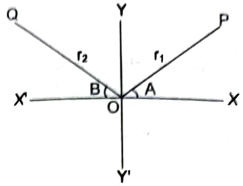
➥ Please refer to 2057 Q.No. 11a

89. 2059 Q.No. 11 a

Prove by vector method:
sin (A + B) = sin A cos B + cos A sin B

SOLUTION

Let XOX' and YOY' be two mutual perpendicular straight lines representing x-axis and y-axis respectively. Let ∠XOP = and ∠QOX' = B so that ∠POQ = π – (A + B)



Also, let OP = r_1 and OQ = r_2.

Then the coordinates of P and Q are (r_1 cos A, $r_1$sin A)

and (r_2 cos (π – B), r_2 sin (π – B)) = (–r_2 cos B, r_2 sin B).

So, $\vec{OP}$ = (r_1 cos A, r_1 sin A)
= (r_1 cos A, r_1 sin A, 0)

& $\vec{OQ}$ = (–r_2 cos B, r_2 sin B) = (–r_2 cos B, r_2 sin B, 0)

Now,

r_1 cos A	r_1 sin A	0	r_1 cos A	r_1 sin A	
–r_2 cos B	r_2 sin B	0	–r_2 cos B	r_2 sin B	

$\vec{OP} \times \vec{OQ}$ = (0, 0, $r_1 r_2$ cos A sin B + $r_1 r_2$ sin A cos B)

$|\vec{OP} \times \vec{OQ}|$ = $r_1 r_2$(sin A cos B + cos A sin B)

Since π – (A + B) is the angle between OP and OQ,

$$\sin[\pi - (A+B)] = \frac{|\vec{OP} \times \vec{OQ}|}{|\vec{OP}|\,|\vec{OQ}|}$$

$$= \frac{r_1 r_2(\sin A \cos B + \cos A \sin B)}{r_1 r_2}$$

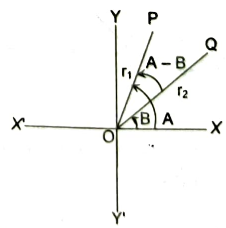
∴ sin (A + B) = sin A cos B + cos A sin B

90. 2060 Q.No. 11 a

Prove by vector method:
sin (A – B) = sin A cos B – cos A sin B [4]

SOLUTION

Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let ∠XOQ = B and ∠XOP = A so that ∠QOP = A – B. Again, let OP = r_1 and OQ = r_2. Then the coordinates of P and Q are



(r_1 cos A, r_1 sin A) and (r_2 cos B, r_2 sin B) respectively. So,

$\vec{OP}$ = (r_1 cos A, r_1 sin A) = (r_1 cos A, r_1 sin A, 0)

$\vec{OQ}$ = (r_2 cos B, r_2 sin B) = (r_2 cos B, r_2 sin B, 0)

Now,

r_2 cos B	r_2 sinB	0	r_2 cos B	r_2 sin B	
r_1 cos A	r_1 sin A	0	r_1 cos A	r_1 sin A	

$\vec{OQ} \times \vec{OP}$=(0, 0, $r_1 r_2$ sin A cos B – $r_1 r_2$ cos A sin B)

$|\vec{OQ} \times \vec{OP}|$ = $r_1 r_2$ (sin A cos B – cos A sin B)

Since (A – B) is the angle between OQ and OP, so

$$\sin(A-B) = \frac{|\vec{OQ} \times \vec{OP}|}{|\vec{OQ}|\,|\vec{OP}|}$$

$$= \frac{r_1 r_2(\sin A \cos B - \cos A \sin B)}{r_1 r_2}$$

= sin A cos B – cos A sin B

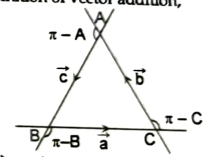
91. 2061 Q.No. 11 a

Prove, in any triangle, by vector method that:
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$ [4]

SOLUTION

In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, and $\vec{AB} = \vec{c}$

By definition of vector addition,



$\vec{AB} = \vec{AC} + \vec{CB}$

or, $\vec{c} = -\vec{b} - \vec{a}$

or, $\vec{a} + \vec{b} + \vec{c} = 0$ … (i)

From (i)

$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$

or, $0 + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$ (∵ $\vec{a} \times \vec{a} = 0$)

or, $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ …(ii)

Again, from (i)

$\vec{a} \times \vec{b} + \vec{b} \times \vec{b} + \vec{c} \times \vec{b} = 0$

or, $\vec{a} \times \vec{b} + 0 = -\vec{c} \times \vec{b}$

or, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ …(iii)

From (ii) and (iii),

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Taking modulus on each side, we have

$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or, ab sin (π – C) = bc sin (π – A) = ca sin (π – B)

or, ab sin c = bc sin A = ca sin B

or, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

∴ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

92. 2062 Q.No. 11 a

Prove vectorically that:
cos (A – B) = cos A cos B + sin A sin B [4]

➥ Please refer to 2057 Q.No. 11a

93. 2063 Q.No. 11 a

Using vector method, prove in any triangle that: a = b cos C + c cos B [4]

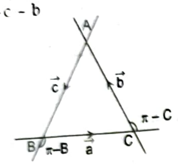
SOLUTION

In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$

By definition of vector addition, we have

$\vec{BC} = \vec{BA} + \vec{AC}$

or, $\vec{a} = -\vec{c} - \vec{b}$



Multiplying both sides scalarly by $\vec{a}$, we have

$\vec{a}\cdot\vec{a} = -\vec{a}\cdot\vec{c} - \vec{a}\cdot\vec{b}$

or, $a^2 = -ac\cos(\pi - B) - ab\cos(\pi - C)$

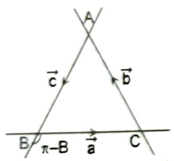
or, $a^2 = ac\cos B + ab\cos C$

∴ $a = c\cos B + b\cos C$

94. 2064 Q.No. 11 a

Using vector method, prove in any triangle, that: $b^2 = c^2 + a^2 - 2ac\cos B$ [4]

SOLUTION



In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$

By definition of vector addition, we have

$\vec{CA} = \vec{CB} + \vec{BA}$

or, $\vec{b} = -\vec{a} - \vec{c}$

or, $b^2 = (-\vec{a} - \vec{c})^2 = (\vec{a} + \vec{c})^2$

$= a^2 + 2\vec{a}\cdot\vec{c} + c^2$

$= a^2 + c^2 + 2ac\cos(\pi - B)$

∴ $b^2 = a^2 + c^2 - 2ac\cos B$

95. 2065 Q.No 11 a

By using vectors, prove that in any ΔABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ [4]

➥ Please refer to 2061 Q.No. 11a

96. 2066 C Q.No. 11 a

Prove by vector method: cos (A – B) = cos A cos B + sin A sin B [4]

➥ Please refer to 2057 Q.No. 11a

97. 2066 Q.No.11 a

Use vector method to prove that, in any triangle ABC, a = b cos C + c cos B. [4]

➥ Please refer to 2063 Q.No. 11a

98. 2067 Q.No. 11a

Prove by vector method: cos (A + B) = cos A cos B – sin A sin B. [4]

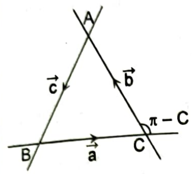
➥ Please refer to Model Set II, Q.No. 10

99. 2068 Q.No. 11 a

Using vector method, prove that: $c^2 = a^2 + b^2 - 2ab\cos C$ [4]

SOLUTION

In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$.



By definition of vector addition,

$\vec{AB} = \vec{AC} + \vec{CB}$

or, $\vec{c} = -\vec{b} - \vec{a}$

or, $c^2 = (-\vec{b} - \vec{a})^2$

$= (\vec{a} + \vec{b})^2$

$= a^2 + 2\vec{a}\cdot\vec{b} + b^2$

$= a^2 + b^2 + 2ab\cos(\pi - C)$

∴ $c^2 = a^2 + b^2 - 2ab\cos C$

100. 2069 (Set A) Old Q.No. 11a

Prove by vector method that: sin (A – B) = sinA cosB – cosA sinB. [4]

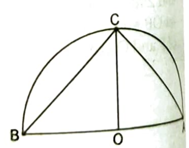
➥ Please refer to 2060 Q.No. 11a

101. 2069 Old (Set B) Q.No. 9a

Prove, analytically that the angle in a semi-circle is a right angle. [4]

SOLUTION

Let ACB be an angle in the semi-circle. Let O be the origin.



Let $\vec{OA} = \vec{a}$. Then

$\vec{OB} = -\vec{OA} = -\vec{a}$

Let $\vec{OC} = \vec{c}$

Then,

$\vec{CA} = \vec{CO} + \vec{OA} = -\vec{c} + \vec{a} = \vec{a} - \vec{c}$

and $\vec{CB} = \vec{CO} + \vec{OB} = -\vec{c} - \vec{a}$

Now,

$\vec{CA}\cdot\vec{CB} = (\vec{a} - \vec{c})\cdot(-\vec{a} - \vec{c}) = a^2 - c^2$

$= 0$ (∵ OA = OC)

∴ ∠ACB = 90°

102. 2069 Old (Set B) Q.No. 10a

Show that the three points whose position vectors are: $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{i} - 4\vec{j} - 4\vec{k}$ form the sides of a right angled triangle. Also, find the remaining two angles. [4]

SOLUTION

In ΔABC, let the position vectors of A, B and C be $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} - 3\vec{j} - 5\vec{k}$ and $3\vec{i} - 4\vec{j} - 4\vec{k}$ respectively. Let O be the origin. Then,

$\vec{OA} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{OB} = \vec{i} - 3\vec{j} - 5\vec{k}$

$\vec{OC} = 3\vec{i} - 4\vec{j} - 4\vec{k}$

Now,

$\vec{AB} = \vec{OB} - \vec{OA} = -\vec{i} - 2\vec{j} - 6\vec{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 2\vec{i} - \vec{j} + \vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = \vec{i} - 3\vec{j} - 5\vec{k}$

$AB = |\vec{AB}| = \sqrt{(-1)^2 + (-2)^2 + (-6)^2} = \sqrt{41}$

$BC = |\vec{BC}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$

$AC = |\vec{AC}| = \sqrt{1^2 + (-3)^2 + (-5)^2} = \sqrt{35}$

Here, $BC^2 + AC^2 = 6 + 35 = 41 = (\sqrt{41})^2 = AB^2$

∴ ∠C = 90°

Hence, ABC is a right angled triangle.

Now,

$$\cos A = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{-1 + 6 + 30}{\sqrt{41}\sqrt{35}} = \sqrt{\frac{35}{41}}$$

∴ $A = \cos^{-1}\sqrt{\frac{35}{41}}$

$$\cos B = \frac{\vec{BC}\cdot\vec{BA}}{|\vec{BC}|\,|\vec{BA}|} = \frac{2 - 2 + 6}{\sqrt{6}\sqrt{41}} = \sqrt{\frac{6}{41}}$$

∴ $B = \cos^{-1}\sqrt{\frac{6}{41}}$

Hence, the remaining two angle are $\cos^{-1}\sqrt{\frac{35}{41}}$ and $\cos^{-1}\sqrt{\frac{6}{41}}$.

103. 2069 Old (Set B) Q.No. 11a

Prove vectorically that in any triangle. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ [4]

➥ Please refer to 2061 Q.No. 11a

104. 2070 (Old) Q.No. 11 a

Prove by vector method: cos (A + B) = cos A cos B - sin A sin B [4]

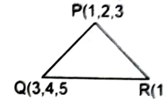
➥ Please refer to Model Set II, Q.No. 10

105. 2071 Old Q.No. 10 b

Show that the area of the triangle PQR whose vertices are P(1, 2, 3), Q(3, 4, 5) and R (1, 4, 7) is $2\sqrt{6}$ sq. units. [4]

SOLUTION

Let P(1, 2, 3), Q (3, 4, 5) and R(1, 4, 7) be three vertices of ΔPQR. Let O be the origin. Then,



$\vec{OP} = (1, 2, 3)$

$\vec{OQ} = (3, 4, 5)$

$\vec{OR} = (1, 4, 7)$

$\vec{PQ} = \vec{OQ} - \vec{OP} = (2, 2, 2)$

$\vec{PR} = \vec{OR} - \vec{OP} = (0, 2, 4)$

Now,

$\begin{matrix} 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 4 & 0 & 2 \end{matrix}$

$\vec{PQ}\times\vec{PR} = (8 - 4, 0 - 8, 4 - 0)$
$= (4, -8, 4)$

$|\vec{PQ}\times\vec{PR}| = \sqrt{4^2 + (-8)^2 + 4^2} = \sqrt{96} = 4\sqrt{6}$

∴ Area of $\Delta PQR = \frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{1}{2}(4\sqrt{6})$

$= 2\sqrt{6}$ sq. units.

106. 2071 Old Q.No. 11 a

Prove by vector method: cos (A + B) = cosA cosB – sinA sinB [4]

➥ Please refer to Model Set II, Q.No. 10

107. 2075 Set B Q.No. 7a

State and prove the sine law by the vector method. [4]

SOLUTION

Statement: In any triangle ABC, the sine law states that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

Proof: Please refer to 2061 Q.No. 11a

108. 2075 Set B Q.No. 7a OR

Prove that if θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}\cdot\vec{b} = ab\cos\theta$.

➥ Please refer to Model Set I Q.No 8b OR

6 MARKS QUESTIONS

109. 2069 (Set B) Q.No. 10

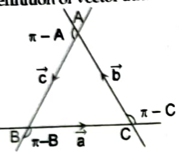
Define vector product of two vectors. Interpret the vector product of two vectors geometrically. Prove by vector method that: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ [6]

SOLUTION

Vector product of two vectors: The vector (cross) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is defined by $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$, and $\vec{AB} = \vec{c}$

By definition of vector addition,



$\vec{AB} = \vec{AC} + \vec{CB}$

or, $\vec{c} = -\vec{b} - \vec{a}$

or, $\vec{a} + \vec{b} + \vec{c} = 0$ ... (i)

From (i)

$\vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = 0$

or, $0 + \vec{a} \times \vec{b} = -\vec{a} \times \vec{c}$ $(\because \vec{a} \times \vec{a} = 0)$

or, $\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(ii)

Again, from (i)

$\vec{a} \times \vec{b} + \vec{b} \times \vec{b} + \vec{c} \times \vec{b} = 0$

or, $\vec{a} \times \vec{b} + 0 = -\vec{c} \times \vec{b}$

or, $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ...(iii)

From (ii) and (iii),

$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$

Taking modulus on each side, we have

$|\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$

or, ab sin (π – C) = bc sin (π – A) = ca sin (π – B)

or, ab sin c = bc sin A = ca sin B

or, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

110. 2069 (Set A) Q.No. 10

Define scalar product of two vectors. Prove by the method of vectors that:
cos (A – B) = cosA cosB + sinA sinB. [6]

➥ **First Part:** Please refer to Model Set II Q.No. 10

Second Part: Please refer to 2057 Q.No. 11a

111. 2070 Set C Q.No. 10

Define vector product of two vectors. Using vector method, prove that: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

➥ Please refer to 2069 Set B Q.No. 10

112. 2070 Set D Q.No. 10

Define scalar product of two vectors. Prove by vector method that:
cos (A – B) = cos A cos B + sin A sin B. [6]

➥ **First part:**
Please refer to Model Set II, Q.No. 10
Second part:
Please refer to 2057 Q.No. 11a

113. 2070 Supp. Q.No. 11

Define scalar product of two vectors. Prove by vector method that
cos(A+B) = cos A cos B – sin A sin B. [6]

➥ Please refer to Model Set II, Q.No. 10

114. 2070 Supp. Q.No. 11 OR

Define a vector product. Find a unit vector perpendicular to the plane of $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = \vec{i} - 2\vec{j} + \vec{k}$. Also compute the sine of the angle between them. [6]

SOLUTION

Vector product of two vectors: The vector (cross) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is defined by $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Here, $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \vec{k}$$

$= -3\vec{i} - 3\vec{j} - 3\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{(-3)^2 + (-3)^2 + (-3)^2} = 3\sqrt{3}$

Unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is

$$\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-3\vec{i} - 3\vec{j} - 3\vec{k}}{3\sqrt{3}} = \frac{1}{\sqrt{3}}(-\vec{i} - \vec{j} - \vec{k})$$

Also,

$|\vec{a}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}$

$|\vec{b}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

If θ be the angle between $\vec{a}$ and $\vec{b}$ then

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|} = \frac{3\sqrt{3}}{\sqrt{6}\sqrt{6}} = \frac{\sqrt{3}}{2}$$

115. 2071 Set C Q.No. 10

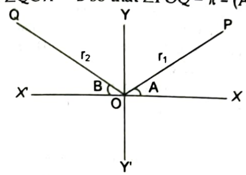
Define vector product of two vectors. Prove by vector method that
sin(A + B) = sin A cosB + cosA sinB.

SOLUTION

Vector product of two vectors: The vector (cross) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is defined by $\vec{a} \times \vec{b} = |\vec{a}|\ |\vec{b}| \sin\theta\ \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Second part

Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let ∠ XOP = A and ∠QOX' = B so that ∠POQ = π – (A + B).



Also, let OP = r_1 and OQ = r_2.

Then the coordinates of P and Q are (r_1 cos A, $r_1$sin A) and (r_2 cos (π – B), r_2 sin (π – B)) = (–r_2 cos B, r_2 sin B).

So, $\vec{OP}$ = (r_1 cos A, r_1 sin A)
= (r_1 cos A, r_1 sin A, 0)

& $\vec{OQ}$ = (–r_2 cos B, r_2 sin B) = (–r_2 cos B, r_2 sin B, 0)

Now,

r_1 cos A r_1 sin A   0 r_1 cos A r_1 sin A
–r_2 cos B r_2 sin B   0   –r_2 cos B r_2 sin B

$\vec{OP} \times \vec{OQ}$ = (0, 0, $r_1 r_2$ cos A sin B + $r_1 r_2$ sin A cos B)

$|\vec{OP} \times \vec{OQ}| = r_1 r_2$(sin A cos B + cos A sin B)

Since π – (A + B) is the angle between OP and OQ, so

$$\sin[\pi - (A + B)] = \frac{|\vec{OP} \times \vec{OQ}|}{|\vec{OP}|\,|\vec{OQ}|} = \frac{r_1 r_2 (\sin A \cos B + \cos A \sin B)}{r_1 r_2}$$

∴ sin (A + B) = sin A cos B + cos A sin B

116. 2071 Set D Q.No. 10

Define scalar product of two vectors. Prove by vector method that [6]
cos (A + B) = cosAcosB – sinAsinB

➥ Please refer to Model Set II, Q.No. 10

117. 2071 Supp. Q.No. 10

Define the scalar product of two vectors. Prove by vector method:
Cos (A+B) = CosA CosB – SinA SinB [6]

➥ Please refer to Model Set II, Q.No. 10

118. 2072 Set C Q.No. 10

Define Vector product of two Vectors. Prove by Vector method sin(A + B) = sinA cosB + cosA sinB [6]

➥ Please refer to 2071 Set C Q.No. 10

119. 2072 Set D Q.No. 10

Define Vector product of two Vectors. Prove by Vector method that in any triangle ABC,
$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ [6]

➥ Please refer to 2069 (Set B) Q.No. 10

120. 2072 Set E Q.No. 10

Define scalar product of two vectors. Give the geometrical interpretation of the scalar product of two vectors. In any triangle prove vectorically that a² = b² + c² – 2bc cos A [6]

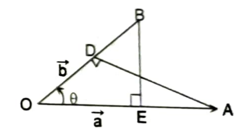
SOLUTION

Scalar product of two vectors: Scalar product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined by $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta = ab\cos\theta$ where θ is the angle between the two vectors.

Geometrical Interpretation

Let, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$. Let ∠AOB = θ. Draw BE perpendicular to OA and AD perpendicular to OB.

Now,



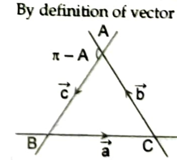
$\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}| \cos\theta$
= ab cos θ
= (OA) (OB) cos θ
= (OA) (OB cos θ)
= (OA) (OE)
= (magnitude of $\vec{a}$) (projection of $\vec{b}$ on $\vec{a}$)

Similarly, $\vec{a}.\vec{b}$ = (magnitude of $\vec{b}$) (projection of $\vec{a}$ on $\vec{b}$)

Next part,

In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$

By definition of vector addition,



$\vec{BC} = \vec{BA} + \vec{AC}$

or, $\vec{a} = -\vec{c} - \vec{b}$

or, $a^2 = (-\vec{c} - \vec{b})^2$

$= (\vec{c} + \vec{b})^2 = b^2 + 2\vec{b}.\vec{c} + c^2$

$= b^2 + 2bc\cos(\pi - A) + c^2 = b^2 + c^2 - 2bc\cos A$

121. 2072 Supp Q.No. 10

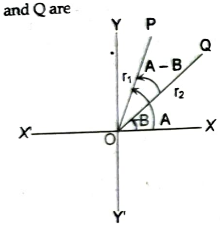
Define vector product of two vectors. Using vector method prove that sin (A – B) = sin A cos B – cos A sin B. [6]

SOLUTION

Vector product of two vectors: The vector (cross) product of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$, is defined by $\vec{a}\times\vec{b} = |\vec{a}|\ |\vec{b}|\ \sin\theta\ \hat{n}$ where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

Second part

Let XOX' and YOY' be two mutually perpendicular straight lines representing x-axis and y-axis respectively. Let ∠XOQ = B and ∠XOP = A so that ∠QOP = A – B. Again, let OP = r_1 and OQ = r_2. Then the coordinates of P and Q are



(r_1 cos A, r_1 sin A) and (r_2 cos B, r_2 sin B) respectively. So,

$\vec{OP} = (r_1\cos A, r_1\sin A) = (r_1\cos A, r_1\sin A, 0)$

$\vec{OQ} = (r_2\cos B, r_2\sin B) = (r_2\cos B, r_2\sin B, 0)$

Now,

$r_2\cos B$	$r_2\sin B$	0	$r_2\cos B$		$r_2\sin B$
$r_1\cos A$	$r_1\sin A$	0	$r_1\cos A$		$r_1\sin A$

$\vec{OQ}\times\vec{OP} = (0, 0, r_1 r_2\sin A\cos B - r_1 r_2\cos A\sin B)$

$|\vec{OQ}\times\vec{OP}| = r_1 r_2(\sin A\cos B - \cos A\sin B)$

Since (A – B) is the angle between OQ and OP, so

$$\sin(A - B) = \frac{|\vec{OQ}\times\vec{OP}|}{|\vec{OQ}|\ |\vec{OP}|}$$

$$= \frac{r_1 r_2(\sin A\cos B - \cos A\sin B)}{r_1 r_2}$$

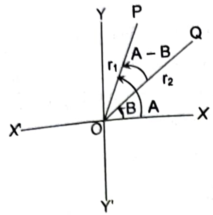
$= \sin A\cos B - \cos A\sin B$

122. 2073 Set C Q.No. 10

Define scalar product of two vectors. Prove vectorically that cos(A–B) = cos A cos B + sin A sin B. [6]

SOLUTION

Scalar product of two vectors: Scalar (dot) product of two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}.\vec{b}$ is defined by $\vec{a}.\vec{b} = |\vec{a}|\ |\vec{b}|\cos\theta = ab\cos\theta$ where θ is the angle between the two vectors.



Next part

Please refer to 2057 Q.No. 11a

123. 2073 Set D Q.No. 10

Define scalar product of two vectors. Give the geometrical interpretation of the scalar product of two vectors. Prove vectorically that,

$b^2 = c^2 + a^2 - 2ca\cos B$ [6]

➥ **First and Second Part:**

Please refer to Model Set II Q.No. 10

Last Part:

Please refer to 2064 Q.No. 11a

124. 2073 Supp Q.No. 10

Define vector product of two vectors, prove by vector method

sin (A– B) = sin A cos B – cos A sin B. [6]

➥ Please refer to 2072 Supp Q.No. 10

125. 2074 Set A Q.No. 10

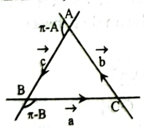
Using vectors prove that [6]

i. **$b^2 = c^2 + a^2 - 2ca\cos B$**

ii. **c = a cos B + b cos A for any triangle ABC.**

SOLUTION

i. In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$



By definition of vector addition, we have

$\vec{CA} = \vec{CB} + \vec{BA}$

or, $\vec{b} = -\vec{a} - \vec{c}$

or, $b^2 = (-\vec{a} - \vec{c})^2 = (\vec{a} + \vec{c})^2$

$= a^2 + 2\vec{a}.\vec{c} + c^2$

$= a^2 + c^2 + 2ac\cos(\pi - B)$

∴ $b^2 = a^2 + c^2 - 2ac\cos B$

ii. In ΔABC, let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{b}$

By the definition of vector addition,

$\vec{AB} = \vec{AC} + \vec{BC}$

or, $\vec{AB} = -\vec{CA} - \vec{BC}$

or, $\vec{c} = -\vec{b} - \vec{a}$

Multiplying each term scalarly by $\vec{c}$, we have

$\vec{c}.\vec{c} = -\vec{b}.\vec{c} - \vec{a}.\vec{c}$

or, $c^2 = -bc\cos(\pi - A) - ac\cos(\pi - B)$

or, $c^2 = -bc(-\cos A) - ac(-\cos B)$

or, $c^2 = bc\cos A + ac\cos B$

∴ $c = b\cos A + a\cos B$.

126. 2074 Set B Q.No. 10

Define Scalar product of two vectors. Prove vectorically

Cos(A + B) = CosA. CosB - SinA. SinB. [6]

➥ Please refer to Model Set II, Q.No. 10

127. 2074 Supp Q.No. 10

Define scalar product of two vectors. Prove by vector method that cos(A – B) = cosA.cosB + sinA.sinB.

➥ **First Part:** Please refer to Model Set II Q.No. 10

Second Part: Please refer to 2057 Q.No. 11a

128. 2075 Set A Q.No. 10

Define vector product of two vectors. Interpret the vector product of two vectors geometrically. Prove, in any triangle, by vector method that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. [6]

➥ Please refer to 2069 Set B Q.No. 10

129. 2075 Set C Q.No. 10

Define a dot product. Interpret it geometrically. Let A(1, 0, –1), B(–1, 2, 0), C(2, 0, –3) and D(3, –2, –1) are four points. Show that the projection of AB on CD is equal to projection of CD on AB. Also, show that their inclination is $\cos^{-1}\left(\frac{-4}{9}\right)$. [6]

SOLUTION

First and second part:

Please refer to 2072 Set E Q.No. 10

Next part:

Given points are A(1, 0, –1), B(–1, 2, 0) C(2, 0, –3) and D (3, –2, –1). Let O be the origin. Then,

$\vec{OA} = (1, 0, -1)$

$\vec{OB} = (-1, 2, 0)$

$\vec{OC} = (2, 0, -3)$

$\vec{OD} = (3, -2, -1)$

Now,

$\vec{AB} = \vec{OB} - \vec{OA} = (-1, 2, 0) - (1, 0, -1)$

$= (-1 - 1, 2 - 0, 0 + 1)$

$= (-2, 2, 1)$

And,

$\vec{CD} = \vec{OD} - \vec{OC} = (3, -2, -1) - (2, 0, -3)$

$= (3 - 2, -2 - 0, -1 + 3)$

$= (1, -2, 2)$ [6]

$\vec{AB}.\vec{CD} = (-2, 2, 1).(1, -2, 2)$

$= -2 - 4 + 2 = -4$

$AB = |\vec{AB}| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{9} = 3$

$CD = |\vec{CD}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$

Projection of $\vec{AB}$ on $\vec{CD}$ $= \frac{\vec{AB}.\vec{CD}}{|\vec{CD}|} = -\frac{4}{3}$

Again,

Projection of $\vec{CD}$ on $\vec{AB}$ $= \frac{\vec{CD}.\vec{AB}}{|\vec{AB}|} = -\frac{4}{3}$

∴ Projection of $\vec{AB}$ on $\vec{CD}$ = Projection of $\vec{CD}$ on $\vec{AB}$.

If θ be the angle between $\vec{AB}$ and $\vec{CD}$ then,

$$\cos\theta = \frac{\vec{AB}.\vec{CD}}{|\vec{AB}|\ |\vec{CD}|} = \frac{-4}{(3)(3)} = \frac{-4}{9}$$

∴ $\theta = \cos^{-1}\left(-\frac{4}{9}\right)$

UNIT 7

DERIVATIVE AND ITS APPLICATION

A. DERIVATIVES
I. DERIVATIVE BY FIRST PRINCIPLE OR DEFINITION

4 MARKS QUESTIONS

1. 2058 Q.No. 10 b

Find from first principles, the derivatives of $e^{\tan x}$. [4]

SOLUTION

Let $f(x) = e^{\tan x}$

$\therefore f(x+h) = e^{\tan(x+h)}$

We have, $f'(x) = \lim_{h \to 0} \dfrac{e^{\tan(x+h)} - e^{\tan x}}{h}$ ...(i)

Put $\tan x = y$ & $\tan(x+h) = y + k$

so that $k = \tan(x+h) - \tan x$

As $h \to 0, k \to 0$

Now, from (i)

$$\frac{d}{dx}(e^{\tan x}) = \lim_{h \to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{h \to 0} \frac{e^y(e^k - 1)}{h}$$

$$= e^y \lim_{k \to 0} \frac{e^k - 1}{k} \cdot \lim_{h \to 0} \frac{k}{h}$$

$$= e^y . 1 . \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= e^y \cdot \lim_{h \to 0} \left\{ \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h} \right\}$$

$$= e^y \lim_{h \to 0} \left\{ \frac{\sin(x+h)\cos x - \cos(x+h) . \sin x}{h \cos(x+h) \cos x} \right\}$$

$$= e^y \lim_{h \to 0} \left\{ \frac{\sin(x+h-x)}{h} \cdot \frac{1}{\cos(x+h)\cos x} \right\}$$

$$= e^y . 1 . \frac{1}{\cos x . \cos x} = e^{\tan x} \sec^2 x$$

2. 2059 Q.No. 10 b

Find, from definition, the derivative of $e^{\tan x}$. [4]

➥ Please refer to 2058 Q.No. 10b

3. 2060 Q.No. 10 b

Find from first principles, the derivative of $e^{\sin x}$. [4]

SOLUTION

Let $f(x) = e^{\sin x}$

$\therefore f(x+h) = e^{\sin(x+h)}$

We have, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\therefore \dfrac{d}{dx}(e^{\sin x}) = \lim_{h \to 0} \dfrac{e^{\sin(x+h)} - e^{\sin x}}{h}$ ...(i)

Put $\sin x = y$ and $\sin(x+h) = y + k$.

Then, $k = \sin(x+h) - \sin x$

As $h \to 0, k \to 0$

Then, from (i)

$$\frac{d}{dx}(e^{\sin x}) = \lim_{h \to 0} \frac{e^{y+k} - e^y}{h}$$

$$= \lim_{k \to 0} \left\{ \frac{e^y(e^k - 1)}{k} \cdot \frac{k}{h} \right\}$$

$$= e^y . 1 . \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= e^y \lim_{h \to 0} \left\{ \frac{2\cos\left(\frac{x+h+x}{2}\right) . \sin\left(\frac{x+h-x}{2}\right)}{h} \right\}$$

$$= e^y . \lim_{h \to 0} \left\{ 2\cos\left(x + \frac{h}{2}\right) \frac{\sin\frac{h}{2}}{\frac{h}{2} \times 2} \right\}$$

$$= e^y . \cos x . 1 = e^{\sin x} . \cos x$$

4. 2061 Q.No. 10 b

Find from first principles, the derivative of $e^{\sqrt{x}}$. [4]

SOLUTION

Let $f(x) = e^{\sqrt{x}}$

$\therefore f(x+h) = e^{\sqrt{x+h}}$

We have, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\therefore \dfrac{d}{dx}\left(e^{\sqrt{x}}\right) = \lim_{h \to 0} \dfrac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h}$ ...(i)

Put $\sqrt{x} = y$ and $\sqrt{x+h} = y + k$

so that $k = \sqrt{x+h} - \sqrt{x}$

As $h \to 0, k \to 0$

Then from (i), $\dfrac{d}{dx}\left(e^{\sqrt{x}}\right) = \lim_{h \to 0} \dfrac{e^{y+k} - e^y}{h}$

$$= \lim_{k \to 0} \left\{ \frac{e^y(e^k - 1)}{k} \times \frac{k}{h} \right\}$$

$$= e^y . 1 . \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= e^y . \lim_{h \to 0} \left\{ \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right\}$$

$$= e^y . \lim_{h \to 0} \left\{ \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \right\}$$

$$= e^y \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = e^{\sqrt{x}} . \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} e^{\sqrt{x}}$$

5. 2062 Q.No. 10 b

Find from first principles, the derivative of $\tan^{-1}x$. [4]

SOLUTION

Let $f(x) = \tan^{-1}x$

$\therefore f(x+h) = \tan^{-1}(x+h)$

We have, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\therefore \dfrac{d}{dx}(\tan^{-1} x) = \lim_{h \to 0} \dfrac{\tan^{-1}(x+h) - \tan^{-1}x}{h}$ ...(i)

Put $\tan^{-1}x = y$ & $\tan^{-1}(x+h) = y + k$

Then, $x = \tan y$ & $(x+h) = \tan(y+k)$

$\therefore h = \tan(y+k) - \tan y$

Also, as $h \to 0, k \to 0$

Now, from (i)

$$\frac{d}{dx}(\tan^{-1}x) = \lim_{h \to 0} \frac{y + k - y}{h}$$

$$= \lim_{h \to 0} \frac{k}{h}$$

$$= \lim_{k \to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= \lim_{k \to 0} \left\{ \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}} \right\}$$

$$= \lim_{k \to 0} \left\{ \frac{k}{\frac{\sin(y+k)\cos y - \cos(y+k)\sin y}{\cos(y+k)\cos y}} \right\}$$

$$= \lim_{k \to 0} \left\{ \frac{k \cos(y+k)\cos y}{\sin(y+k-y)} \right\}$$

$$= \lim_{k \to 0} \left\{ \frac{\cos(y+k)\cos y}{\frac{\sin y}{k}} \right\}$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

6. 2063 Q.No. 10 b

Find from first principles, the derivative of $\log \tan x$. [4]

SOLUTION

Let $f(x) = \log(\tan x)$

$\therefore f(x+h) = \log\tan(x+h)$

We have, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\dfrac{d}{dx}(\log\tan x) = \lim_{h \to 0} \dfrac{\log\tan(x+h) - \log\tan x}{h}$ ...(i)

Put $\tan x = y$ and $\tan(x+h) = y + k$

so that $k = \tan(x+h) - \tan x$

As $h \to 0, k \to 0$

Now, from (i)

$$\frac{d}{dx}(\log\tan x) = \lim_{h \to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(\frac{y+k}{h}\right)}{h}$$

$$= \lim_{k \to 0} \left\{ \frac{\log\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \cdot \frac{\frac{k}{y}}{h} \right\}$$

$$= 1 . \frac{1}{y} \cdot \lim_{h \to 0} \frac{k}{h}$$

$$= \frac{1}{y} \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$= \frac{1}{y} \lim_{h \to 0} \left[\frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos}}{h} \right]$$

$$= \frac{1}{y} \lim_{h \to 0} \left[\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x} \right]$$

$$= \frac{1}{y} \lim_{h \to 0} \frac{\sin(x+h-x)}{h\cos(x+h)\cos x}$$

$$= \frac{1}{y} \lim_{h \to 0} \left\{ \frac{\sin h}{h} \cdot \frac{1}{\cos(x+h)\cos x} \right\}$$

$$= \frac{1}{y} . 1 . \frac{1}{\cos x . \cos x} = \frac{1}{\tan x} \cdot \frac{1}{\cos^2 x} = \frac{1}{\sin x \cos x}$$

$$= \frac{2}{2\sin x \cos x} = 2\,\text{cosec}\,2x$$

7. 2064 Q.No. 10 b

Find from first principles, the derivative of $\log \sin x$. [4]

SOLUTION

Let $f(x) = \log\sin x$

$\therefore f(x+h) = \log\sin(x+h)$

We have, $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\therefore \dfrac{d}{dx}(\log\sin x) = \lim_{h \to 0} \dfrac{\log\sin(x+h) - \log\sin x}{h}$ ... (i)

Put $\sin x = y$ and $\sin(x+h) = y + k$,

so that $k = \sin(x+h) - \sin x$

As $h \to 0, k \to 0$

Then from (i)

$$\frac{d}{dx}[\log(\sin x)] = \lim_{h \to 0} \frac{\log(y+k) - \log y}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{k \to 0} \left\{ \frac{\log\left(1 + \frac{k}{y}\right)}{\frac{k}{y}} \cdot \frac{\frac{k}{y}}{h} \right\}$$

$$= 1 . \frac{1}{y} \lim_{h \to 0} \frac{k}{h}$$

$$= \frac{1}{y} \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$=\frac{1}{y}\lim_{h\to 0}\frac{2\cos\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$=\frac{1}{y}\lim_{h\to 0}\left\{2\cos\left(x+\frac{h}{2}\right)\frac{\sin\frac{h}{2}}{\frac{h}{2}\times 2}\right\}$$

$$=\frac{1}{y}\cos x\,.\,1=\frac{1}{\sin x}.\cos x=\cot x$$

8. 2065 Q.No 10 b

Find from first principles, the derivative of a^x [4]

SOLUTION

Let $f(x) = a^x = e^{\log a^x} = e^{x\log a}$

$\therefore$ $f(x + h) = e^{(x+h)\log a}$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore \frac{d}{dx}(a^x) = \lim_{h\to 0}\frac{e^{(x+h)\log a}-e^{x\log a}}{h} \quad \ldots (i)$$

Put $x \log a = y$ and $(x + h) \log a = y + k$, so that $k = (x + h) \log a - x \log a$

As $h \to 0$, $k \to 0$

Now, from (i)

$$\frac{d}{dx}(a^x) = \lim_{h\to 0}\frac{e^{y+k}-e^y}{h}$$

$$=\lim_{h\to 0}\left\{\frac{e^y(e^k-1)}{k}.\frac{k}{h}\right\}$$

$$=e^y.1\lim_{h\to 0}\frac{(x+h)\log a - x\log a}{h}$$

$$=e^y\lim_{h\to 0}\frac{(x+h-x)\log a}{h}$$

$$=e^y.\log a = e^{x\log a}.\log a = a^x\log a$$

9. 2066 C Q.No. 10 b

Find from first principle, the derivative of: $\log\tan x$: [4]

➥ Please refer to 2063 Q.No. 10b

10. 2066 Q.No. 10 b

Find the derivative of Sin (log x) from first principles. [4]

SOLUTION

Let $f(x) = \sin(\log x)$

$\therefore$ $f(x + h) = \sin\log(x + h)$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore \frac{d}{dx}(\sin\log x) = \lim_{h\to 0}\frac{\sin\log(x+h)-\sin\log x}{h} \quad \ldots (i)$$

Put $\log x = y$ and $\log(x + h) = y + k$

so that $k = \log(x + h) - \log x$.

As $h \to 0$, $k \to 0$.

Then, from (i)

$$\frac{d}{dx}(\sin\log x) = \lim_{h\to 0}\frac{\sin(y+k)-\sin y}{h}$$

$$=\lim_{h\to 0}\frac{2\cos\left(\frac{y+k+y}{2}\right)\sin\left(\frac{y+k-y}{2}\right)}{h}$$

$$=\lim_{k\to 0}\left[2\cos\left(y+\frac{k}{2}\right).\frac{\sin\frac{k}{2}}{\frac{k}{2}}\times\frac{\frac{k}{2}}{h}\right]$$

$$=\cos y\,.\,1\lim_{h\to 0}\frac{k}{h}$$

$$=\cos y\cdot\lim_{h\to 0}\frac{\log(x+h)-\log x}{h}$$

$$=\cos y\cdot\lim_{h\to 0}\frac{\log\left(\frac{x+h}{x}\right)}{h}$$

$$=\cos y\cdot\lim_{h\to 0}\left\{\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}.x}\right\}$$

$$=\cos y\,.\,1.\frac{1}{x}=\frac{1}{x}\cos(\log x)$$

11. 2067 Q.No. 10b

Find from first principles, the derivative of $\sin^{-1}x$. [4]

SOLUTION

Let $f(x) = \sin^{-1}x$

$\therefore$ $f(x + h) = \sin^{-1}(x + h)$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore \frac{d}{dx}(\sin^{-1}x) = \lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h} \quad \ldots (i)$$

Put $y = \sin^{-1}x$ and $y + k = \sin^{-1}(x + h)$

so that $x = \sin y$ and $x + h = \sin(y + k)$

Then, $h = \sin(y + k) - \sin y$. As $h \to 0$, $k \to 0$

Then from (i)

$$\frac{d}{dx}(\sin^{-1}x) = \lim_{h\to 0}\frac{y+k-y}{h}$$

$$=\lim_{k\to 0}\left\{\frac{k}{\sin(y+k)-\sin y}\right\}$$

$$=\lim_{k\to 0}\left\{\frac{k}{2\cos\left(\frac{y+k+y}{2}\right)\sin\left(\frac{y+k-y}{2}\right)}\right\}$$

$$=\lim_{k\to 0}\left\{\frac{k}{2\cos\left(y+\frac{k}{2}\right).\sin\frac{k}{2}}\right\}$$

$$=\lim_{k\to 0}\left\{\frac{1}{2\cos\left(y+\frac{k}{2}\right).\frac{\sin\frac{k}{2}}{\frac{k}{2}\times 2}}\right\}$$

$$=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}$$

12. 2068 Q.No. 10b

Find from first principle, the derivative of: $e^{\sin x}$ [4]

➥ Please refer to 2060 Q.No. 10b

13. 2069 (Set A) Old Q.No. 10b

Find from first principle, the derivative of $e^{\cos x}$. [4]

SOLUTION

Let $f(x) = e^{\cos x}$

$\therefore$ $f(x + h) = e^{\cos(x+h)}$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore \frac{d}{dx}(e^{\cos x}) = \lim_{h\to 0}\frac{e^{\cos(x+h)}-e^{\cos x}}{h} \quad \ldots (i)$$

Put $\cos x = y$ and $\cos(x + h) = y + k$

so that $k = \cos(x + h) - \cos x$

As $h \to 0$, $k \to 0$

Then, from (i)

$$\frac{d}{dx}(e^{\cos x}) = \lim_{h\to 0}\frac{e^{y+k}-e^y}{h}$$

$$=\lim_{k\to 0}\left\{\frac{e^y(e^k-1)}{k}.\frac{k}{h}\right\}$$

$$=e^y.1.\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$=e^y\lim_{h\to 0}\left\{\frac{\sin\left(\frac{x+h+x}{2}\right).\sin\left(\frac{x-x-h}{2}\right)}{h}\right\}$$

$$=e^y\lim_{h\to 0}\left\{\frac{2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{-h}{2}\right)}{\left(\frac{-h}{2}\right).(-2)}\right\}$$

$$=e^y.\sin x.(-1) = -e^{\cos x}.\sin x$$

14. 2069 (Set B) Q.No. 11 Or

Find from first principle, the derivative of $e^{\sin x}$. [4]

➥ Please refer to 2060 Q.No. 10b

15. 2069 Old (Set B) Q.No. 10b

Find, from definition the derivative of log $(\sin\frac{x}{a})$. [4]

SOLUTION

Let $f(x) = \log\left(\sin\frac{x}{a}\right)$

$\therefore$ $f(x + h) = \log\sin\frac{(x+h)}{a}$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{g}$

$$\therefore \frac{d}{dx}\left(\log\sin\frac{x}{a}\right) = \lim_{h\to 0}\frac{\log\sin\left(\frac{x+h}{a}\right)-\log\left(\sin\frac{x}{a}\right)}{h} \quad \ldots (i)$$

Put $\sin\frac{x}{a} = y$ & $\sin\left(\frac{x+h}{a}\right) = y + k$

so that $k = \sin\left(\frac{x+h}{a}\right) - \sin\frac{x}{a}$

Also, as $h \to 0$, $k \to 0$

Now, from (i)

$$\frac{d}{dx}\left(\ln\sin\frac{x}{a}\right)=\lim_{h\to 0}\frac{\ln(y+k)-\ln y}{h}$$

$$=\lim_{h\to 0}\frac{\log\left(\frac{y+k}{y}\right)}{h}$$

$$=\lim_{k\to 0}\left\{\frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}}.\frac{k/y}{h}\right\}$$

$$=1.\frac{1}{y}.\lim_{h\to 0}\frac{k}{h}$$

$$=\frac{1}{y}\lim_{h\to 0}\frac{\sin\left(\frac{x+h}{a}\right)-\sin\frac{x}{a}}{h}$$

$$=\frac{1}{y}\lim_{h\to 0}\left\{\frac{2\cos\left(\frac{\frac{x+h}{a}+\frac{x}{a}}{2}\right)\sin\left(\frac{\frac{x+h}{a}-\frac{x}{a}}{2}\right)}{h}\right\}$$

$$=\frac{1}{y}\lim_{h\to 0}\left\{2\cos\left(\frac{x}{a}+\frac{h}{2a}\right).\frac{\sin\frac{h}{2a}}{\frac{h}{2a}.2a}\right\}$$

$$=\frac{1}{y}.\cos\left(\frac{x}{a}\right).1.\frac{1}{a}$$

$$=\frac{1}{\sin\frac{x}{a}}\cos\frac{x}{a}.\frac{1}{a}=\frac{1}{a}\cot\frac{x}{a}$$

16. 2070 Supp. Q.No. 6 b

Find, from first principles, the derivative of $\sin x^2$. [4]

SOLUTION

Let $f(x) = \sin x^2$

$\therefore$ $f(x + h) = \sin(x + h)^2$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore \frac{d}{dx}(\sin x^2) = \lim_{h\to 0}\frac{\sin(x+h)^2-\sin x^2}{h}$$

$$=\lim_{h\to 0}\left[\frac{2\cos\left\{\frac{(x+h)^2+x^2}{2}\right\}.\sin\left\{\frac{(x+h)^2-x^2}{2}\right\}}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{2\cos\left(x^2+hx+\frac{h^2}{2}\right).\sin\left(hx+\frac{h^2}{2}\right)}{h}\right]$$

$$=\lim_{h\to 0}\left[2\cos\left(x^2+hx+\frac{h^2}{2}\right).\frac{\sin h\left(x+\frac{h}{2}\right)}{h\left(x+\frac{h}{2}\right)}.\left(x+\frac{h}{2}\right)\right]$$

$$=2\cos x^2.1.x=2x\cos x^2$$

17. 2071 Old Q.No. 9 b

Find the derivative of $e^{\tan x}$ from first principles. [4]

➥ Please refer to 2058 Q.No. 10b

6 MARKS QUESTIONS

18. 2069 (Set A) Q.No. 11 or

Find from first principle, the derivative of $\sin(\log x)$ [6]

➥ Please refer to 2066 Q.No. 10b

19. 2070 (Old) Q.No. 10 b

Find the derivative of log (tan x) by first principle. [6]

➥ Please refer to 2063 Q.No. 10b

20. 2070 Set C Q.No. 11 or

Find from first principles, the derivative of log (tan$x$). [6]

➥ Please refer to 2063 Q.No. 10b

21. 2070 Set D Q.No. 11 Or

Find from first principles, the derivative of x^x. [6]

SOLUTION

Let $f(x) = x^x = e^{\log x^x} = e^{x\log x}$

$\therefore f(x+h) = e^{(x+h)\log(x+h)}$

We have, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$\therefore \frac{d}{dx}(x^x) = \lim_{h\to 0} \frac{e^{(x+h)\log(x+h)} - e^{x\log x}}{h}$ ... (i)

Put $y = x\log x$ and $(x+h)\log(x+h) = y + k$ so that $k = (x+h)\log(x+h) - x\log x$

As $h \to 0, k \to 0$

Now from (i)

$\frac{d}{dx}(x^x) = \lim_{h\to 0} \frac{e^{y+k} - e^y}{h}$

$= \lim_{k\to 0} \left\{\frac{e^y(e^k - 1)}{k} \times \frac{k}{h}\right\}$

$= e^y . 1 . \lim_{h\to 0} \frac{(x+h)\log(x+h) - x\log}{h}$

$= e^y \lim_{h\to 0} \left\{\frac{x\log(x+h) + h\log(x+h) - x\log x}{h}\right\}$

$= e^y \lim_{h\to 0} \left\{\frac{x\log\left(\frac{x+h}{h}\right)}{h} + \frac{h\log(x+h)}{h}\right\}$

$= e^y \lim_{h\to 0} \left\{\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} + \log(x+h)\right\}$

$= e^y(1 + \log x) = x^x(1 + \log x)$

22. 2071 Set C Q.No. 11 OR

Find from first principle, the derivative of $\tan^{-1}x$. [6]

➥ Please refer to 2062 Q.No. 10b

23. 2071 Set D Q.No. 11 OR

Find from first principle, the derivative of: $\ln\left(\sin\frac{x}{a}\right)$. [6]

➥ Please refer to 2069 (Old) Set B Q.No. 10b

24. 2071 Supp. Q.No. 11

Find, from first principles, the derivative of $\ln(\cos\sqrt{x})$ [6]

SOLUTION

Let $f(x) = \ln(\cos\sqrt{x})$

$f(x+h) = \ln\cos\sqrt{x+h}$

We have, $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$\frac{d}{dx}(\ln\cos\sqrt{x}) = \lim_{h\to 0}\frac{\ln\cos\sqrt{x+h} - \ln\cos\sqrt{x}}{h}$ ... (i)

Put $\cos\sqrt{x} = y$ & $\cos\sqrt{x+h} = y + k$

Then, $k = \cos\sqrt{x+h} - \cos\sqrt{x}$

As $h\to 0, k\to 0$

Then, from (i)

$\frac{d}{dx}(\ln\cos\sqrt{x}) = \lim_{h\to 0}\frac{\ln(y+k) - \ln y}{h}$

$= \lim_{h\to 0}\frac{\ln\left(\frac{y+k}{y}\right)}{h}$

$= \lim_{k\to 0}\left\{\frac{\ln\left(1+\frac{k}{y}\right)}{\frac{k}{y}}\cdot\frac{\frac{k}{y}}{h}\right\}$

$= 1 . \frac{1}{y}\lim_{h\to 0}\frac{k}{h}$

$= \frac{1}{y}\lim_{h\to 0}\frac{\cos\sqrt{x+h} - \cos\sqrt{x}}{h}$

$= \frac{1}{y}\lim_{h\to 0}\frac{2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right).\sin\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}{h}$

$= \frac{1}{y}\lim_{h\to 0}\left\{2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\cdot\frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}\cdot\frac{\sqrt{x}-\sqrt{x+h}}{2h}\right\}$

$= \frac{1}{y}\lim_{h\to 0}\left\{\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\cdot\frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}\cdot\frac{\sqrt{x}-\sqrt{x+h}}{h}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right\}$

$= \frac{1}{y}\lim_{h\to 0}\left\{\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\cdot\frac{\sin\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}{\left(\frac{\sqrt{x}-\sqrt{x+h}}{2}\right)}\cdot\frac{(x-x-h)}{h(\sqrt{x}+\sqrt{x+h})}\right\}$

$= \frac{1}{y}.\sin\sqrt{x} . 1 . \frac{(-1)}{2\sqrt{x}} = \frac{1}{\cos\sqrt{x}} . \sin\sqrt{x} . \left(\frac{-1}{2\sqrt{x}}\right)$

$= -\frac{1}{2\sqrt{x}}\tan\sqrt{x}$

25. 2072 Set C Q.No. 11 OR

Find, from first principles, the derivative of $x\ln x$. [6]

SOLUTION

Let $f(x) = x\ln x$

We have $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$\therefore \frac{d}{dx}(x\ln x) = \lim_{h\to 0}\frac{(x+h)\ln(x+h) - x\ln x}{h}$

$= \lim_{x\to 0}\frac{h\ln(x+h) + x\ln(x+h) - x\ln x}{h}$

$= \lim_{h\to 0}\left[\frac{x\{\ln(x+h) - \ln x\}}{h} + \frac{h\ln(x+h)}{h}\right]$

$= \lim_{h\to 0}\left\{\frac{x}{h}\ln\left(\frac{x+h}{x}\right) + \ln(x+h)\right\}$

$= \lim_{h\to 0}\left\{\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} + \ln(x+h)\right\} = 1 + \ln x$

26. 2072 Set D Q.No. 11 OR

Find from first principles the derivative of $\ln\cos^{-1}x$. [6]

➥ Please see Model Set II., Q.No. 11 or

27. 2072 Set E Q.No. 11 OR

Find from first principles, the derivative of $\sin x^2$.

➥ Please refer to 2070 Supp. Q.No. 6b

28. 2072 Supp Q.No. 11 OR

Find from first principles, the derivative of log (tanx). [6]

➥ Please refer to 2063 Q.No. 10b

29. 2073 Set C Q.No. 11

Find the first principles the derivative of log(sec x) [6]

SOLUTION

Let $f(x) = \log\sec x$

Then $f(x+h) = \log\sec(x+h)$

We have,

$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

or, $\frac{d}{dx}(\log\sec x) = \lim_{h\to 0}\frac{\log\sec(x+h) - \log\sec x}{h}$ ...(i)

Put $y = \sec x$ and $y + k = \sec(x+h)$ such that as $h\to 0, k\to 0$

Then from (i),

$\frac{d}{dx}(\log\sec x) = \lim_{h\to 0}\left\{\frac{\log(y+k) - \log y}{k}\cdot\frac{k}{h}\right\}$

$= \lim_{h\to 0}\left\{\frac{\log\left(\frac{y+k}{y}\right)}{k}\cdot\frac{k}{h}\right\}$

$= \lim_{k\to 0}\left\{\frac{\log\left(1+\frac{k}{y}\right)}{\frac{k}{y}\cdot y}\right\}\lim_{h\to 0}\left\{\frac{\sec(x+h) - \sec x}{h}\right\}$

$= \frac{1}{y}\lim_{h\to 0}\frac{\left(\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right)}{h}$

$= \frac{1}{y}\lim_{h\to 0}\frac{\cos x - \cos(x+h)}{h\cos(x+h)\cos x}$

$= \frac{1}{y}\lim_{h\to 0}\left\{\frac{2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x+h-x}{2}\right)}{h\cos(x+h)\cos x}\right\}$

$= \frac{1}{y}\lim_{h\to 0}\left\{\frac{2\sin\left(x+\frac{h}{2}\right)}{\cos(x+h)\cos x}\cdot\frac{\sin\frac{h}{2}}{\frac{h}{2}\cdot 2}\right\}$

$= \frac{1}{y}\cdot\frac{\sin x}{\cos x\cdot\cos x}\cdot 1 = \frac{1}{\sec x}\cdot\frac{\sin x}{\cos x\cdot\cos x} = \tan x$

30. 2073 Set D Q.No. 11 OR

Find from first principles, the derivative of sin (logx). [6]

➥ Please refer to 2066 Q.No. 10b

31. 2074 Set A Q.No. 11

Find, from first principles, the derivative of $\ln\left(\cos\frac{x}{a}\right)$. [6]

SOLUTION

Let $f(x) = \ln(\cos x/a)$

$\therefore f(x+h) = \ln\cos\left(\frac{x+h}{a}\right)$

By definition of derivative, we have,

$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$

$\therefore \frac{d}{dx}\left(\ln\cos\frac{x}{a}\right) = \lim_{h\to 0}\frac{\ln\cos\left(\frac{x+h}{a}\right) - \ln\cos\frac{x}{a}}{h}$ ... (i)

Put $y = \cos\frac{x}{a}$ & $y + k = \cos\left(\frac{x+h}{a}\right)$

so that $k = \cos\left(\frac{x+h}{a}\right) - \cos\frac{x}{a}$

Also, when $h\to 0, k\to 0$.

Then, from (i)

$\frac{d}{dx}\left(\ln\cos\frac{x}{a}\right) = \lim_{h\to 0}\frac{\ln(y+k) - \ln y}{h}$

$= \lim_{h\to 0}\frac{\ln\left(\frac{y+k}{y}\right)}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{k}{y}\right)}{h}$

$= \lim_{h\to 0}\frac{\ln\left(1+\frac{k}{y}\right)}{k/y}\cdot\lim_{h\to 0}\frac{k/y}{h}$

$= 1 . \lim_{h\to 0}\frac{\cos\left(\frac{x+h}{a}\right) - \cos\frac{x}{a}}{y.h}$

$$= \frac{1}{y}\lim_{h\to 0}\frac{\left\{-2\sin\dfrac{\left(\frac{x+h}{a}+\frac{x}{a}\right)}{2}.\sin\dfrac{\left(\frac{x+h}{a}-\frac{x}{a}\right)}{2}\right\}}{h}$$

$$= \frac{-2}{y}\lim_{h\to 0}\frac{\sin\left(\frac{x+h}{2a}+\frac{x}{2a}\right)\sin\left(\frac{h}{2a}\right)}{h}$$

$$= \frac{-2}{y}\lim_{h\to 0}\left\{\sin\left(\frac{2x+h}{2a}\right).\frac{\sin\frac{h}{2a}}{\frac{h}{2a}.2a}\right\}$$

$$= \frac{-2}{y}.\sin\left(\frac{2x}{2a}\right).1.\frac{1}{2a} = -\frac{1}{ay}\sin\left(\frac{x}{a}\right)$$

$$= \frac{1}{a.\cos x/a}.\sin\left(\frac{x}{a}\right) = -\frac{1}{a}\tan\frac{x}{a}$$

32. 2074 Set B Q.No. 11

Find from definition the derivative of log (tanx.) [6]

➥ Please refer to 2063 Q.No. 10b

33. 2074 Supp Q.No. 11 OR

Find from first principles, the derivative of sinx². [6]

➥ Please refer to 2070 Supp. Q.N. 6b

34. 2075 Set A Q.No. 11 OR

Find from first principles, the derivative of logsinx. [6]

➥ Please refer to 2064 Q.No. 10b

35. 2075 Set B Q.No. 10

Find, from first principles, the derivative of ln (sin x²). [6]

SOLUTION

Let $f(x) = ln\ (\sin x^2)$

$f(x+h) = ln\ (\sin (x+h)^2)$

By definition, we have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\therefore \frac{d}{dx}(ln\ \sin x^2) = \lim_{h\to 0}\frac{ln\ \sin(x+h)^2 - ln\ \sin x^2}{h} \quad \ldots (i)$$

Put $y = \sin x^2$ and $y + k = \sin (x+h)^2$

So that, $k = \sin (x+h)^2 - \sin x^2$.

As $h\to 0, k\to 0$. Then (i) becomes

$$\frac{d}{dx}(ln\ \sin x^2)$$

$$= \lim_{h\to 0}\frac{ln(y+k) - ln y}{h}$$

$$= \lim_{h\to 0}\frac{ln\left(\frac{y+k}{y}\right)}{h}$$

$$= \lim_{k\to 0}\left[\frac{ln(1+k/y)}{k/y}\right].\lim_{h\to 0}\left[\frac{k/y}{h}\right]$$

$$= \lim_{h\to 0} 1.\frac{\sin (x+h)^2 - \sin x^2}{y.h}$$

$$= \frac{1}{y}\lim_{h\to 0}\frac{2.\cos\frac{(x+h)^2+x^2}{2}.\sin\frac{(x+h)^2-x^2}{2}}{h}$$

$$= \frac{1}{y}\lim_{h\to 0}\frac{2.\cos\frac{(x+h)^2+x^2}{2}.\sin\frac{x^2+2xh+h^2-x^2}{2}}{h}$$

$$= \frac{1}{y}\lim_{h\to 0} 2.\cos\frac{(x+h)^2+x^2}{2}.\frac{\sin\frac{(2x+h)h}{2}}{\frac{(2x+h)h}{2}.\frac{2}{2x+h}}$$

$$= \frac{1}{y}.2.\cos x^2.1.\frac{1}{\frac{2}{2x}} = \frac{1}{\sin x^2}.2x\cos x^2$$

$= 2x\cot x^2$

36. 2075 Set C Q.No. 11 OR

Find from first principles, the derivative of cosx².

SOLUTION

Let,

$f(x) = \cos x^2$

$f(x+h) = \cos (x+h)^2$

By definition of derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\therefore \frac{d}{dx}(\cos x^2)$$

$$= \lim_{h\to 0}\frac{\cos (x+h)^2 - \cos x^2}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\left\{\frac{(x+h)^2+x^2}{2}\right\}.\sin\left\{\frac{(x+h)^2-x^2}{2}\right\}}{h}$$

$$= -\lim_{h\to 0}\frac{2\sin\left\{\frac{(x+h)^2+x^2}{2}\right\}.\sin\left\{\frac{x^2+2xh+h^2-x^2}{2}\right\}}{h}$$

$$= -\lim_{h\to 0} 2\sin\left\{\frac{(x+h)^2+x^2}{2}\right\}.\frac{\sin\frac{(2x+h)h}{2}}{\frac{(2x+h)h}{2}}\times\frac{2x+h}{2}$$

$$= -2\sin\left(\frac{x^2+x^2}{2}\right).1.\frac{2x}{2} = -2x\sin x^2$$

II. DERIVATIVE USING FORMULA

2 MARKS QUESTIONS

37. 2067 Q.No. 2 b

Show that $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$. [2]

SOLUTION

$$\text{L.H.S.} = \lim_{x\to 0}\frac{\log(1+x)}{x}$$

$$= \lim_{x\to 0}\frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\right)$$

$$= \lim_{x\to 0}\left(1 - \frac{x}{2} + \frac{x^2}{3} - \ldots\right) = 1 = \text{R.H.S}$$

38. 2058 Q.No. 2 b

Find the derivative of $x^{\cos hx}$. [2]

SOLUTION

Let $y = x^{\cos hx}$

Taking log on both sides, we get,

$\log y = \cos hx \log x$

Differentiating both sides with respect to 'x'

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\cos hx \log x)$$

or, $\frac{d}{dy}(\log y).\frac{dy}{dx} = \cos hx\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\cos hx)$

or, $\frac{1}{y}\frac{dy}{dx} = \cos hx.\frac{1}{x} + \log x.\sin hx$

or, $\frac{dy}{dx} = y\left(\frac{1}{x}\cos hx + \log x \sin hx\right)$

$= x^{\cos hx}\left(\frac{1}{x}\cos hx + \log x\sin hx\right)$

39. 2059 Q.No. 2 a

Find the derivative of $e^{\cos h^{-1}x}$. [2]

SOLUTION

Let $y = e^{\cos h^{-1}x}$

Differentiating both sides with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos h^{-1}x})$$

$$= \frac{d(e^{\cos h^{-1}x})}{d(\cos h^{-1}x)}.\frac{d(\cos h^{-1}x)}{dx} = e^{\cos h^{-1}x}.\frac{1}{\sqrt{x^2-1}}$$

$$= \frac{e^{\cos h^{-1}x}}{\sqrt{x^2-1}}$$

40. 2060 Q.No. 2 b

Find the derivative of $2\tan^{-1}\left(\tan h\frac{x}{2}\right)$. [2]

SOLUTION

Let $y = 2\tanh^{-1}\left(\tan\frac{x}{2}\right)$

Differentiating both sides with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left\{2\tan h^{-1}\left(\tan\frac{x}{2}\right)\right\}$$

$$= 2\frac{d\left\{\tan h^{-1}\left(\tan\frac{x}{2}\right)\right\}}{d\left(\tan\frac{x}{2}\right)}.\frac{d\left(\tan\frac{x}{2}\right)}{d\left(\frac{x}{2}\right)}.\frac{d\left(\frac{x}{2}\right)}{dx}$$

$$= 2.\frac{1}{1-\tan^2\frac{x}{2}}.\sec^2\frac{x}{2}.\frac{1}{2}$$

$$= \frac{\sec^2\frac{x}{2}}{1-\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}} = \frac{\sec^2\frac{x}{2}.\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{1}{\cos x} = \sec x$$

41. 2061 Q.No. 2 b

Find the derivative of: $x^{\cos h\, x/a}$ [2]

SOLUTION

Let $y = x^{\cosh x/a}$

Taking log on both sides, we get,

$\log y = \cosh\frac{x}{a}\log x$

Differentiating both sides with respect to 'x'

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left(\cosh\frac{x}{a}\log x\right)$$

or, $\frac{d(\log y)}{dy}.\frac{dy}{dx} = \cosh\frac{x}{a}\frac{d}{dx}(\log x) + \log x\frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}.\frac{d\left(\frac{x}{a}\right)}{dx}$

or, $\frac{1}{y}\frac{dy}{dx} = \cosh\frac{x}{a}.\frac{1}{x} + \log x\sinh\frac{x}{a}.\frac{1}{a}$

or, $\frac{dy}{dx} = y\left(\frac{1}{x}\cosh\frac{x}{a} + \frac{1}{a}\log x.\sinh\frac{x}{a}\right)$

$\therefore \frac{dy}{dx} = x^{\cos h\frac{x}{a}}\left(\frac{1}{x}\cosh\frac{1}{x} + \frac{1}{a}\log x.\sinh\frac{x}{a}\right)$

42. 2062 Q.No. 2 b

Find the derivative of $\log\left(\sin h\frac{x}{a}\right)$. [2]

SOLUTION

Let $y = \log\left(\sin h\frac{x}{a}\right)$

Differentiating both sides with respect to 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log\sinh\frac{x}{a}\right)$$

$$= \frac{d\left(\log\sinh\frac{x}{a}\right)}{d\left(\sinh\frac{x}{a}\right)}.\frac{d\left(\sinh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}.\frac{d\left(\frac{x}{a}\right)}{dx}$$

$$= \frac{1}{\sinh\frac{x}{a}}.\cosh\frac{x}{a}.\frac{1}{a} = \frac{1}{a}\coth\frac{x}{a}$$

43. 2063 Q.No. 2 b

Find the derivative of $x^{\cos h^2(x/a)}$ [2]

SOLUTION

Let $y = x^{\cosh^2\left(\frac{x}{a}\right)}$

Taking log on both sides, we get,

$\log y = \cosh^2\left(\frac{x}{a}\right).\log x$

Differentiating both sides with respect to 'x'

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left\{\cos h^2\left(\frac{x}{a}\right).\log x\right\}$$

or, $\frac{d}{dy}(\log y)\cdot\frac{dy}{dx} = \cosh^2\left(\frac{x}{a}\right)\frac{d}{dx}(\log x) +$

$\log x\,\frac{d\left(\cosh\frac{x}{a}\right)^2}{d\left(\cosh\frac{x}{a}\right)}\cdot\frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}\cdot\frac{d\left(\frac{x}{a}\right)}{dx}$

or, $\frac{1}{y}\frac{dy}{dx} = \cosh^2\left(\frac{x}{a}\right)\cdot\frac{1}{x} + \log x\,.\,2\cosh\frac{x}{a}\,.\,\sinh\frac{x}{a}\cdot\frac{1}{a}$

or, $\frac{dy}{dx} = y\left\{\frac{1}{x}\cosh^2\left(\frac{x}{a}\right) + \frac{1}{a}\log\sinh\frac{2x}{a}\right\}$

$\therefore\ \frac{dy}{dx} = x^{\cosh^2\frac{x}{a}}\left\{\frac{1}{x}\cosh^2\left(\frac{x}{a}\right) + \frac{1}{a}\log\sinh\frac{2x}{a}\right\}$

44. 2064 Q.No. 2 b

Find the derivative of: $\left(\sinh\frac{x}{a}\right)^{x^2}$ [2]

SOLUTION

Let $y = \left(\sinh\frac{x}{a}\right)^{x^2}$

Taking log on both sides, we get,

$\log y = x^2\log\left(\sinh\frac{x}{a}\right)$

Differentiating both sides with respect to 'x'

$\frac{d}{dy}(\log y) = \frac{d}{dx}\left\{x^2\log\left(\sinh\frac{x}{a}\right)\right\}$

or, $\frac{d}{dy}(\log y)\cdot\frac{dy}{dx} = x^2\frac{d}{dx}\left(\log\sinh\frac{x}{a}\right) + \log\sinh\frac{x}{a}\cdot\frac{d}{dx}(x^2)$

or, $\frac{1}{y}\frac{dy}{dx} = x^2\,\frac{d\left(\log\sinh\frac{x}{a}\right)}{d\left(\sinh\frac{x}{a}\right)}\cdot\frac{d\left(\sinh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}\cdot\frac{d\left(\frac{x}{a}\right)}{dx} +$

$\log\sinh\frac{x}{a}\,.\,2x$

or, $\frac{1}{y}\frac{dy}{dx} = x^2\,.\,\frac{1}{\sinh\frac{x}{a}}\,.\,\cosh\frac{x}{a}\cdot\frac{1}{a} + 2x.\log\sinh\frac{x}{a}$

or, $\frac{dy}{dx} = y\left(\frac{x^2}{a}\coth\frac{x}{a} + 2x\log\sinh\frac{x}{a}\right)$

$= \left(\sinh\frac{x}{a}\right)^{x^2}\left(\frac{x^2}{a}\coth\frac{x}{a} + 2x\log\sinh\frac{x}{a}\right)$

45. 2065 Q.No 2 b

Find the derivative of Arc tan sin hx. [2]

SOLUTION

Let y $=$ Arc tan sin hx

$= \tan^{-1}\sinh x$

Differentiating both sides with respect to 'x'

$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}\sinh x)$

$= \frac{d(\tan^{-1}\sinh x)}{d(\sinh x)}\cdot\frac{d(\sinh x)}{dx}$

$= \frac{1}{1+\sinh^2 x}\cdot\cosh x = \frac{\cosh x}{\cosh^2 x} = \text{sech}\,x$

46. 2066 Q.No. 2 b

Find the derivative of $2\tanh^{-1}\left(\tan\frac{1}{2}x\right)$ [2]

➥ Please refer to 2060 Q.No. 2b

47. 2066 C Q.No. 2 b

Find the derivative of: $\log\left(\sinh\frac{x}{a}\right)$ [2]

➥ Please refer to 2062 Q.No. 2b

48. 2067 Q.No. 2b

Find the derivative of $x^{\cosh\frac{x}{a}}$ [2]

➥ Please refer to 2061 Q.No. 2b

49. 2068 Q.No. 2b

Find the derivative of $\left(\cosh\frac{x}{a}\right)^{\log x}$ [2]

SOLUTION

Let $y = \left(\cosh\frac{x}{a}\right)^{\log x}$

Taking log on both sides, we get,

$\log y = \log x.\log\left(\cosh\frac{x}{a}\right)$

Differentiating both sides with respect to 'x'

$\frac{d}{dx}(\log y) = \frac{d}{dx}\left\{\log x.\log\left(\cosh\frac{x}{a}\right)\right\}$

$\frac{d}{dy}(\log y)\frac{dy}{dx} = \log x\frac{d}{dx}\left\{\log\left(\cosh\frac{x}{a}\right)\right\} +$

$\log\left(\cosh\frac{x}{a}\right)\cdot\frac{d}{dx}(\log x)$

or, $\frac{1}{y}\frac{dy}{dx} = \log x\,\frac{d\left\{\log\left(\cosh\frac{x}{a}\right)\right\}}{d\left(\cosh\frac{x}{a}\right)}\cdot\frac{d\left(\cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}$

$\frac{d(x/a)}{dx} + \log\left(\cosh\frac{x}{a}\right)\frac{1}{x}$

or, $\frac{dy}{dx} = y\left\{\log x.\frac{1}{\cosh\frac{x}{a}}\sinh\frac{x}{a}\cdot\frac{1}{a} + \frac{1}{x}\log\left(\cosh\frac{x}{a}\right)\right\}$

$\therefore\ \frac{dy}{dx} = \left(\cosh\frac{x}{a}\right)^{\log x}\left\{\frac{1}{a}\log x\,.\,\tanh\frac{x}{a} + \frac{1}{x}\log\left(\cosh\frac{x}{a}\right)\right\}$

50. 2069 (Set A) Old Q.No. 2b

Find the derivative of $\left(\sinh\frac{x}{a}\right)^{x^2}$ [2]

➥ Please refer to 2064 Q.No. 2b

51. 2069 Old (Set B) Q.No. 2a

Find the derivative of $(\cosh x)^{\sinh^{-1}x}$ [2]

SOLUTION

Let $y = (\cosh x)^{\sinh^{-1}x}$

Taking log on both sides, we get,

$\log y = \sinh^{-1}x\log\cosh x$

Differentiating both sides with respect to 'x'

$\frac{d}{dy}(\log y) = \frac{d}{dx}(\sinh^{-1}x\log\cosh x)$

or, $\frac{d}{dy}(\log y).\frac{dy}{dx} = \sinh^{-1}x\frac{d}{dx}(\log\cosh x) +$

$\log\cosh x\frac{d}{dx}(\sinh^{-1}x)$

or, $\frac{1}{y}\frac{dy}{dx} = \sinh^{-1}x\,.\,\frac{d(\log\cosh x)}{d(\cosh x)}\cdot\frac{d(\cosh x)}{dx} +$

$\log\cosh x\frac{1}{\sqrt{1+x^2}}$

or, $\frac{dy}{dx} = y\left\{\sinh^{-1}x\,.\,\frac{1}{\cosh x}\cdot\sinh x + \log\cosh x\frac{1}{\sqrt{1+x^2}}\right\}$

$= (\cosh x)^{\sinh^{-1}x}\left\{\sinh^{-1}x\,.\,\tanh x + \frac{1}{\sqrt{1+x^2}}\log\cosh x\right\}$

52. 2070 (Old) Q.No. 2 c

Find $\frac{dy}{dx}$ when $y = \text{sech}(\tan^{-1}x)$. [2]

SOLUTION

Here, $y = \text{sech}(\tan^{-1}x)$

Differentiating both sides with respect to 'x'

$\frac{dy}{dx} = \frac{d}{dx}\{\text{sech}(\tan^{-1}x)\}$

$= \frac{d\{\text{sech}(\tan^{-1}x)\}}{d(\tan^{-1}x)}\cdot\frac{d(\tan^{-1}x)}{dx}$

$= -\text{sech}(\tan^{-1}x)\,.\,\tanh(\tan^{-1}x).\frac{1}{1+x^2}$

$= \frac{-\text{sech}(\tan^{-1}x)\,.\,\tanh(\tan^{-1}x)}{1+x^2}$

53. 2070 Supp. Q.No. 3 a

Find the derivative of $(\ln x)^{\sinh x}$ [2]

➥ Please see Model Set I, Q.No. 3a

54. 2071 Old Q.No. 2 b

Find the derivative of log sinh x. [2]

SOLUTION

Let $y = \log\sinh x$

Differentiating both sides with respect to 'x'

$\frac{dy}{dx} = \frac{d}{dx}(\log\sinh x) = \frac{d(\log\sinh x)}{d(\sinh x)}\cdot\frac{d(\sinh x)}{dx}$

$= \frac{1}{\sinh x}\cdot\cosh x = \coth x$

55. 2071 Supp. Q.No. 3a

Find the derivative of $(\sec x)^{\tan x}$. [2]

SOLUTION

Let $y = (\sec x)^{\tan x}$

Taking 'log' on both sides, we get,

$\log y = \tan x.\log\sec x$

Differentiating both sides with respect to 'x'

$\frac{d}{dx}(\log y) = \frac{d}{dx}(\tan x.\log\sec x)$

or, $\frac{d}{dy}(\log y)\cdot\frac{dy}{dx} = \tan x\frac{d}{dx}(\log\sec x) + \log\sec x\frac{d}{dx}(\tan x)$

or, $\frac{1}{y}\frac{dy}{dx} = \tan x\frac{d(\log\sec x)}{d(\sec x)}\cdot\frac{d(\sec x)}{dx} + \log\sec x.\sec^2 x$

or, $\frac{1}{y}\frac{dy}{dx} = \tan x\,.\,\frac{1}{\sec x}\,.\,\sec x\,.\,\tan x + \sec^2 x\,.\,\log\sec x$

or, $\frac{dy}{dx} = y\,(\tan^2 x + \sec^2 x\,.\,\log\sec x)$

$\therefore\ \frac{dy}{dx} = (\sec x)^{\tan x}(\tan^2 x + \sec^2 x\,.\,\log\sec x)$

56. 2075 Set B Q.No. 3a

Find the derivative of $x^{\sinh x}$. [2]

SOLUTION

Let $y = x^{\sinh x}$

Taking log on both sides, we get,

$\log y = \sinh x\log x$

Differentiating both sides w.r. to x,

$\frac{d}{dx}(\log y) = \frac{d}{dx}(\sinh x.\log x)$

or, $\frac{d}{dy}(\log y).\frac{dy}{dx} = \sinh x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sinh x)$

or, $\frac{1}{y}\cdot\frac{dy}{dx} = \sinh x.\frac{1}{x} + \log x.\cosh x$

or, $\frac{dy}{dx} = y\left(\frac{\sinh x}{x} + \log x\cosh x\right)$

$\therefore\ \frac{dy}{dx} = x^{\sinh x}\left(\frac{\sinh x}{x} + \log x\cosh x\right)$

4 MARKS QUESTIONS

57. 2057 Q.No. 10 b

Find the derivative of $\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}$ [4]

SOLUTION

Let $y = \left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}$

Taking log on both sides, we get,

$\log y = nx\log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)$

Differentiating both sides with respect to 'x'

$\frac{d}{dx}(\log y) = \frac{d}{dx}\left\{nx\log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}$

or, $\frac{d}{dy}(\log y)\,.\,\frac{dy}{dx} = nx\frac{d}{dx}\left\{\log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\} +$

$\log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\frac{d}{dx}(nx)$

or, $\frac{1}{y}\frac{dy}{dx} = nx\,\frac{d\left\{\log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right\}}{d\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}\cdot\frac{d\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}{d\left(\frac{x}{a}\right)}$

$\cdot\frac{d\left(\frac{x}{a}\right)}{dx} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right).\,n$

or, $\frac{1}{y}\frac{dy}{dx} = \left[nx\left\{\frac{1}{\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)}\cdot\left(\cosh\frac{x}{a} + \sinh\frac{x}{a}\right)\cdot\frac{1}{a}\right\} + \log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\cdot n\right]$

or, $\frac{dy}{dx} = y\left[\frac{nx}{a} + n\log\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)\right]$

$$\therefore \frac{dy}{dx} = n\left(\sinh\frac{x}{a} + \cosh\frac{x}{a}\right)^{nx}\left\{\frac{x}{a} + \log\left(\sinh\frac{x}{a} + \cos\frac{x}{a}\right)\right\}$$

B. APPLICATION OF DERIVATIVES

I. DIFFERENTIALS, TARGET & NORMAL

2 MARKS QUESTIONS

1. 2057 Q.No. 5 c

Find the slope and inclination with x-axis of the tangent of the curve $2y = 2 - x^2$ at $x = 1$. [2]

SOLUTION

Given curve is $2y = 2 - x^2$

Differentiating both sides with respect to 'x'

$$2\frac{dy}{dx} = 0 - 2x$$

$$\frac{dy}{dx} = -x$$

At $x = 1$, $\frac{dy}{dx} = -1$

i.e. Slope (m) = -1

If θ be the inclination of the tangent with x-axis, then $\tan\theta = -1 = \tan\frac{3\pi}{4}$

$\therefore \theta = \frac{3\pi}{4}$

2. 2058 Q.No. 5 c

Find the slope and inclination with x-axis of the tangent of: $x^2 + y^2 = 36$ at (0, 6). [2]

SOLUTION

Given, $x^2 + y^2 = 36$

Differentiating both sides with respect to 'x'

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

At (0, 6); $\frac{dy}{dx} = \frac{-0}{6} = 0$

$\therefore$ Slope (m) = 0

If θ be the inclination of the tangent with x-axis, then $\theta = 0 = \tan 0°$.

$\therefore \theta = 0°$

3. 2060 Q.No. 5 c

Find the points on the curve $4y = x^4 - 8x^2$ where the tangents are paralleled to the x-axis. [2]

SOLUTION

Given, $4y = x^4 - 8x^2$

$$4\frac{dy}{dx} = 4x^3 - 16x$$

$$\frac{dy}{dx} = x^3 - 4x$$

For the tangent to be parallel to the x-axis, $\frac{dy}{dx} = 0$

i.e. $x^3 - 4x = 0$

or, $x(x^2 - 4) = 0$

$\therefore x = 0, 2, -2$

When $x = 0$ $\quad 4y = 0 - 8 \times 0$

$\Rightarrow y = 0$

When $x = 2$ $\quad 4y = 2^4 - 8 \times 2^2$

$\Rightarrow y = -4$

When $x = -2$ $\quad 4y = (-2)^4 - 8(-2)^2 \Rightarrow y = -4$

$\therefore$ Required points are (0,0), (2, −4) and (−2, −4)

4. 2061 Q.No. 2 c

Find the slope and inclination with the x-axis of the tangent of $y = -3x - x^4$ at $x = -1$. [2]

SOLUTION

Here, $y = -3x - x^4$

$$\frac{dy}{dx} = -3 - 4x^3$$

At $x = -1$, $\frac{dy}{dx} = -3 - 4(-1)^3 = 1$

$\therefore$ Slope (m) = 1

If θ be the inclination of the tangent with x-axis, then

$\tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$.

5. 2062 Q.No. 2 c

Find the angle of intersection of the curves $4y = x^2 + 12$ and $y^2 = 8x$ at (2, 4) [2]

SOLUTION

Given curves are

$4y = x^2 + 12$ ...(i)

and $y^2 = 8x$ ...(ii)

From (i)

$$4\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{x}{2}$$

At (2,4), $\frac{dy}{dx} = \frac{2}{2} = 1$

i.e. $m_1 = 1$

Again from (ii)

$$2y\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = \frac{4}{y}$$

At (2, 4), $\frac{dy}{dx} = \frac{4}{4} = 1$

If θ be the angle of intersection, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{1 - 1}{1 + 1 \times 1} = 0 = \tan 0°$$

$\therefore \theta = 0°$

6. 2063 Q.No. 2 c

A circular copper plate is heated so that radius increases from 5cm to 5.06cm. Find approximate increase in area.

SOLUTION

Let 'r' be the radius and 'A' be the area of circular plate. Given,

$r = 5$ cm, $\Delta r = dr = 5.06 - 5 = 0.06$

We have,

$A = \pi r^2$

$dA = 2\pi r\, dr = 2\pi \times 5 \times 0.06 = 0.6\pi$ cm²

$\therefore$ Approximate increase in area $= 0.6\pi$ cm²

7. 2064 Q.No. 2 c

Find the points on the curve $y = x^3 - 3x^2 + 1$ where the tangents are parallel to x-axis. [2]

SOLUTION

Here, $y = x^3 - 3x^2 + 1$

$$\frac{dy}{dx} = 3x^2 - 6x$$

For tangent parallel to x-axis, $\frac{dy}{dx} = 0$.

$\Rightarrow 3x^2 - 6x = 0$

$\Rightarrow 3x(x - 2) = 0$

$\Rightarrow x = 0, 2$

When $x = 0$, $y = 0^3 - 3 \cdot 0^2 + 1 = 1$

When $x = 2$, $y = 2^3 - 3 \cdot 2^2 + 1 = -3$

$\therefore$ The required points are (0, 1) and (2, −3).

8. 2065 Q.No 2 c

Find where the tangent is parallel to the x-axis for the curve $y = x^3 - 3x^2 - 9x + 15$. [2]

SOLUTION

Given, $y = x^3 - 3x^2 - 9x + 15$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

For the tangent to be parallel to the x-axis, we have $\frac{dy}{dx} = 0$

i.e. $3x^2 - 6x - 9 = 0$

or, $x^2 - 2x - 3 = 0$

or, $(x - 3)(x + 1) = 0$

$\therefore x = -1, 3$

When $x = -1$, $y = (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 15$

$= -1 - 3 + 9 + 15 = 20$

When $x = 3$, $y = 3^3 - 3 \times 3^2 - 9 \times 3 + 15 = -12$

$\therefore$ Required points are (−1, 20) and (3, −12)

9. 2066 C Q.No. 2 c

Find the points on the curve $x^2 + y^2 = 16$ at which the tangents are parallel to y-axis. [2]

SOLUTION

Given, $x^2 + y^2 = 16$ ... (i)

Differentiating both sides with respect to 'x'

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(16)$$

$$2x + 2y\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = \frac{-x}{y}$

or, $\frac{dx}{dy} = \frac{-y}{x}$

For the tangent parallel to y-axis, we have

$\frac{dx}{dy} = 0$

i.e. $\frac{-y}{x} = 0$

$\Rightarrow y = 0$

Then from (i)

$x^2 + 0 = 16$

or, $x^2 = (\pm 4)^2$

$\therefore x = \pm 4$

Required points are (4, 0) and (−4, 0)

10. 2066 Q.No. 2 c

Find the angle of intersection of the curves $y^2 = x^3$ and $y = 2x$ at the point (0, 0). [2]

SOLUTION

Given curves are

$y^2 = 2x$ ...(i)

$y^2 = x^3$

$y = x^{3/2}$ ...(ii)

Now, differentiating both sides of (ii) with respect to 'x'

$$\frac{dy}{dx} = 2$$

At (0, 0); $\frac{dy}{dx} = 0$

i.e. $m_1 = 0$

Again, differentiating both sides of (i) with respect to 'x'

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

At (0, 0); $\frac{dy}{dx} = 0$

i.e. $m_2 = 0$

If θ be the angle of intersection then,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2 - 0}{1 + 2 \times 0} = 2$$

$\therefore \theta = \tan^{-1}(2)$

11. 2067 Q.No. 2c

Find the angle of intersection between the curves $y = x^2$ and $6y = 7 - x^3$ at (1, 1). [2]

SOLUTION

Given curves are

$y = x^2$ ...(i)

and $6y = 7 - x^3$ ...(ii)

Differentiating (i) with respect to 'x'

$$\frac{dy}{dx} = 2x$$

At (1, 1); $\frac{dy}{dx} = 2 \times 1 = 2$

i.e. $m_1 = 2$

Again, differentiating (ii) both sides with respect to 'x'

$$6\frac{dy}{dx} = -3x^2$$

or, $\frac{dy}{dx} = \frac{-x^2}{2}$

At (1, 1); $\frac{dy}{dx} = \frac{-1}{2}$

i.e. $m_2 = \frac{-1}{2}$

If θ be the angle of intersection, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2 + \frac{1}{2}}{1 + 2 \cdot \left(\frac{-1}{2}\right)} = \left(\frac{\frac{5}{2}}{0}\right) = \infty = \tan 90°$$

$\therefore \theta = 90°$

12. 2068 Q.No. 5c

At what angle does the curve $y(1+x) = x$ cut the x-axis? [2]

SOLUTION

Given, $y(1+x) = x$ ...(i)

The curve meets the x-axis where y = 0. So, putting y = 0 in (i), we get

From (i)

$$y = \frac{x}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x)\frac{dx}{dx} - x\frac{d}{dx}(1+x)}{(1+x)^2} = \frac{(1+x).1 - x}{(1+x)^2}$$

$$= \frac{1}{(1+x)^2}$$

At x = 0, $\frac{dy}{dx} = \frac{1}{(1+0)^2} = 1$

If θ be the angle made by tangent with x-axis

then, $\tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

13. 2070 Set D Q.No. 3 a

Find the equation of the tangent to the curve.

$y = 2x^3 - 5x^2 + 8$ at (2, 4) [2]

SOLUTION

Given curve is $y = 2x^3 - 5x^2 + 8$

Differentiating both sides with respect to 'x'

$$\frac{dy}{dx} = 6x^2 - 10x$$

At (2, 4), $\frac{dy}{dx} = 6 \times 2^2 - 10 \times 2 = 4$

i.e. slope (m) = 4

The equation of tangent at (2, 4) and having slope 4 is

$y - y_1 = m(x - x_1)$

or, $y - 4 = 4(x - 2)$

or, $y - 4 = 4x - 8$

or, $4x - y - 4 = 0$

$\therefore 4x - y = 4$

14. 2070 (Old) Q.No. 6 c

Find the points on circle $x^2 + y^2 = 16$ at which tangents are parallel to y- axis. [2]

➥ Please refer to 2066 C Q.No. 2c

15. 2070 Supp. Q.No. 3 b

Find the slope of the tangent to the curve.

$y = x^3 + 2x^2 + 3x - 10$ at (-3, 2). [2]

SOLUTION

Here, $y = x^3 + 2x^2 + 3x - 10$

$$\frac{dy}{dx} = 3x^2 + 4x + 3$$

$\frac{dy}{dx}$ at (-3, 2) $= 3 \times (-3)^2 + 4 \times (-3) + 3$

$= 27 - 12 + 3 = 18$

$\therefore$ Slope of tangent at (-3, 2) = $\frac{dy}{dx}$ at (-3, 2) = 18.

16. 2071 Old Q.No. 5 c

Find the points on the curve $y = x^3 - 3x^2 + 1$ where the tangents are parallel to the x-axis. [2]

➥ Please refer to 2064 Q.No. 2c

17. 2071 Set C Q.No. 3 a

Find the points on the curve $y = x^3 - 3x^2 + 1$ where the tangent is parallel to the x-axis. [2]

➥ Please refer to 2064 Q.No. 2c

18. 2072 Set E Q.No. 3a

Find the points on the circle $x^2 + y^2 = 16$ at which the tangents are parallel to X–axis. [2]

SOLUTION

Here, $x^2 + y^2 = 16$ ... (i)

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = \frac{-x}{y}$$

For tangent parallel to x-axis, $\frac{dy}{dx} = 0$

i.e. $\frac{-x}{y} = 0$

$\Rightarrow x = 0$

Then from (i)

$0 + y^2 = 16$

$y = \pm 4$

$\therefore$ Required points are (0, 4) and (0, – 4)

19. 2074 Set A Q.No. 3a

Find the points on the curve $x^2 + y^2 = 36$ at which the tangents are parallel to the y-axis. [2]

SOLUTION

Given, curve is $x^2 + y^2 = 36$ ... (i)

Differentiating both sides w.r.to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(36)$$

or, $2x + 2y\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = \frac{-x}{y}$

For the tangent parallel to y- axis, we have,

$$\frac{dx}{dy} = 0$$

or, $\frac{-y}{x} = 0 \quad \therefore y = 0$

Putting the value of y in (i), we get,

$x^2 = 36$

or, $x = \pm 6$.

Hence, the required points are (6, 0) & (-6, 0)

20. 2074 Set B Q.No. 3a

Find the equation of the normal of the curve $y = 2x^3 - 5x^2 + 8$ at (2, 4).

SOLUTION

Given,

$y = 2x^3 - 5x^2 + 8$

$$\frac{dy}{dx} = 6x^2 - 10x$$

$\frac{dy}{dx}$ at (2, 4) $= 6 \times 2^2 - 10 \times 2 = 4$

Slope of normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{1}{4}$

The equation of normal at $(x_1, y_1) = (2, 4)$ is

$y - y_1 = m(x - x_1)$

or, $y - 4 = -\frac{1}{4}(x - 2)$

or, $4y - 16 = -x + 2$

$\therefore x + 4y = 18$

21. 2074 Supp Q.No. 3a

Find the equation of the normal to the curve $y = 2x^3 - 5x^2 + 8$ at the point (2, 4). [2]

➥ Please refer to 2074 Set B Q.N. 3a

22. 2075 Set C Q.No. 3a

Find the equation of the tangent to $y = x^3 - 2x^2 + 4$ at (2, 4). [2]

SOLUTION

Given,

$y = x^3 - 2x^2 + 4$

$$\frac{dy}{dx} = 3x^2 - 4x$$

$\therefore \frac{dy}{dx}$ at (2, 4) $= 3 \times 2^2 - 4 \times 2 = 4$

The equation of tangent at $(x_1, y_1) = (2, 4)$ is

$y - y_1 = m(x - x_1)$

or, $y - 4 = 4(x - 2)$

or, $y - 4 = 4x - 8$

or, $4x - y - 8 + 4 = 0$

$\therefore 4x - y - 4 = 0.$

II. HOSPITAL'S RULE

2 MARKS QUESTIONS

23. 2069 (Set A) Q.No. 3a

Using L Hospital's rule, evaluate: $\lim_{x\to 0}\frac{e^x - x - 1}{x^2}$. [2]

SOLUTION

$\lim_{x\to 0}\frac{e^x - x - 1}{x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{e^x - 1}{2x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

24. 2069 (Set B) Q.No. 3a

Using L Hospital's rule, evaluate: $\lim_{x\to 0}\frac{x - \sin x}{x^3}$ [2]

SOLUTION

$\lim_{x\to 0}\frac{x - \sin x}{x^3}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{1 - \cos x}{3x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{\sin x}{6x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{\cos x}{6}$

$= \frac{\cos 0}{6} = \frac{1}{6}$

25. 2070 Set C Q.No. 3 a

Using L Hospital's rule, evaluate: $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$. [2]

SOLUTION

$\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{\sec^2 x - 1}{1 - \cos x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{2\sec x.\sec x\tan x}{\sin x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{\sin x}$

$= \lim_{x\to 0}\frac{2(\sec^2 x.\sec^2 x + \tan x.2\sec x.\sec x.\tan x)}{\cos x}$

$= \lim_{x\to 0}\frac{2(\sec^4 x + 2\sec^2 x\tan^2 x)}{\cos x} = \frac{2(1+0)}{1} = 2$

26. 2071 Set D Q.No. 3 a

Using L Hospital's rule, evaluate:

$\lim_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$ [2]

SOLUTION

$\lim_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{\sin^2 x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{e^x + e^{-x} + 2\sin x}{2\sin x\cos x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{e^x - e^{-x} + 2\sin x}{\sin 2x}$

$= \lim_{x\to 0}\frac{e^x - e^{-x} + 2\cos x}{2\cos 2x}$

$= \frac{e^0 + e^{-0} + 2\cos 0}{2\cos 0} = \frac{1 + 1 + 2}{2} = 2$

27. 2072 Set C Q.No. 3a

Evaluate, using L' Hospital's rule: $\text{Lt}_{x\to 0}\frac{\tan ax}{\tan bx}$. [2]

SOLUTION

$\lim_{x\to 0}\frac{\tan ax}{\tan bx}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0}\frac{a\sec^2 ax}{b\sec^2 bx} = \frac{a.1}{b.1} = \frac{a}{b}$

28. 2072 Set D Q.No. 3b

Using L' Hospital's rule, evaluate: $\text{Lt}_{x\to 0}\frac{e^x - x - 1}{x^2}$. [2]

➥ Please refer to 2069 (Set A) Q.No. 3a

29. 2072 Supp Q.No. 3a

Using L Hospital's rule, evaluate:

$\lim\limits_{x\to0} \dfrac{xe^x - \log(1+x)}{x^2}$ **[2]**

SOLUTION

$\lim\limits_{x\to0} \dfrac{xe^x - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to0} \dfrac{xe^x + e^x - \frac{1}{1+x}}{2x}$ $\left(\frac{0}{0}\text{ form}\right)$

$= \lim\limits_{x\to0} \dfrac{xe^x + e^x + e^x + \frac{1}{(x+1)^2}}{2}$

$= \dfrac{0+1+1+1}{2} = \dfrac{3}{2}$

30. 2073 Set C Q.No. 3a

Using L Hospital's rule, evaluate: $\lim\limits_{x\to0} \dfrac{x - \sin x \cos x}{x^3}$ **[2]**

SOLUTION

$\lim\limits_{x\to0} \dfrac{x - \sin x\cos x}{x^3}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{x - \frac{1}{2}\sin 2x}{x^3}$

$= \lim\limits_{x\to0} \dfrac{1-\cos 2x}{3x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{2\sin 2x}{6x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{\sin 2x}{3x}$

$= \lim\limits_{x\to0} \dfrac{2\cos 2x}{3} = \dfrac{2\times1}{3} = \dfrac{2}{3}$

31. 2073 Set D Q.No. 3a

Using L Hospital's rule, evaluate:

$\lim\limits_{x\to0} \dfrac{x - \sin x\cos x}{x^3}$ **[2]**

➥ Please refer to 2073 Set C Q.No. 3a

32. 2073 Supp Q.No. 3a

Evaluate, using L' Hospital's rule: $\lim\limits_{x\to0} \dfrac{\ln(\tan x)}{\ln x}$.

SOLUTION

$\lim\limits_{x\to0} \dfrac{\ln(\tan x)}{\ln x}$ $\left[\frac{\infty}{\infty}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{\frac{1}{\tan x}\cdot\sec^2 x}{\frac{1}{x}}$

$= \lim\limits_{x\to0} \dfrac{x\sec^2 x}{\tan x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{x.2\sec x.\sec x\tan x + \sec^2 x}{\sec^2 x} = \dfrac{0+1}{1} = 1$

33. 2074 Set A Q.No. 4a

Using L Hospital's rule, evaluate: $\lim\limits_{x\to0} \dfrac{x^2 - \sin^2 x}{x^2}$ **[2]**

SOLUTION

$\lim\limits_{x\to0} \dfrac{x^2 - \sin^2 x}{x^2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{2x - 2\sin x\cos x}{2x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x\to0} \dfrac{2x - \sin 2x}{2x}$

$= \lim\limits_{x\to0} \dfrac{2 - 2\cos 2x}{2}$

$= \dfrac{2 - 2\cos 0}{2} = \dfrac{2 - 2\times1}{2} = \dfrac{2-2}{2} = \dfrac{0}{2} = 0.$

34. 2075 Set A Q.No. 3a

Using L Hospital's rule, evaluate: $\lim\limits_{x\to0} \dfrac{\tan x - x}{x - \sin x}$. **[2]**

➥ Please refer to 2070 Set C Q.No. 3a

III. ROLLE'S THEOREM AND MEAN VALUE THEOREM

4 MARKS QUESTIONS

1. 2070 Supp. Q.No. 6 b OR

State Rolle's theorem. Verify that the function $f(x) = x(x-3)^2$ on [0, 3] satisfies conditions of Rolle's theorem and find c prescribed in the theorem. **[4]**

SOLUTION

First Part

Statement of Rolle's theorem:

If a function f(x) is

(a) continuous in [a, b]

(b) differentiable in (a, b)

(c) f(a) = f(b)

then there exists at least one point c ∈ (a, b) such that f'(c) = 0.

Last part

Here, $f(x) = x(x-3)^2, x \in [0, 3]$

$= x(x^2 - 6x + 9) = x^3 - 6x^2 + 9x$

Since f(x) is a polynomial function, so it is continuous in [0, 3]

Again, $f'(x) = 3x^2 - 12x + 9$ which exists for all $x \in (0, 3)$

So, f(x) is differentiable in (0, 3)

And

$f(0) = 0^3 - 6\times0^2 + 9\times0 = 0$

$f(3) = 3^3 - 6\times3^2 + 9\times3 = 0$

∴ f(0) = f(3)

All the conditions of Rolle's theorem are satisfied. So there exists at least one $c \in (0, 3)$ such that $f'(c) = 0$

⇒ $3c^2 - 12c + 9 = 0$

⇒ $c^2 - 4c + 3 = 0$

⇒ $(c-1)(c-3) = 0$

⇒ $c = 1, 3$

Here, $c = 1 \in (0, 3)$ but $c = 3 \notin (0, 3)$

Hence Rolle's theorem is verified.

6 MARKS QUESTIONS

2. 2069 (Set A) Q.No. 11

State mean value theorem. Interpret it geometrically. Verify mean value theorem for the function $f(x) = x^3 + x^2 - 6x$ in [–1, 4]. **[6]**

SOLUTION

Statement of mean value theorem

If a function f(x) is

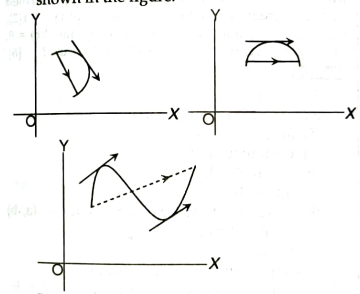
(a) continuous in the closed interval [a, b]

(b) differentiable in the open interval (a, b)

then there exists at least one c ∈ (a, b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Second part

Geometrically, Lagrange's mean value theorem says that in a continuous curve, in which tangent can be drawn at every point, there is at least one point where the tangent is parallel to the secant joining the end points as shown in the figure.



Last part

Here, $f(x) = x^3 + x^2 - 6x$

Since f(x) is a polynomial function, so it is continuous in [–1, 4].

Again, $f'(x) = 3x^2 + 2x - 6$ which exists for all $x \in (-1, 4)$.

∴ f(x) is differentiable in (–1, 4).

So both conditions of mean value theorem are satisfied. Hence there exits at least one c in (–1, 4) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

⇒ $3c^2 + 2c - 6 = \dfrac{56-6}{4-(-1)}$

⇒ $3c^2 + 2c - 6 = 10$

⇒ $3c^2 + 2c - 16 = 0$

⇒ $3c^2 + 8c - 6c - 16 = 0$

⇒ $c(3c+8) - 2(3c+8) = 0$

⇒ $(3c+8)(c-2) = 0$

⇒ $c = \dfrac{-8}{3}, 2$

Here, $c = \dfrac{-8}{3} \notin (-1, 4)$ but $c = 2 \in (-1, 4)$

Hence, mean value theorem is verified.

3. 2069 (Set B) Q.No. 11

State mean value theorem. Interpret it geometrically. Verify mean value theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in [1, 4]. **[6]**

SOLUTION

First part:

Please refer to 2069 (Set A) Q.No.11

Next part:

Here, $f(x) = (x-1)(x-2)(x-3)$ in [1, 4]

$= x^3 - 6x^2 + 11x - 6$

Since f(x) is a polynomial function, so it is continuous in [1, 4]

Again, $f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (1, 4)$, so it is differentiable in (1, 4).

Hence, both the conditions of mean value theorem are satisfied. So there exists at least one c ∈ (1, 4) such that

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

But $f(b) = f(4) = 4^3 - 6\cdot4^2 + 11\cdot4 - 6 = 6$

$f(a) = f(1) = 1^3 - 6\cdot1^2 + 11\cdot1 - 6 = 0$

∴ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

⇒ $3c^2 - 12c + 11 = \dfrac{6-0}{4-1}$

⇒ $3c^2 - 12c + 11 = 2$

⇒ $3c^2 - 12c + 9 = 0$

⇒ $c^2 - 4c + 3 = 0$

⇒ $(c-1)(c-3) = 0$

⇒ $c = 1, 3$

But $c = 1 \notin (1, 4)$ and $c = 3 \in (1, 4)$

Hence mean value theorem is verified.

4. 2070 Set C Q.No. 11

State Rolle's theorem. Interpret it geometrically. Verify Rolle's Theorem for the function $f(x) = x(x-3)^2$ for $x \in [0, 3]$. **[6]**

SOLUTION

First Part

Statement of Rolle's theorem:

If a function f(x) is

(a) continuous in [a, b]

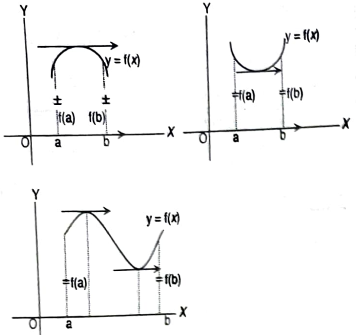
(b) differentiable in (a, b)

(c) f(a) = f(b)

then there exists at least one point c ∈ (a, b) such that f'(c) = 0.

Second Part

If all the conditions of Rolle's theorem are satisfied then there is at least one point c ∈(a, b) where the tangent is parallel to x-axis.



Last part

Here, $f(x) = x(x-3)^2, x \in [0, 3]$

$= x(x^2 - 6x + 9) = x^3 - 6x^2 + 9x$

Since $f(x)$ is a polynomial function, so it is continuous in [0, 3]

Again, $f'(x) = 3x^2 - 12x + 9$ which exists for all $x \in (0, 3)$

So, $f(x)$ is differentiable in (0, 3)

And

$f(0) = 0^3 - 6 \times 0^2 + 9 \times 0 = 0$

$f(3) = 3^3 - 6 \times 3^2 + 9 \times 3 = 0$

$\therefore f(0) = f(3)$

All the conditions of Rolle's theorem are satisfied. So there exists at least one $c \in (0, 3)$ such that $f'(c) = 0$

$\Rightarrow 3c^2 - 12c + 9 = 0$

$\Rightarrow c^2 - 4c + 3 = 0$

$\Rightarrow (c-1)(c-3) = 0$

$\Rightarrow c = 1, 3$

Here, $c = 1 \in (0, 3)$ but $c = 3 \notin (0, 3)$

Hence Rolle's theorem is verified.

5. 2070 Set D Q.No. 11

State mean value theorem. Interpret it geometrically. Verify mean value theorem for the function $f(x) = x(x-1)^2$ in [0, 2]. [6]

SOLUTION

First part:

Please refer to 2069 (Set A) Q.No. 11

Next part:

Here, $f(x) = x(x-1)^2, x \in [0, 2]$

$= x(x^2 - 2x + 1) = x^3 - 2x^2 + x$

Since $f(x)$ is a polynomial function, so it is continuous in [0, 2].

Again, $f'(x) = 3x^2 - 4x + 1$ which exists for all x in (0, 2)

$\therefore f(x)$ is differentiable in (0, 2)

Hence both the conditions of mean value theorem are satisfied. So, there exists at least one $c \in (0, 2)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Here, $f(b) = f(2) = 2^3 - 2 \times 2^2 + 2 = 2$

$f(a) = f(0) = 0^3 - 2 \times 0^2 + 0 = 0$

Now, $f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow 3c^2 - 4c + 1 = \frac{2 - 0}{2 - 0}$

$\Rightarrow c(3c - 4) = 0$

$\Rightarrow c = 0, \frac{4}{3}$

Here, $c = 0 \notin (0, 2)$ but $c = \frac{4}{3} \in (0, 2)$

Hence, mean value theorem is verified.

6. 2071 Set C Q.No. 11

State mean value theorem. Interpret it geometrically. Verify the mean value theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in [1, 4]. [6]

➥ Please refer to 2069 (Set B) Q.No. 11

7. 2071 Set D Q.No. 11

State Rolle's Theorem. Interpret it geometrically. Verify Rolle's Theorem for the function $f(x) = x(x-3)^2$ for $x \in [0, 3]$ [6]

➥ Please refer to 2070 Set C Q.No. 11

8. 2071 Supp. Q.No. 11 OR

State Rolle's theorem. Verify that the function f(x)=sinx + cosx on [0, 2π) satisfies the conditions of Rolle's theorem and find the constant c prescribed by the theorem. If $f(x) = (x-1)^{-2}$, show that f(0) = f(2), but there is no number c in (0,2) such that f'(c) = 0. Why does not this contradict Rolle's theorem? [6]

SOLUTION

First part

Statement of Rolle's theorem:

If a function f(x) is

(a) continuous in [a, b]

(b) differentiable in (a, b)

(c) f(a) = f(b)

then there exists at least one point $c \in (a, b)$ such that f'(c) = 0.

Next part

Here, $f(x) = \sin x + \cos x$ in $[0, 2\pi]$

Since for all values of x such that $0 \le x \le 2\pi$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[0, 2\pi]$

Also $f'(x) = \cos x - \sin x$ which exists for all $x \in (0, 2\pi)$

So, $f(x)$ is differentiable in $(0, 2\pi)$

& $f(0) = \sin 0 + \cos 0 = 1$

$f(2\pi) = \sin 2\pi + \cos 2\pi = 1$

$\therefore f(0) = f(2\pi)$

$\therefore$ All the conditions of Rolle's theorem are satisfied. So there exists at least one $c \in (0, 2\pi)$ such that $f'(c) = 0$

$\Rightarrow \cos c - \sin c = 0$

$\Rightarrow \sin c = \cos c$

$\Rightarrow \tan c = 1 = \tan \frac{\pi}{4}$

$\Rightarrow c = \frac{\pi}{4} \in (0, 2\pi)$

Hence, Rolle's theorem is satisfied.

Again,

$f(x) = (x-1)^{-2} = \frac{1}{(x-1)^2}$

$f(0) = \frac{1}{(0-1)^2} = 1$

& $f(2) = \frac{1}{(2-1)^2} = 1$

$\therefore f(0) = f(2)$

The function $f(x) = \frac{1}{(x-1)^2}$ is not defined at $x = 1$.

So it is not continuous in [0, 2]. So, the all conditions of Rolle's theorem are not satisfied and hence, there is no number $c \in (0, 2)$ such that $f'(c) = 0$

9. 2072 Set C Q.No. 11

State Rolle's theorem. Verify Rolle's theorem for the function $f(x) = 2x^2 - 3x + 1$ in $\left[\frac{1}{2}, 1\right]$ [6]

SOLUTION

Statement of Rolle's theorem:

If a function f(x) is

(a) continuous in [a, b]

(b) differentiable in (a, b)

(c) f(a) = f(b)

then there exists at least one point $c \in (a, b)$ such that f'(c) = 0.

Here $f(x) = 2x^2 - 3x + 1$ in $\left[\frac{1}{2}, 1\right]$

Since $f(x)$ is a polynomial function, so it is continuous in $\left[\frac{1}{2}, 1\right]$.

Also, $f'(x) = 4x - 3$ which exists for all $x \in \left(\frac{1}{2}, 1\right)$

So, $f(x)$ is differentiable in $\left(\frac{1}{2}, 1\right)$

& $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^2 - 3 \times \frac{1}{2} + 1 = 0$

$f(1) = 2 \times 1^2 - 3 \times 1 + 1 = 0$

$\therefore f\left(\frac{1}{2}\right) = f(1)$

Hence, all the conditions of Rolle's theorem are satisfied. So, there exists at least one $c \in \left(\frac{1}{2}, 1\right)$ such that $f'(c) = 0$

i.e. $4c - 3 = 0$

or, $c = \frac{3}{4} \in \left(\frac{1}{2}, 1\right)$

Hence, Rolle's theorem is verified.

10. 2072 Set D Q.No. 11

State Mean Value theorem. Verify the mean value theorem for the function $f(x) = \sqrt{x^2 - 4}$, $x \in [2, 4]$. [6]

SOLUTION

Statement of mean value theorem:

If a function $f(x)$ is

(a) continuous in the closed interval [a, b]

(b) differentiable in the open interval (a, b)

then there exists at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Here, $f(x) = \sqrt{x^2 - 4}$, $x \in [2, 4]$

For every value of x such that $2 \le x \le 4$, $f(x)$ has a definite value, so $f(x)$ is continuous in [2, 4].

Also, $f'(x) = \frac{x}{\sqrt{x^2 - 4}}$ which exists for all $x \in (2, 4)$

$\therefore f(x)$ is differentiable in (2, 4)

Hence, both conditions of mean value theorem are satisfied. So there exists at least one c in (2, 4) such that

$f'(c) = \frac{f(b) - f(a)}{b - a}$

But, $f(b) = f(4) = \sqrt{4^2 - 4} = 2\sqrt{3}$

$f(a) = f(2) = \sqrt{2^2 - 4} = 0$

$\therefore f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3}}{4 - 2}$

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \sqrt{3}$

$\Rightarrow c^2 = 3c^2 - 12$

$\Rightarrow c^2 = 6$

$\Rightarrow c = \pm\sqrt{6}$

Here, $c = -\sqrt{6} \notin (2, 4)$ but $c = \sqrt{6} \in (2, 4)$.

Hence, mean value theorem is satisfied.

11. 2072 Set E Q.No. 11

State Rolle's theorem. What is the geometrical interpretation of Rolle's theorem. Verify Rolle's theorem for the function $f(x) = \sqrt{1 - x^2}$, $x \in [-1, 1]$. [6]

SOLUTION

First Part: Statement of Rolle's theorem:

If a function f(x) is

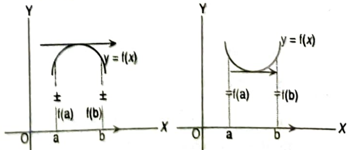
(a) continuous in [a, b]

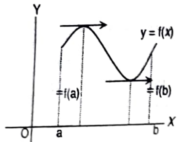
(b) differentiable in (a, b)

(c) f(a) = f(b)

then there exists at least one point $c \in (a, b)$ such that f'(c) = 0.

Second Part: If all the conditions of Rolle's theorem are satisfied then there is at least one point $c \in (a, b)$ where the tangent is parallel to x-axis.





Here, $f(x) = \sqrt{1-x^2}, x \in [-1, 1]$

For every value of x such that $-1 \le x \le 1$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[-1, 1]$

Again, $f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$ which exists for all x in $(-1, 1)$

So, $f(x)$ is differentiable in $(-1, 1)$

Also, $f(1) = \sqrt{1-1^2} = 0$

$f(-1) = \sqrt{1-(-1)^2} = 0$

$f(1) = f(-1)$

All the conditions of Rolle's theorem are satisfied. Hence, there exists at least one $c \in (-1, 1)$ such that $f'(c) = 0$

$\Rightarrow \frac{-c}{\sqrt{1-c^2}} = 0$

$\Rightarrow c = 0 \in (-1, 1)$

Hence Rolle's theorem is verified.

12. 2072 Supp Q.No. 11

State Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function $f(x) = \sin x, x \in [0, \pi]$. Also find a point in the curve represented by given function where the tangent is parallel to x-axis. [6]

SOLUTION

First part and Second part:

Please refer to 2072 Set E Q.No. 11

Last Part

Here, $f(x) = \sin x$

For all $x \in [0, \pi]$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[0, \pi]$

Again, $f(x) = \cos x$ which exists for all $x \in (0, \pi)$.

$\therefore$ $f(x)$ is differentiable in $(0, \pi)$.

Also, $f(0) = \sin 0 = 0$

$f(\pi) = \sin \pi = 0$

$\therefore$ $f(0) = f(\pi)$

$\therefore$ all the conditions of Rolle's theorem are satisfied.

Hence, there exists at least a point $c \in (0, \pi)$ such that $f'(c) = 0$

or, $\cos c = 0 = \cos \frac{\pi}{2}$

$\therefore$ $c = \frac{\pi}{2} \in (0, \pi)$

Hence, Rolle's theorem is verified.

$c = \frac{\pi}{2}$ is the x-coordinate of the point at which tangent is parallel to x-axis.

Put $x = \frac{\pi}{2}$ in $y = f(x) = \sin x$, we have

$y = \sin\frac{\pi}{2} = 1$

$\therefore$ Required point is $\left(\frac{\pi}{2}, 1\right)$

13. 2073 Set C Q.No. 11 OR

State Rolle's theorem. Using Rolle's theorem find a point on the curve $f(x) = \cos 2x$ where the tangent is parallel to x-axis on $[-\pi, \pi]$.

SOLUTION

Rolle's Theorem

If a function f(x) is

i. continuous in [a, b],

ii. differentiable in (a, b) and

iii. f(a) = f(b)

then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$

Next Part

Here, $f(x) = \cos 2x, x \in [-\pi, \pi]$

i. For all $x \in [-\pi, \pi]$, $f(x)$ has a definite value, so $f(x)$ is continuous in $[-\pi, \pi]$.

ii. Again, $f'(x) = -2\sin 2x$, which exists for all $x \in (-\pi, \pi)$.

$\therefore$ $f(x)$ is derivable in $(-\pi, \pi)$

iii. Again, $f(-\pi) = \cos 2(-\pi) = \cos 2\pi = 1$ and $f(\pi) = \cos 2\pi = 1$

$\therefore$ $f(-\pi) = f(\pi)$

Thus, all conditions of Rolle's theorem are satisfied. So, there exists at least one value $c \in (-\pi, \pi)$ such that

$f'(c) = 0$

$\Rightarrow -2\sin 2c = 0$

$\Rightarrow \sin 2c = 0 = \sin 0, \sin\pi, \sin(-\pi)$

$\Rightarrow c = 0, \frac{\pi}{2}, -\frac{\pi}{2} \in (-\pi, \pi)$

$\therefore$ $c = 0$ or $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ is the x-coordinate of the point where the tangent is parallel to x-axis.

When $x = 0$, $f(0) = \cos 0 = 1$

When $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = \cos 2\left(\frac{\pi}{2}\right) = \cos\pi = -1$

When $x = -\frac{\pi}{2}$, $f\left(-\frac{\pi}{2}\right) = \cos 2\left(-\frac{\pi}{2}\right) = \cos\pi = -1$

$\therefore$ Required point is $(0, 1)$ or $\left(\frac{\pi}{2}, -1\right)$ or $\left(-\frac{\pi}{2}, -1\right)$

14. 2073 Set D Q.No. 11

State Mean Value Theorem. Interpret it geometrically. Verify Lagrange's mean value theorem for the function $f(x) = x(x-1)^2$ in $[0, 2]$. [6]

➥ Please refer to 2070 Set D Q.No. 11

15. 2073 Supp Q.No. 11

Using Mean Value theorem, find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1). [6]

SOLUTION

The value of x ranges from 3 to 4.

So, $y = f(x) = (x-3)^2, x \in [3, 4]$

Since f(x) is a polynomial function, it is continuous in [3, 4].

Again, $f'(x) = 2x - 6$ which exists for all $x \in (3, 4)$.

So, f(x) is differentiable in (3, 4).

Both the conditions of mean value theorem are satisfied. Hence by mean value theorem,

there exists at least one $c \in (3, 4)$ such that

$f'(c) = \frac{f(b) - f(a)}{b - a}$

or, $f'(c) = \frac{f(4) - f(3)}{4 - 3}$

or, $2c - 6 = \frac{(4-3)^2 - (3-3)^2}{1}$

or, $2c - 6 = \frac{1}{1} = 1$

or, $2c = 1 + 6$

$\therefore$ $c = \frac{7}{2} \in (3, 4)$

So, $c = \frac{7}{2}$ is the x - coordinate of the point at which the tangent is parallel to the chord joining the given points.

Put $x = \frac{7}{2}$ in $y = (x-3)^2$, we get $y = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$

$\therefore$ Required point $= \left(\frac{7}{2}, \frac{1}{4}\right)$.

16. 2074 Set A Q.No. 11 OR

State the Mean value theorem. Use it to find a point on the parabola $f(x) = (x-3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1). [6]

➥ **First part:** Please refer to 2072 Set D Q.No. 11

Second part: Please refer to 2073 supp Q.No. 11

17. 2074 Set B Q.No. 11 OR

Define Lagrange's Mean value theorem. Also verify the theorem for the function: $f(x) = 2x^2 - 10x + 29$ in $[2, 7]$. [6]

SOLUTION

First part: Please refer to 2072 Set D Q.No.11

Next part:

Here, $f(x) = 2x^2 - 10x + 29$

Since, f'(x) is a polynomial function, it is continuous in [2, 7].

Also, $f'(x) = 4x - 10$, which exists for all $x \in (2, 7)$

i.e. f(x) is differentiable in (2, 7)

Hence all the conditions of mean value theorem are satisfied.

So, there exists at least one $c \in (2, 7)$ such that

$f'(c) = \frac{f(b) - f(a)}{b - a}$

or, $f'(c) = \frac{f(7) - f(2)}{7 - 2}$

or, $4c - 10 = \frac{(2\times7^2 - 10\times7 + 29) - (2\times2^2 - 10\times2 + 29)}{5}$

or, $4c - 10 = \frac{57 - 17}{5} = \frac{40}{5} = 8$

or, $4c = 18$

$\therefore$ $c = \frac{9}{2}$

Clearly $c = \frac{9}{2} \in (2, 7)$.

Hence Lagrange's mean value theorem is verified.

18. 2074 Supp Q.No. 11

State mean value theorem. Interpret it geometrically. Verify mean value theorem for the function $f(x) = x(x-1)^2$ in $[0, 2]$.

➥ **First Part:** Please refer to 2069 Set A Q.N. 11

Second Part: Please refer to 2070 Set D Q.N. 11

19. 2075 Set A Q.No. 11

State Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function $f(x) = (x-1)(x-2)(x-3)$ in $[1, 3]$. [6]

SOLUTION

First two parts:

Please refer to 2070 Set C Q.No. 11

Last part:

Here,

$f(x) = (x-1)(x-2)(x-3)$

$= x^3 - 6x^2 + 11x - 6$

Here, f(x) is a polynomial function, so it is continuous in [1, 3].

Again,

$f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (1, 3)$, so it is differentiable in (1, 3).

Also,

$f(1) = (1-1)(1-2)(1-3) = 0$

$f(3) = (3-1)(3-2)(3-3) = 0$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Hence, there exists at least a point $c \in (1, 3)$ such that $f'(c) = 0$

or, $3c^2 - 12c + 11 = 0$

or, $c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4.3.11}}{2.3}$

$= \frac{12 \pm \sqrt{12}}{6}$

$= 2.577, 1.423$

Clearly $c = 2.5577, 1.423 \in (1, 3)$

Hence Rolle's theorem is verified.

20. 2075 Set B Q.No. 10 OR

State the mean value theorem and interpret it geometrically. Verify that the function $f(x) = \sqrt{x}$ on [1, 4] satisfies conditions of the mean value theorem and find C prescribed in the theorem. [6]

SOLUTION

First part: Please refer to 2069 Set A Q.No. 11

Second part:

Here, $f(x) = \sqrt{x}$ in [1, 4]

For every value of x such that $1 \le x \le 4$, f(x) has a definite value, so f(x) is continuous in [1, 4].

Again, $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ which exists for all x such that $1 < x < 4$.

∴ f(x) is differentiable in (1, 4).

Thus, both conditions of MVT are satisfied.

So, there exists at least one C in (1, 4) such that

$$f'(C) = \frac{f(b) - f(a)}{b - a}$$

or, $f'(C) = \frac{f(4) - f(1)}{4 - 1}$

or, $\frac{1}{2\sqrt{C}} = \frac{\sqrt{4} - \sqrt{1}}{3}$

or, $\frac{1}{2\sqrt{C}} = \frac{2 - 1}{3}$

or, $\frac{1}{\sqrt{C}} = \frac{2}{3}$

or, $\sqrt{C} = \frac{3}{2}$

$C = \frac{9}{4} \in (1, 4)$

∴ Required value of C is $\frac{9}{4}$.

21. 2075 Set C Q.No. 11

State the mean value theorem. Interpret it geometrically. Verify the mean value theorem for the function f(x) = (x − 1) (x − 2) (x − 3) in [1, 4]. **[6]**

➥ **First Part:** Please refer to 2069 Set A Q.N. 11

Second Part: Please refer to 2070 Set D Q.N. 11

UNIT 8

ANTIDERIVATIVES

2 MARKS QUESTIONS

1. 2057 Q.No. 3 a

Prove: $\int \operatorname{cosec} x \, dx = \log \left| \tan \frac{x}{2} \right| + c$ **[2]**

SOLUTION

$$\int \operatorname{cosec} x \, dx = \int \frac{1}{\sin x} dx$$

$$= \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \times \frac{\sec^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} dx$$

$$= \frac{1}{2} \int \frac{\sec^2 \frac{x}{2}}{\tan \frac{x}{2}} dx$$

Put $\tan \frac{x}{2} = y$

$\frac{1}{2} \sec^2 \frac{x}{2} \, dx = dy$

Then,

$$\int \operatorname{cosec} x \, dx = \int \frac{dy}{y}$$

$$= \log |y| + C$$

$$= \log \left| \tan \frac{x}{2} \right| + C$$

2. 2058 Q.No. 3 a

Prove that: $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + c.$ **[2]**

SOLUTION

Put $x = a \sin\theta$

$dx = a \cos\theta \, d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos\theta \, d\theta}{\sqrt{a^2 - a^2 \sin^2\theta}} = \int \frac{a \cos\theta \, d\theta}{a \cos\theta}$$

$$= \int d\theta = \theta + C = \sin^{-1} \frac{x}{a} + C$$

3. 2060 Q.No. 3 a

Evaluate: $\int \frac{dx}{\sqrt{a^2 - x^2}}$. **[2]**

➥ Please refer to 2058 Q.No. 3a

4. 2061 Q.No. 3 a

Evaluate: $\int \sqrt{\frac{1+x}{1-x}} \, dx$ **[2]**

SOLUTION

Let $I = \int \sqrt{\frac{1+x}{1-x}} \, dx$

$$= \int \sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}} \, dx = \int \frac{1+x}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx - \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1}\left(\frac{x}{1}\right) - \frac{1}{2} \cdot 2\sqrt{1-x^2} + C$$

$$= \sin^{-1} x - \sqrt{1-x^2} + C$$

5. 2062 Q.No. 3 a

Integrate: $\int \frac{dx}{\sqrt{2ax - x^2}}$ **[2]**

SOLUTION

Let $I = \int \frac{dx}{\sqrt{2ax - x^2}}$

$$= \int \frac{dx}{\sqrt{a^2 - a^2 + 2ax - x^2}}$$

$$= \int \frac{dx}{\sqrt{a^2 - (x^2 - 2ax + a^2)}}$$

$$= \int \frac{dx}{\sqrt{a^2 - (x-a)^2}} = \sin^{-1}\left(\frac{x-a}{a}\right) + C$$

6. 2063 Q.No. 3 a

Integrate: $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $(\beta > \alpha)$ **[2]**

➥ Please refer to Model Set II, Q.No. 3b

7. 2064 Q.No. 3 a

Evaluate: $\int \frac{dx}{x^2-16}$ [2]

SOLUTION

$= \int \frac{dx}{x^2-4^2} = \frac{1}{2\times 4}\log\left(\frac{x-4}{x+4}\right) + C$

$= \frac{1}{8}\log\left(\frac{x-4}{x+4}\right) + C$

8. 2065 Q.No 3 a

Evaluate: $\int \frac{dx}{e^x+e^{-x}}$ [2]

SOLUTION

Let $I = \int \frac{dx}{e^x+e^{-x}} = \int \frac{dx}{e^x+\frac{1}{e^x}} = \int \frac{e^x\,dx}{(e^x)^2+1}$

Put $e^x = y$

$e^x dx = dy$

$\therefore\ I = \int \frac{dy}{y^2+1} = \frac{1}{1}\tan^{-1}y + C$

$\left[\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right]$

$= \tan^{-1}(e^x) + C$

9. 2066 Q.No. 3 a

Integrate: $\int \frac{dx}{\sqrt{2ax+x^2}}$ [2]

SOLUTION

Let $I = \int \frac{dx}{2ax+x^2} = \int \frac{dx}{x^2+2ax+a^2-a^2}$

$= \int \frac{dx}{(x+a)^2-a^2}$

Put $y = x+a$

$dy = dx$

$\therefore\ I = \int \frac{dy}{y^2-a^2} = \log(y+\sqrt{y^2-a^2}) + C$

$= \log(x+a+\sqrt{(x+a)^2-a^2}) + C$

$= \log(x+a+\sqrt{2ax+x^2}) + C$

10. 2066 C Q.No. 3 a

Evaluate: $\int \frac{dx}{\sqrt{2ax-x^2}}$ [2]

Please refer to 2062 Q.No. 3a

11. 2067 Q.No. 3a

Evaluate $\int \frac{1}{x^2} e^{\frac{-1}{x}}\,dx$ [2]

SOLUTION

Let $I = \int \frac{1}{x^2} e^{\frac{-1}{x}}\,dx$

Put $y = \frac{-1}{x}$

$dy = \frac{1}{x^2}dx$

$\therefore\ I = \int e^y dy = e^y + C = e^{\frac{-1}{x}} + C$

12. 2068 Q.No. 3a

Evaluate: $\int \frac{dx}{\sqrt{2ax-x^2}}$ [2]

Please refer to 2062 Q.No. 3a

13. 2069 (Set A) Q.No. 3b

Evaluate: $\int \frac{dx}{\sqrt{2ax-x^2}}$. [2]

Please refer to 2062 Q.No. 3a

14. 2069 (Set A) Old Q.No. 3a

Show that: $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ [2]

Please refer to 2058 Q.No. 3a

15. 2069 (Set B) Q.No. 3b

Evaluate: $\int \frac{dx}{e^x+e^{-x}}$ [2]

Please refer to 2065 Q.No. 3a

16. 2070 Set C Q.No. 3 b

Evaluate: $\int \frac{2x+3}{4x^2+1}dx$. [2]

SOLUTION

Let $I = \int \frac{2x+3}{4x^2+1}dx$

$= \int \frac{2x}{4x^2+1}dx + \int \frac{3}{4x^2+1}dx$

$= \frac{1}{4}\int \frac{8x}{4x^2+1}dx + \frac{3}{4}\int \frac{1}{x^2+\frac{1}{4}}dx$

$= \frac{1}{4}\int \frac{8x}{4x^2+1}dx + \frac{3}{4}\int \frac{1}{x^2+\left(\frac{1}{2}\right)^2}dx$

$= \frac{1}{4}\log(4x^2+1) + \frac{3}{4}\times\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x}{\frac{1}{2}}\right) + C$

$= \frac{1}{4}\log(4x^2+1) + \frac{3}{2}\tan^{-1}(2x) + C$

17. 2070 Set D Q.No. 3 b

Evaluate: $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$, $(\beta > \alpha)$. [2]

Please refer to Model Set II, Q.No. 3b

18. 2070 (Old) Q.No. 3 a

Evaluate: $\int \frac{dx}{e^x+e^{-x}}$ [2]

Please refer to 2065 Q.No. 3a

19. 2070 Supp. Q.No. 3 c

Find the integral $\int(2x-5)\sqrt{x^2-5x+1}\,.\,dx$ [2]

SOLUTION

Put $x^2-5x+1 = y$

$(2x-5)\,dx = dy$

$\therefore\ \int(2x-5)\sqrt{x^2-5x+1}\,dx = \int\sqrt{y}\,dy$

$= \int y^{1/2}dy = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}(x^2-5x+1)^{\frac{3}{2}} + C$

20. 2071 Set C Q.No. 3 b

Evaluate: $\int \frac{6x+1}{x^2+9}dx$. [2]

SOLUTION

Let $I = \int \frac{6x+1}{x^2+9}dx = \int \frac{6x}{x^2+9}dx + \int \frac{1}{x^2+9}dx$

$= I_1 + I_2$ (say)

$I_1 = \int \frac{6x}{x^2+9}dx$

Put $y = x^2+9$

$dy = 2x\,dx$

$\therefore\ I_1 = \int \frac{3\,dy}{y} = 3\log y + C_1 = 3\log(x^2+9) + C_1$

$I_2 = \int \frac{dx}{x^2+3^2} = \frac{1}{3}\tan^{-1}\frac{x}{3} + C_2$

$\therefore\ I = I_1 + I_2$

$= 3\log(x^2+9) + C_1 + \frac{1}{3}\tan^{-1}\frac{x}{3} + C_2$

$= 3\log(x^2+9) + \frac{1}{3}\tan^{-1}\frac{x}{3} + C$,

where $C = C_1 + C_2$.

21. 2071 Set D Q.No. 3 b

Evaluate: $\int \frac{dx}{\sqrt{2ax+x^2}}$ [2]

Please refer to 2066 Q.No. 3a

22. 2071 Old Q.No. 3 a

Show that: $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$ [2]

Please refer to 2058 Q.No. 3a

23. 2071 Supp. Q.No. 3b

Find the integral $\int \frac{dx}{3-2x-x^2}$ [2]

SOLUTION

$= \int \frac{dx}{3-2x-x^2} = \int \frac{dx}{3-(x^2+2x)}$

$= \int \frac{dx}{3-(x^2+2x+1-1)} = \int \frac{dx}{3-(x+1)^2+1}$

$= \int \frac{dx}{2^2-(x+1)^2} = \frac{1}{2\cdot 2}\log\left(\frac{2+x+1}{2-x-1}\right) + C$

$= \frac{1}{4}\log\left(\frac{3+x}{1-x}\right) + C$

24. 2072 Set C Q.No. 3b

Evaluate: $\int \frac{dx}{1-2\cos x}$ [2]

SOLUTION

Let $I = \int \frac{dx}{1-2\cos x}$

$= \int \frac{dx}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right) - 2\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$

$= \int \frac{dx}{3\sin^2\frac{x}{2}-\cos^2\frac{x}{2}} = \int \frac{dx}{3\sin^2\frac{x}{2}-\cos^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$

$= \int \frac{\sec^2\frac{x}{2}}{3\tan^2\frac{x}{2}-1}dx$

Put $\sqrt{3}\tan\frac{x}{2} = y$

$\frac{\sqrt{3}}{2}\sec^2\frac{x}{2}dx = dy$

$\therefore\ \sec^2\frac{x}{2}dx = \frac{2}{\sqrt{3}}dy$

$\therefore\ I = \frac{2}{\sqrt{3}}\int \frac{dy}{y^2-1^2} = \frac{2}{\sqrt{3}}\cdot\frac{1}{2\cdot 1}\log\left(\frac{y-1}{y+1}\right) + C$

$= \frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}\tan\frac{x}{2}-1}{\sqrt{3}\tan\frac{x}{2}+1}\right) + C$

25. 2072 Set D Q.No. 3a

Compute the integral $\int \frac{\coth x\,dx}{\sinh x - 9\,\text{cosech}\,x}$ [2]

SOLUTION

Let $I = \int \frac{\coth x\,dx}{\sinh x - 9\,\text{cosech}\,x}dx$

$$= \int \frac{\frac{\cosh x}{\sinh x}}{\sinh x - \frac{9}{\sinh x}} = \int \frac{\frac{\cosh x}{\sinh x}}{\frac{\sinh^2 x - 9}{\sinh x}} dx$$

$$= \int \frac{\cosh x}{\sinh^2 x - 9} dx$$

Put $y = \sinh x$

Then, $dy = \cosh x\, dx$

$$\therefore \quad I = \int \frac{dy}{y^2 - 9} = \int \frac{dy}{y^2 - 3^2} = \frac{1}{2 \cdot 3} \log \frac{y-3}{y+3} + C$$

$$= \frac{1}{6} \log \left(\frac{\sinh x - 3}{\sinh x + 3}\right) + C$$

26. 2072 Set E Q.No. 3b

Evaluate: $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$ $(\beta > \alpha)$ [2]

➥ Please refer to Model Set II, Q.No. 3b

27. 2072 Supp Q.No. 3b

Evaluate: $\int \sqrt{2ax - x^2}\, dx$ [2]

SOLUTION

$$\int \sqrt{2ax - x^2}\, dx = \int \sqrt{a^2 - a^2 + 2ax - x^2}\, dx$$

$$= \int \sqrt{a^2 - (x^2 - 2ax + a^2)}\, dx$$

$$= \int \sqrt{a^2 - (x-a)^2}\, dx$$

$$= \frac{(x-a)\sqrt{a^2 - (x-a)^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + C$$

$$= \frac{1}{2}(x-a)\sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) + C$$

28. 2073 Set C Q.No. 3b

Evaluate: $\int \frac{dx}{x + \sqrt{x^2 - 1}}$. [2]

SOLUTION

$$\int \frac{dx}{x + \sqrt{x^2-1}} = \int \frac{dx}{x + \sqrt{x^2-1}} \times \frac{x - \sqrt{x^2-1}}{x - \sqrt{x^2-1}}$$

$$= \int \frac{x - \sqrt{x^2-1}}{x^2 - x^2 + 1} dx = \int (x - \sqrt{x^2-1})\, dx$$

$$= \int x\,dx - \int \sqrt{x^2-1}\, dx$$

$$= \frac{x^2}{2} - \left[\frac{x\sqrt{x^2-1}}{2} - \frac{1}{2}\log(x + \sqrt{x^2-1})\right] + C$$

$$\left[\because \int \sqrt{x^2 - a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log(x + \sqrt{x^2-a^2}) + C\right]$$

$$= \frac{x^2}{2} - \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\log(x + \sqrt{x^2-1}) + C$$

29. 2073 Set D Q.No. 3b

Evaluate: $\int \frac{dx}{x + \sqrt{x^2-1}}$. [2]

➥ Please refer to 2073 Set C Q.No. 3b

30. 2073 Supp Q.No. 3b

Evaluate: $\int \frac{\sin 2x}{(\sin x + \cos x)^2} dx$ [2]

SOLUTION

$$\text{Let } I = \int \frac{\sin 2x}{(\sin x + \cos x)^2}$$

$$= \int \frac{2 \sin x \cos x}{(\sin x + \cos)^2} dx$$

$$= \int \frac{2 \sin x \cos x + 1 - 1}{(\sin x + \cos x)^2} dx$$

$$= \int \frac{\sin^2 x + \cos^2 x + 2 \sin x \cos - 1}{(\sin x + \cos x)^2} dx$$

$$= \int \frac{(\sin x + \cos x)^2 - 1}{(\sin x + \cos x)^2} dx$$

$$= \int \left\{1 - \frac{1}{(\sin x + \cos x)^2}\right\} dx$$

$$= \int dx - \int \frac{1}{(\sin x + \cos x)^2} dx$$

$= x - I_1$ (say)

$$I_1 = \int \frac{1}{(\sin x + \cos x)^2} \times \frac{\sec^2 x}{\sec^2 x} dx = \int \frac{\sec^2 x\, dx}{(1 + \tan x)^2}$$

Put $1 + \tan x = y$

$\sec^2 x\, dx = dy$

$$\therefore \quad I_1 = \int \frac{dy}{y^2} = \int y^{-2} dy = \frac{y^{-2+1}}{(-2+1)} + C$$

$$= \frac{-1}{y} + C = -\frac{1}{(1 + \tan x)} + C$$

$$\therefore \quad I = x + \frac{1}{1 + \tan x} + C = x + \frac{1}{1 + \tan x} + C$$

31. 2074 Set A Q.No. 3b

Find the integral $\int \frac{dx}{\sqrt{x^2 - 6x + 13}}$. [2]

SOLUTION

$$\int \frac{dx}{\sqrt{x^2 - 6x + 13}}$$

$$= \int \frac{dx}{\sqrt{x^2 - 2.x.3 + 3^2 - 3^2 + 13}}$$

$$= \int \frac{dx}{\sqrt{(x-3)^2 + 4}}$$

$$= \int \frac{dx}{\sqrt{(x-3)^2 + 2^2}}$$

$$= \log\{(x-3) + \sqrt{(x-3)^2 + 2^2}\} + c$$

$$= \log(x - 3 + \sqrt{x^2 - 6x + 13}) + c$$

32. 2074 Set B Q.No. 3b

Evaluate: $\int \frac{3x}{(x-a)(x-b)} dx$. [2]

SOLUTION

Let, $\frac{3x}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$ ...(i)

$$= \frac{A(x-b) + B(x-a)}{(x-a)(x-b)}$$

or, $3x = A(x-b) + B(x-a)$ ...(ii)

Put $x = a$ in (ii), then

$3a = A(a-b)$

$A = \frac{3a}{a-b}$

Again, put $x = b$ in (ii), then

$3b = B(b-a)$

$B = \frac{3b}{b-a} = \frac{-3b}{b-a}$

Then, from (i)

$$\frac{3x}{(x-a)(x-b)} = \frac{3a}{(a-b)(x-a)} - \frac{3b}{(a-b)(x-b)}$$

Now, $\int \frac{3x}{(x-a)(x-b)} dx$

$$= \int \left\{\frac{3a}{(a-b)(x-a)} - \frac{3b}{(a-b)(x-b)}\right\} dx$$

$$= \frac{3a}{a-b} \int \frac{1}{x-a} dx - \frac{3b}{a-b} \int \frac{1}{x-b} dx$$

$$= \frac{3a}{a-b} \log(x-a) - \frac{3b}{a-b} \log(x-b) + C$$

$$= \frac{3}{a-b} \{a \log(x-a) - b \log(x-b)\} + C$$

33. 2074 Supp Q.No. 3b

Evaluate: $\int \frac{dx}{e^x + e^{-x}}$. [2]

➥ Please refer to 2065 Q.N. 3a

34. 2075 Set A Q.No. 3b

Evaluate: $\int \frac{dx}{\sqrt{2ax - x^2}}$. [2]

➥ Please refer to 2062 Q.No. 3a

35. 2075 Set B Q.No. 3b

Find the integral $\int \frac{dx}{1 + 3\cos^2 x}$. [2]

SOLUTION

Let

$$I = \int \frac{dx}{1 + 3\cos^2 x}$$

$$= \int \frac{dx}{1 + 3\cos^2 x} \times \frac{\sec^2 x}{\sec^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{\sec^2 x + 3}$$

$$= \int \frac{\sec^2 x\, dx}{1 + \tan^2 x + 3}$$

$$= \int \frac{\sec^2 x\, dx}{\tan^2 x + 4}$$

Put $z = \tan x$

$dz = \sec^2 x\, dx$.

Then,

$$I = \int \frac{dz}{z^2 + 4}$$

$$= \int \frac{dz}{z^2 + 2^2}$$

$$= \frac{1}{2} \tan^{-1} \frac{z}{2} + C$$

$$= \frac{1}{2} \tan^{-1}\left(\frac{\tan x}{2}\right) + C$$

36. 2075 Set C Q.No. 3b

Evaluate: $\int \frac{dx}{\sqrt{1 + e^{-2x}}}$. [2]

SOLUTION

Let

$$I = \int \frac{dx}{\sqrt{1 + e^{-2x}}} = \int \frac{dx}{\sqrt{1 + \frac{1}{e^{2x}}}} = \int \frac{dx}{\sqrt{\frac{e^{2x} + 1}{e^{2x}}}}$$

$$= \int \frac{dx}{\frac{\sqrt{e^{2x} + 1}}{e^x}} = \int \frac{e^x dx}{\sqrt{(e^x)^2 + 1}}$$

Let $y = e^x$

$dy = e^x\, dx$.

Then,

$$I = \int \frac{dy}{\sqrt{y^2 + 1}}$$

$$= \log(y + \sqrt{y^2 + 1}) + C$$

$$= \log(e^x + \sqrt{(e^x)^2 + 1}) + C$$

$$= \log(e^x + \sqrt{e^{2x} + 1}) + C$$

37. 2075 Set C Q.No. 3c

Evaluate: $\int \frac{2x - 11}{x^2 + x - 2} dx$. [2]

SOLUTION

$$\int \frac{2x - 11}{x^2 + x - 2} dx$$

$$= \int \frac{2x + 1 - 1 - 11}{x^2 + x - 2} dx$$

$$= \int \left(\frac{2x+1}{x^2+x-2} - \frac{12}{x^2+x-2}\right)dx$$

$$= \int \frac{2x+1}{x^2+x-2}dx - \int \frac{12}{x^2+2x.\frac{1}{2}+\frac{1}{4}-\frac{1}{4}-2}dx$$

$$= \int \frac{2x+1}{x^2+x-2}dx - 12\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\frac{9}{4}}$$

$$= \int \frac{2x+1}{x^2+x-2}dx - 12\int \frac{dx}{\left(x+\frac{1}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

$$= \log(x^2+x-2) - 12\frac{1}{2.\frac{3}{2}}\log\left(\frac{x+\frac{1}{2}-\frac{3}{2}}{x+\frac{1}{2}+\frac{3}{2}}\right)+C$$

$$= \log(x^2+x-2) - 4\log\left(\frac{x-1}{x+2}\right)+C$$

Alternatively

Let,

$$\frac{2x-11}{x^2-x-2} = \frac{2x-11}{(x+2)(x-1)} = \frac{A}{x+2}+\frac{B}{x-1} \quad \text{... (i)}$$

or, $\frac{2x-11}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$

or, $2x-11 = A(x-1)+B(x+2)$ ... (2)

Put x = 1 in (2), we get

$2-11 = B(1+2)$

$B = -3$

Again, put x = −2 in (2), we get,

$-4-11 = A(-2-1)$

$A = 5$

Then from (1)

$$\frac{2x-11}{x^2+x-2} = \frac{5}{x+2} - \frac{3}{x-1}$$

Now,

$$\int \frac{2x-11}{x^2+x-2}dx = \int\left(\frac{5}{x+2}-\frac{3}{x-1}\right)dx$$

$$= 5\int\frac{1}{x+2}dx - 3\int\frac{1}{x-1}dx.$$

$$= 5\log(x+2) - 3\log(x-1) + C$$

4 MARKS QUESTIONS

38. 2057 Q.No. 11 b

Integrate: $\int \frac{dx}{a+b\cos x}$ when a > b **[4]**

Please refer to Model Set II, Q.No. 7a

39. 2058 Q.No. 11 b

Integrate: $\int\sqrt{\frac{1+x}{1-x}}\,dx$ **[4]**

SOLUTION

Let I = $\int\sqrt{\frac{1+x}{1-x}}\,dx$

$$= \int\sqrt{\frac{(1+x)(1+x)}{(1-x)(1+x)}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \int\frac{1}{\sqrt{1-x^2}}dx + \int\frac{x}{\sqrt{1-x^2}}dx$$

$$= \int\frac{1}{\sqrt{1-x^2}}dx - \frac{1}{2}\int\frac{-2x}{\sqrt{1-x^2}}dx$$

$$= \sin^{-1}\left(\frac{x}{1}\right) - \frac{1}{2}\cdot 2\sqrt{1-x^2}+C$$

$$= \sin^{-1}x - \sqrt{1-x^2}+C$$

40. 2059 Q.No. 11 b

$\int\frac{dx}{a+b\cos x}$, a < b **[4]**

SOLUTION

Here, $\int\frac{dx}{a+b\cos x}$

$$= \int\frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int\frac{1}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}\times\frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}dx$$

$$= \int\frac{\sec^2\frac{x}{2}\,dx}{(a+b)-(b-a)\tan^2\frac{x}{2}} \quad \text{...(i)}$$

Put $\sqrt{b-a}\tan\frac{x}{2} = y$

$\frac{1}{2}\sqrt{b-a}\sec^2\frac{x}{2}dx = dy$

Then the integral (i) becomes

$$\frac{2}{\sqrt{b-a}}\int\frac{dy}{(a+b)-y^2}$$

$$= \frac{2}{\sqrt{b-a}}\times\frac{1}{2\sqrt{b+a}}\log\frac{\sqrt{b+a}+y}{\sqrt{b+a}-y}+C$$

$$= \frac{1}{\sqrt{b^2-a^2}}\log\left(\frac{\sqrt{b+a}+\sqrt{b-a}\tan\frac{x}{2}}{\sqrt{b+a}-\sqrt{b-a}\tan\frac{x}{2}}\right)+C$$

41. 2060 Q.No. 11 b

Find the value of: $\int\frac{\sin x\,.\cos x}{(\sin x+\cos x)^2}dx$ **[4]**

SOLUTION

Let I = $\int\frac{\sin x\,.\cos x}{(\sin x+\cos x)^2}$

$$= \frac{1}{2}\int\frac{2\sin x\cos x}{(\sin x+\cos)^2}dx$$

$$= \frac{1}{2}\int\frac{2\sin x\cos x+1-1}{(\sin x+\cos x)^2}dx$$

$$= \frac{1}{2}\int\frac{\sin^2x+\cos^2x+2\sin x\cos-1}{(\sin x+\cos x)^2}dx$$

$$= \frac{1}{2}\int\frac{(\sin x+\cos x)^2-1}{(\sin x+\cos x)^2}dx$$

$$= \frac{1}{2}\int\left\{1-\frac{1}{(\sin x+\cos x)^2}\right\}dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int\frac{1}{(\sin x+\cos x)^2}dx$$

$$= \frac{1}{2}x - I_1 \text{ (say)}$$

$$I_1 = \frac{1}{2}\int\frac{1}{(\sin x+\cos x)^2}\times\frac{\sec^2 x}{\sec^2 x}dx$$

$$= \frac{1}{2}\int\frac{\sec^2x\,dx}{(1+\tan x)^2}$$

Put $1+\tan x = y$

$\sec^2x\,dx = dy$

∴ $I_1 = \frac{1}{2}\int\frac{dy}{y^2}$

$$= \frac{1}{2}\int y^{-2}dy = \frac{1}{2}\cdot\frac{y^{-2+1}}{(-2+1)}+C$$

$$= \frac{-1}{2y} = -\frac{1}{2(1+\tan x)}+C$$

∴ $I = \frac{1}{2}x + \frac{1}{2(1+\tan x)}+C$

$$= \frac{1}{2}\left(x+\frac{1}{1+\tan x}\right)+C$$

42. 2061 Q.No. 11 b

Find the value of: $\int\frac{dx}{3\sin x-4\cos x}$ **[4]**

SOLUTION

Let I = $\int\frac{dx}{3\sin x-4\cos x}$

$$= \int\frac{dx}{3\times2\sin\frac{x}{2}\cos\frac{x}{2}-4\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int\frac{dx}{4\sin^2\frac{x}{2}+6\sin\frac{x}{2}\cos\frac{x}{2}-4\cos^2\frac{x}{2}}\times\frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int\frac{\sec^2\frac{x}{2}\,dx}{4\tan^2\frac{x}{2}+6\tan\frac{x}{2}-4}$$

$$= \frac{1}{4}\int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+\frac{3}{2}\tan\frac{x}{2}-1}$$

Put $\tan\frac{x}{2} = y$

$\frac{1}{2}\sec^2\frac{x}{2}dx = dy$

$\sec^2\frac{x}{2}dx = 2\,dy$

$$\therefore I = \frac{1}{4}\int\frac{2dy}{y^2+\frac{3}{2}y-1} = \frac{1}{2}\int\frac{dy}{y^2+\frac{3}{2}y-1}$$

$$= \frac{1}{2}\int\frac{dy}{\left(y+\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}$$

$$= \frac{1}{2}\times\frac{1}{2\times\frac{5}{4}}\log\left(\frac{y+\frac{3}{4}-\frac{5}{4}}{y+\frac{3}{4}+\frac{5}{4}}\right)+C$$

$$= \frac{1}{5}\log\left(\frac{2y-1}{2y+4}\right)+C = \frac{1}{5}\log\left(\frac{2\tan\frac{x}{2}-1}{2\tan\frac{x}{2}+4}\right)+C$$

43. 2062 Q.No. 11 b

Integrate: $\int\frac{dx}{\sin x+\cos x}$ **[4]**

SOLUTION

Here, $\int\frac{dx}{\sin x+\cos x}$

$$= \int\frac{dx}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)}$$

$$= \frac{1}{\sqrt{2}}\int\frac{dx}{\cos x\cos\frac{\pi}{4}+\sin x\sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}\int\frac{dx}{\cos\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int\sec\left(x-\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left\{\tan\left(\frac{\pi}{4}+\frac{x-\frac{\pi}{4}}{2}\right)\right\}+C$$

$$\left[\because \int \sec x\, dx = \log \tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + C\right]$$

$$= \frac{1}{\sqrt{2}} \log\left[\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right] + C$$

44. 2063 Q.No. 11 b

Integrate: $\int \frac{dx}{1+\sin x+\cos x}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{1+\sin x+\cos x}$

$$= \int \frac{dx}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right) + 2\sin\frac{x}{2}\cos\frac{x}{2} + \left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}\, dx}{1+\tan\frac{x}{2}} \quad \ldots (i)$$

Put $y = 1+\tan\frac{x}{2}$

$dy = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$2dy = \sec^2\frac{x}{2}\, dx$

Then integral (i) becomes

$$= \frac{1}{2}\int \frac{2\,dy}{y} = \int \frac{dy}{y} = \log y + C = \log\left(1+\tan\frac{x}{2}\right) + C$$

45. 2064 Q.No. 11 b

Integrate: $\int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}\, dx$ **[4]**

SOLUTION

Let $I = \int \frac{\cos x - \sin x}{\sqrt{\sin 2x}}\, dx$

$$= \int \frac{\cos x - \sin x}{\sqrt{2\sin x\cos x}} = \int \frac{\cos x - \sin x}{\sqrt{1+2\sin x\cos x - 1}}\, dx$$

$$= \int \frac{(\cos x - \sin x)\, dx}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x\cos x - 1}}$$

$$= \int \frac{(\cos x - \sin x)\, dx}{(\sin x + \cos x)^2 - 1}$$

Put $\sin x + \cos y = y$

$(\cos x - \sin x)\, dx = dy$

$$\therefore I = \int \frac{dy}{\sqrt{y^2-1}} = \log\left(y + \sqrt{y^2-1}\right) + C$$

$$= \log\left(\sin x + \cos x + \sqrt{(\sin x + \cos x)^2 - 1}\right) + C$$

$$= \log\left(\sin x + \cos x + \sqrt{\sin 2x}\right) + C$$

46. 2065 Q.No 11 b

Integrate: $\int \frac{dx}{3+4\cosh x}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{3+4\cosh x}$

$$= \int \frac{dx}{3\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + 4\left(\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{7\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}} \times \frac{\operatorname{sech}^2\frac{x}{2}}{\operatorname{sech}^2\frac{x}{2}}$$

$$= \int \frac{\operatorname{sech}^2\frac{x}{2}}{\tanh^2\frac{x}{2} + 7}\, dx$$

Put $\tanh\frac{x}{2} = y$

Then, $\frac{1}{2}\operatorname{sech}^2\frac{x}{2}\, dx = dy$

or, $2\, dy = \operatorname{sech}^2\frac{x}{2}\, dx$

$$\therefore I = \int \frac{2\, dy}{y^2+7} = 2\int \frac{dy}{y^2 + (\sqrt{7})^2}$$

$$= 2\cdot\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{y}{\sqrt{7}}\right) + C = \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{\tanh\frac{x}{2}}{\sqrt{7}}\right) + C$$

47. 2066 Q.No.11 b

Integrate: $\int \frac{x^2}{(x+2)(x+3)^2}\, dx.$ **[4]**

SOLUTION

Let $\frac{x^2}{(x+2)(x+3)^2} = \frac{A}{x+2} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$

or, $\frac{x^2}{(x+2)(x+3)^2} = \frac{A(x+3)^2 + B(x+2)(x+3) + C(x+2)}{(x+2)(x+3)^2}$

or, $x^2 = A(x+3)^2 + B(x+2)(x+3) + C(x+2)$

Put $x = -2$ in (i), we get $4 = A$

Again, put $x = -3$ in (i), we get

$9 = C(-3+2)$

$C = -9$

And, put $x = 0$ in (i), we get

$0 = 9A + 6B + 2C$

or, $0 = 9 \times 4 + 6B + 2 \times (-9)$

$\therefore B = -3$

$$\therefore \frac{x^2}{(x+2)(x+3)^2} = \frac{4}{x+2} - \frac{3}{x+3} - \frac{9}{(x+3)^2}$$

Now,

$$\int \frac{x^2}{(x+2)(x+3)^2}$$

$$= 4\int \frac{1}{x+2}\, dx - 3\int \frac{1}{x+3}\, dx - 9\int \frac{1}{(x+3)^2}\, dx$$

$$= 4\log(x+2) - 3\log(x+3) + \frac{9}{x+3} + C$$

48. 2066 C Q.No. 11 b

Evaluate: $\int \frac{dx}{1+\text{Sin}\,x+\text{Cos}\,x}$ **[4]**

➥ Please refer to 2063 Q.No. 11b

49. 2067 Q.No. 10b

Evaluate: $\int \frac{dx}{1+2\,\text{Sin}\,x}$ **[4]**

➥ Please refer to Model Set I, Q.No. 3b

50. 2068 Q.No. 11 b

Evaluate: $\int \frac{dx}{1+\sin x+\cos x}$ **[4]**

➥ Please refer to 2063 Q.No. 11b

51. 2069 (Set A) Q.No. 7a

Evaluate: $\int \frac{dx}{3\sin x - 4\cos x}$ **[4]**

➥ Please refer to 2061 Q.No. 11b

52. 2069 (Set A) Old Q.No. 11b

Evaluate: $\int \frac{dx}{\sin x+\cos x}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{\sin x+\cos x}$

$$= \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}\, dx}{\left(1-\tan\frac{x}{2}\right)^2}$$

Put $y = 1 - \tan\frac{x}{2}$

$dy = -\frac{1}{2}\sec^2\frac{x}{2}\, dx$

$-2dy = \sec^2\frac{x}{2}\, dx$

$$\therefore I = \int \frac{-2dy}{y^2}$$

$$= -2\int y^{-2}\, dy = -2\frac{y^{-1}}{(-1)} + C = \frac{2}{y} + C$$

$$= \frac{2}{1-\tan\frac{x}{2}} + C$$

Alternatively

Let $I = \int \frac{dx}{\sin x + \cos x}$

Put $1 = r\cos\theta$ and $1 = r\sin\theta$
so that $r^2 = 2$

$\therefore r = \sqrt{2}$

Also, $\tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

$$\therefore I = \int \frac{dx}{r\cos\theta\sin x + r\sin\theta\cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{r}\int \operatorname{cosec}(x+\theta)\, dx$$

$$= \frac{1}{r}\log\tan\frac{1}{2}(x+\theta) + C = \frac{1}{\sqrt{2}}\log\tan\frac{1}{2}\left(x+\frac{\pi}{4}\right) + C$$

$$= \frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right) + C$$

53. 2069 (Set B) Q.No. 7a

Evaluate: $\int \frac{dx}{2+3\cos x}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{2+3\cos x}$

$$= \int \frac{dx}{2\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right) + 3\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{5\cos^2\frac{x}{2} - \sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}\, dx}{5-\tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2} = y$

$\frac{1}{2}\sec^2\frac{x}{2}\, dx = dy$

$\sec^2\frac{x}{2}\, dx = 2dy$

$$\therefore I = \int \frac{2dy}{5-y^2} = 2\int \frac{dy}{(\sqrt{5})^2 - y^2}$$

$$= 2 \cdot \frac{1}{2\sqrt{5}} \log\left(\frac{\sqrt{5}+y}{\sqrt{5}-y}\right) + C$$

$$= \frac{1}{\sqrt{5}} \log\left(\frac{\sqrt{5}+\tan\frac{x}{2}}{\sqrt{5}-\tan\frac{x}{2}}\right) + C$$

54. 2069 Old (Set B) Q.No. 11b

Evaluate: $\int (2-x)\sqrt{16-6x-x^2}\,dx$ **[4]**

SOLUTION

Let $I = \int (2-x)\sqrt{16-6x-x^2}\,dx$

$$= \frac{1}{2}\int (4-2x)\sqrt{16-6x-x^2}\,dx$$

$$= \frac{1}{2}\int \{10+(-6-2x)\}\sqrt{16-6x-x^2}\,dx$$

$$= 5\int\sqrt{16-6x-x^2}\,dx + \frac{1}{2}\int(-6-2x)\sqrt{16-6x-x^2}\,dx$$

$$= I_1 + I_2 \text{ (say)}$$

Now,

$$I_1 = 5\int\sqrt{16-6x-x^2}\,dx$$

$$= 5\int\sqrt{5^2-(x+3)^2}\,dx$$

$$= 5\left\{\frac{1}{2}(x+3)\sqrt{5^2-(x+3)^2} + \frac{5^2}{2}\sin^{-1}\left(\frac{x+3}{5}\right)\right\} + C_1$$

$$= \frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + C_1$$

Again,

$$I_2 = \frac{1}{2}\int(-6-2x)\sqrt{16-6x-x^2}\,dx$$

Put $y = 16-6x-x^2$

or, $dy = (-6-2x)\,dx$

Then,

$$I_2 = \frac{1}{2}\int\sqrt{y}\,dy = \frac{1}{2}\cdot\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + C_2$$

$$= \frac{1}{3}(16-6x-x^2)^{\frac{3}{2}} + C_2$$

$\therefore\ I = I_1 + I_2$

$$= \frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + C_1 + \frac{1}{3}(16-6x-x^2)^{\frac{3}{2}} + C_2$$

$$= \frac{5}{2}(x+3)\sqrt{16-6x-x^2} + \frac{125}{2}\sin^{-1}\left(\frac{x+3}{5}\right) + \frac{1}{3}(16-6x-x^2)^{\frac{3}{2}} + C,$$

where $C = C_1 + C_2$

55. 2070 Set C Q.No. 7 a

Evaluate: $\int\frac{dx}{2+\cos x}$ [4]

SOLUTION

Let $I = \int\frac{dx}{2+\cos x}$

$$= \int\frac{dx}{2\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right) + \left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int\frac{dx}{3\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}$$

$$= \int\frac{dx}{3\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+3}$$

Put $\tan\frac{x}{2} = y$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy$$

$$\sec^2\frac{x}{2}\,dx = 2dy$$

Then,

$$I = \int\frac{2dy}{y^2+3} = 2\int\frac{dy}{y^2+(\sqrt{3})^2}$$

$$= 2\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{y}{\sqrt{3}} + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right)$$

56. 2070 Set D Q.No. 7 a

Evaluate: $\int\frac{dx}{1+2\sin x}$ [4]

➡ Please refer to Model Set I, Q.No. 3b

57. 2070 (Old) Q.No. 11 b

Evaluate: $\int\frac{\cos x - \sin x}{\sqrt{\sin 2x}}\,dx$ [4]

➡ Please refer to 2064 Q.No. 11b

58. 2070 Supp. Q.No. 7 a

Find the integral $\int\frac{dx}{3\sin x - 5\cos x}$ [4]

SOLUTION

Let $I = \int\frac{dx}{3\sin x - 5\cos x}$

$$= \int\frac{dx}{3\cdot2\sin\frac{x}{2}\cos\frac{x}{2} - 5\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int\frac{dx}{6\sin\frac{x}{2}\cos\frac{x}{2} - 5\cos^2\frac{x}{2} + 5\sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int\frac{\sec^2\frac{x}{2}\,dx}{5\tan^2\frac{x}{2}+6\tan\frac{x}{2}-5} = \frac{1}{5}\int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}+\frac{6}{5}\tan\frac{x}{2}-1}$$

Put $\tan\frac{x}{2} = y$

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dy$$

$$\sec^2\frac{x}{2}\,dx = 2\,dy$$

$$\therefore\ I = \frac{1}{5}\int\frac{2dy}{y^2+\frac{6}{5}y-1}$$

$$= \frac{2}{5}\int\frac{dy}{y^2+2\cdot y\cdot\frac{3}{5}+\left(\frac{3}{5}\right)^2-\left(\frac{3}{5}\right)^2-1}$$

$$= \frac{2}{5}\int\frac{dy}{\left(y+\frac{3}{5}\right)^2-\left(\frac{\sqrt{34}}{5}\right)^2}$$

$$= \frac{2}{5}\times\frac{1}{2\times\frac{\sqrt{34}}{5}}\log\left(\frac{y+\frac{3}{5}-\frac{\sqrt{34}}{5}}{y+\frac{3}{5}+\frac{\sqrt{34}}{5}}\right) + C$$

$$= \frac{1}{\sqrt{34}}\log\left(\frac{5y+3-\sqrt{34}}{5y+3\sqrt{34}}\right) + C$$

$$= \frac{1}{\sqrt{34}}\log\left(\frac{5\tan\frac{x}{2}+3-\sqrt{34}}{5\tan\frac{x}{2}+3\sqrt{34}}\right) + C$$

Alternative Method

Put $3 = r\cos\theta$ and $5 = r\sin\theta$

so that $r^2 = 34$

$\therefore\ r = \sqrt{34}$

Also, $\tan\theta = \frac{5}{3}$ $\quad\therefore \theta = \tan^{-1}\frac{5}{3}$

$$\therefore\ I = \int\frac{dx}{r\cos\theta\sin x + r\sin\theta\cos x}$$

$$= \frac{1}{r}\int\frac{dx}{\sin(x+\theta)} = \frac{1}{r}\int\operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{r}\log\tan\frac{1}{2}(x+\theta) + C$$

$$= \frac{1}{\sqrt{34}}\log\tan\frac{1}{2}\left(x+\tan^{-1}\frac{5}{3}\right) + C$$

$$= \frac{1}{\sqrt{34}}\log\tan\left(\frac{x}{2}+\frac{1}{2}\tan^{-1}\frac{5}{3}\right) + C$$

59. 2071 Set C Q.No. 7 a

Evaluate: $\int\frac{dx}{1-2\cos x}$ **[4]**

SOLUTION

Let $I = \int\frac{dx}{1-2\cos x}$

$$= \int\frac{dx}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)-2\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \int\frac{dx}{3\sin^2\frac{x}{2}-\cos^2\frac{x}{2}} = \int\frac{dx}{3\sin^2\frac{x}{2}-\cos^2\frac{x}{2}}\times\frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int\frac{\sec^2\frac{x}{2}}{3\tan^2\frac{x}{2}-1}\,dx$$

Put $\sqrt{3}\tan\frac{x}{2} = y$

$$\frac{\sqrt{3}}{2}\sec^2\frac{x}{2}\,dx = dy$$

$$\therefore\ \sec^2\frac{x}{2}\,dx = \frac{2}{\sqrt{3}}\,dy$$

$$\therefore\ I = \frac{2}{\sqrt{3}}\int\frac{dy}{y^2-1^2} = \frac{2}{\sqrt{3}}\cdot\frac{1}{2\cdot1}\log\left(\frac{y-1}{y+1}\right) + C$$

$$= \frac{1}{\sqrt{3}}\log\left(\frac{\sqrt{3}\tan\frac{x}{2}-1}{\sqrt{3}\tan\frac{x}{2}+1}\right) + C$$

60. 2071 Set D Q.No. 7 a

Evaluate: $\int\frac{dx}{1-3\sin x}$ **[4]**

SOLUTION

Let $I = \int\frac{dx}{1-3\sin x}$

$$= \frac{dx}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-6\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int\frac{dx}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-6\sin\frac{x}{2}\cos\frac{x}{2}}\times\frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int\frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2}-6\tan\frac{x}{2}+1}$$

Put $y = \tan\frac{x}{2}$

$$dy = \frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$\therefore \sec^2\frac{x}{2}\,dx = 2dy$

Then,

$$I = \int \frac{2dy}{y^2 - 6y + 1} = 2\int \frac{dy}{(y-3)^2 - (2\sqrt{2})^2}$$

$$= 2 \times \frac{1}{2\times 2\sqrt{2}} \log\left(\frac{y-3-2\sqrt{2}}{y-3+2\sqrt{2}}\right) + C$$

$$= \frac{1}{2\sqrt{2}} \log\left(\frac{\tan\frac{x}{2} - 3 - 2\sqrt{2}}{\tan\frac{x}{2} - 3 + 2\sqrt{2}}\right) + C$$

61. 2071 Old Q.No. 10 a

Integrate $\int \sqrt{\frac{1-x}{1+x}}\,dx$ **[4]**

SOLUTION

Let $I = \int \sqrt{\frac{1-x}{1+x}}\,dx$

$$= \int \sqrt{\frac{(1-x)(1-x)}{(1+x)(1-x)}}\,dx$$

$$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{-x\,dx}{\sqrt{1-x^2}}$$

$= I_1 + I_2$ (say)

$$I_1 = \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}\frac{x}{1} + C_1 = \sin^{-1}x + C_1$$

$$\& \quad I_2 = \int \frac{-x\,dx}{\sqrt{1-x^2}}$$

Put $y = 1 - x^2$

$dy = -2x\,dx$

$\frac{dy}{2} = -x\,dx$

$$\therefore I_2 = \int \frac{\frac{dy}{2}}{\sqrt{y}} = \frac{1}{2}\int y^{-\frac{1}{2}}\,dy = \frac{1}{2} \times \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + C_2$$

$$= \sqrt{y} + C_2 = \sqrt{1-x^2} + C_2$$

$$\therefore I = I_1 + I_2 = \sin^{-1}x + C_1 + \sqrt{1-x^2} + C_2$$

$= \sin^{-1}x + \sqrt{1-x^2} + C$, where $C = C_1 + C_2$

62. 2071 Supp. Q.No. 7a

Find the integral $\int \frac{dx}{5 + 4\sin x}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{5 + 4\sin x}$

$$= \int \frac{dx}{5\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 4\cdot 2\cdot \sin\frac{x}{2}\cos\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{5 + 5\tan^2\frac{x}{2} + 8\tan\frac{x}{2}}$$

$$= \frac{1}{5}\int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2} + \frac{8}{5}\tan\frac{x}{2} + 1}$$

$$= \frac{1}{5}\int \frac{\sec^2\frac{x}{2}\,dx}{\tan^2\frac{x}{2} + 2\cdot\tan\frac{x}{2}\cdot\frac{4}{5} + \left(\frac{4}{5}\right)^2 - \left(\frac{4}{5}\right)^2 + 1}$$

$$= \frac{1}{5}\int \frac{\sec^2\frac{x}{2}\,dx}{\left(\tan\frac{x}{2} + \frac{4}{5}\right)^2 + \frac{9}{25}}$$

Put $y = \tan\frac{x}{2} + \frac{4}{5}$

$dy = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$2dy = \sec^2\frac{x}{2}\,dx$

$$\therefore I = \frac{1}{5}\int \frac{2dy}{y^2 + \left(\frac{3}{5}\right)^2} = \frac{2}{5}\cdot\frac{1}{3/5}\tan^{-1}\left(\frac{y}{3/5}\right) + C$$

$$= \frac{2}{3}\tan^{-1}\left\{\frac{5\left(\tan\frac{x}{2} + \frac{4}{5}\right)}{3}\right\} + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{5\tan\frac{x}{2} + 4}{3}\right) + C$$

63. 2072 Set C Q.No. 7a

Evaluate: $\int \frac{dx}{(x-2)^2(x-3)^3}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{(x-2)^2(x-3)^3}$

Put $x - 2 = z(x - 3)$

$\therefore x = \frac{3z-2}{z-1}$

$$dx = \frac{3(z-1) - (3z-2)}{(z-1)^2}\,dz = \frac{-1}{(z-1)^2}\,dz$$

Also, $\frac{1}{(x-2)^2(x-3)^3} = \frac{1}{z^2(x-3)^5}$

$$= \frac{1}{z^2\left(\frac{3z-2}{z-1} - 3\right)^5} = \frac{(z-1)^5}{z^2}$$

$$\therefore I = \int \frac{(z-1)^5}{z^2} \times \left\{-\frac{1}{(z-1)^2}\right\} dz$$

$$= -\int \frac{(z-1)^3}{z^2}\,dz = \int \frac{z^3 - 3z^2 + 3z - 1}{z^2}\,dz$$

$$= -\int \left(z - 3 + \frac{3}{z} - \frac{1}{z^2}\right) dz = \int \left(z - 3 + \frac{3}{z} - z^{-2}\right) dz$$

$$= -\left(\frac{z^2}{2} - 3z + 3\log z + \frac{1}{z}\right) + C$$

$$= \frac{-z^2}{2} + 3z - 3\log z - \frac{1}{z} + C$$

$$= \frac{-1}{2}\left(\frac{x-2}{x-3}\right)^2 + \frac{3(x-2)}{x-3} - 3\log\frac{x-2}{x-3} - \frac{x-3}{x-2} + C$$

64. 2072 Set D Q.No. 7a

Evaluate: $\int \frac{dx}{(x-1)^2(x-2)^3}$ **[4]**

SOLUTION

Let $I = \int \frac{dx}{(x-1)^2(x-2)^3}$

Put $x - 1 = z(x - 2)$

$\therefore x = \frac{2z-1}{z-1}$

or, $dx = \frac{(z-1)\cdot 2 - (2z-1)\cdot 1}{(z-1)^2}\,dz$

or, $dx = \frac{2z - 2 - 2z + 1}{(z-1)^2}\,dz$

$\therefore dx = \frac{-1}{(z-1)^2}\,dz$

Also,

$$\frac{1}{(x-1)^2(x-2)^3} = \frac{1}{z^2(x-2)^5} = \frac{1}{z^2\left(\frac{2z-1}{z-1} - 2\right)^5}$$

$$= \frac{(z-1)^5}{z^2}$$

$$\therefore I = \int \frac{(z-1)^5}{z^2} \times \left\{-\frac{1}{(z-1)^2}\right\}$$

$$= -\int \frac{(z-1)^3}{z^2}\,dz = \int \frac{z^3 - 3z^2 + 3z - 1}{z^2}\,dz$$

$$= -\int \left(z - 3 + \frac{3}{z} - \frac{1}{z^2}\right)$$

$$= -\frac{z^2}{2} + 3z - 3\log z - \frac{1}{z} + C$$

$$= \frac{-1}{2}\left(\frac{x-1}{x-2}\right)^2 + 3\left(\frac{x-1}{x-2}\right) - 3\log\left(\frac{x-1}{x-2}\right) - \frac{x-2}{x-1} + C$$

65. 2072 Set E Q.No. 7a

Evaluate: $\int \frac{dx}{2 + 3\cos x}$ **[4]**

➥ Please refer to 2069 Set B Q.No. 7a

66. 2072 Supp Q.No. 7a

Evaluate: $\int \frac{dx}{3\sin x + 4\cos x}$. **[4]**

SOLUTION

$$\int \frac{dx}{3\sin x + 4\cos x}$$

Put $3 = r\cos\alpha$ amd $4 = r\sin\alpha$

so that $r = \sqrt{3^2 + 4^2} = 5$ and $\alpha = \tan^{-1}\frac{4}{3}$

$$\therefore \int \frac{dx}{3\sin x + 4\cos x}$$

$$= \int \frac{dx}{r\cos\alpha\sin x + r\sin\alpha\cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)} = \frac{1}{5}\int \operatorname{cosec}(x+\alpha)\,dx$$

$$= \frac{1}{5}\log\left\{\tan\frac{1}{2}(x+\alpha)\right\} + C$$

$$= \frac{1}{5}\log\left\{\tan\frac{1}{2}\left(x + \tan^{-1}\frac{4}{3}\right)\right\} + C$$

$$= \frac{1}{5}\log\tan\left(\frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{4}{3}\right) + C$$

Alternatively

Let $I = \int \frac{dx}{3\sin x + 4\cos x}$

$$= \int \frac{dx}{3\cdot 2\sin\frac{x}{2}\cos\frac{x}{2} + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int \frac{dx}{4\cos^2\frac{x}{2} + 6\sin\frac{x}{2}\cos\frac{x}{2} - 4\sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{4\left(1 + \frac{3}{2}\tan\frac{x}{2} - \tan^2\frac{x}{2}\right)}$$

$$= \frac{1}{4}\int \frac{\sec^2\frac{x}{2}\,dx}{1 - \left(\tan^2\frac{x}{2} - \frac{3}{2}\tan\frac{x}{2}\right)}$$

$$=\frac14\int\frac{\sec^2\frac x2\,dx}{1-\left\{\tan^2\frac x2-2\cdot\tan\frac x2\cdot\frac34+\left(\frac34\right)^2-\left(\frac34\right)^2\right\}}$$

$$=\frac14\int\frac{\sec^2\frac x2\,dx}{1-\left(\tan\frac x2-\frac34\right)^2+\frac{9}{16}}$$

$$=\frac14\int\frac{\sec^2\frac x2\,dx}{\frac{25}{16}-\left(\tan\frac x2-\frac34\right)^2}$$

Put $y=\tan\frac x2-\frac34$

$dy=\frac12\sec^2\frac x2\,dx$

$2dy=\sec^2\frac x2\,dy$

$$\therefore\ I=\frac14\int\frac{2dy}{\left(\frac54\right)^2-y^2}=\frac12\cdot\frac{1}{2\cdot\frac54}\log\left|\frac{\frac54+y}{\frac54-y}\right|+C$$

$$=\frac15\log\left|\frac{\frac54+\tan\frac x2-\frac34}{\frac54-\tan\frac x2+\frac34}\right|+C=\frac15\log\left|\frac{\frac12+\tan\frac x2}{2-\tan\frac x2}\right|+C$$

$$=\frac15\log\left|\frac{1+2\tan\frac x2}{2-\tan\frac x2}\right|+C$$

67. 2073 Set C Q.No. 7a

Evaluate: $\int\frac{dx}{1-2\cos x}$. **[4]**

➥ Please refer to 2071 Set C Q.No. 7a

68. 2073 Set C Q.No. 7a OR

Evaluate: $\int\frac{x^3\,dx}{2x^4-3x^2-5}$.

SOLUTION

Let $I=\int\frac{x^3\,dx}{2x^4-3x^2-5}$

Put $x^2=z$

or, $2xdx=dz$

$\therefore\ xdx=\frac{dz}{2}$

Now, $I=\int\frac{x^3\,dx}{2x^4-3x^2-5}=\int\frac{x^2\cdot xdx}{2(x^2)^2-3x^2-5}$

$$=\int\frac{z\cdot\frac{dz}{2}}{2z^2-3z-5}=\frac12\int\frac{zdz}{2z^2-3z-5}$$

$$=\frac12\int\frac{z}{(z+1)(2z-5)}dz\quad\ldots(i)$$

Let, $\frac{z}{(z+1)(2z-5)}=\frac{A}{z+1}+\frac{B}{2z-5}=\frac{A(2z-5)+B(z+1)}{(z+1)(2z-5)}$

$\therefore\ z=A(2z-5)+B(z+1)\quad\ldots(ii)$

Putting $z=-1$ in (ii), we get, $A=\frac17$

Again, putting $z=\frac52$ in (ii), we get, $B=\frac57$

$\therefore\ \frac{z}{2z^2-3z-5}=\frac{1}{7(z+1)}+\frac{5}{7(2z-5)}$

Now, from (i)

$$I=\frac12\int\left\{\frac{1}{7(z+1)}+\frac{5}{7(2z-5)}\right\}dz$$

$$=\frac1{14}\int\frac{1}{z+1}dz+\frac{5}{14}\int\frac{1}{2z-5}dz$$

$$=\frac1{14}\log(z+1)+\frac5{14}\cdot\frac12\log(2z-5)+C$$

$$=\frac1{14}\log(z+1)+\frac5{28}\log(2z-5)+C$$

$$=\frac1{14}\log(x^2+1)+\frac5{28}\log(2x^2-5)+C$$

69. 2073 Set D Q.No. 7a

Evaluate: $\int\frac{dx}{1+\sin x+\cos x}$. [4]

➥ Please refer to 2063 Q.No. 11b

70. 2073 Supp Q.No. 7a

Evaluate: $\int\frac{x^2}{x^4-2x^2-15}dx$. [4]

SOLUTION

Put $x^2=y$

Then, $\frac{x^2}{x^4-2x^2-15}=\frac{y}{y^2-2y-15}=\frac{y}{(y+3)(y-5)}$

Let $\frac{y}{(y+3)(y-5)}=\frac{A}{y+3}+\frac{B}{y-5}=\frac{A(y-5)+B(y+3)}{(y+3)(y-5)}$

or, $y=A(y-5)+B(y+3)\quad\ldots(i)$

Put $y=5$ in (i), we get

$5=B.8$

$B=\frac58$

Again,

Put $y=-3$ in $\ldots(i)$

$-3=A(-3-5)$

or, $A=\frac38$

Hence, $\frac{x^2}{x^4-2x^2-15}=\frac{y}{(y+3)(y-5)}$

$$=\frac{A}{(y+3)}+\frac{B}{(y-5)}$$

$$=\frac{3}{8(y+3)}+\frac{5}{8(y-5)}$$

$$=\frac{3}{8(x^2+3)}+\frac{5}{8(x^2-5)}$$

Now, $\int\frac{x^2}{x^4-2x^2-15}dx$

$$=\frac38\int\frac{dx}{x^2+3}+\frac58\int\frac{dx}{x^2-5}$$

$$=\frac38\int\frac{dx}{x^2+(\sqrt3)^2}+\frac58\int\frac{dx}{x^2-(\sqrt5)^2}$$

$$=\frac38\cdot\frac{1}{\sqrt3}\tan^{-1}\frac{x}{\sqrt3}+\frac58\cdot\frac{1}{2\sqrt5}\log\frac{x-\sqrt5}{x+\sqrt5}+C$$

$$=\frac{\sqrt3}{8}\tan^{-1}\frac{x}{\sqrt3}+\frac{\sqrt5}{16}\log\frac{x-\sqrt5}{x+\sqrt5}+C.$$

71. 2074 Set A Q.No. 7a

Evaluate: $\int\frac{dx}{3+5\cosh x}$. **[4]**

SOLUTION

Let $I=\int\frac{dx}{3+5\cos hx}$

$$=\int\frac{dx}{3\left(\cos h^2\frac x2-\sin h^2\frac x2\right)+5\left(\cos h^2\frac x2+\sin h^2\frac x2\right)}$$

$$=\int\frac{dx}{8\cos h^2\frac x2+2\sin h^2\frac x2}$$

$$=\frac12\int\frac{dx}{4\cos h^2\frac x2+\sin h^2\frac x2}\times\frac{\sec h^2\frac x2}{\sec h^2\frac x2}$$

$$=\frac12\int\frac{\sec h^2\frac x2}{4+\tan h^2\frac x2}dx$$

Put $y=\tanh\frac x2$.

$\frac{dy}{dx}=\sec h^2\frac x2\cdot\frac12$

$\Rightarrow\ 2dy=\sec h^2\frac x2\,dx$

Then, $I=\frac12\int\frac{2dy}{4+y^2}$

$$=\int\frac{dy}{y^2+2^2}$$

$$=\frac12\tan^{-1}\left(\frac y2\right)+c=\frac12\tan^{-1}\left(\frac{\tan h\frac x2}{2}\right)+c$$

72. 2074 Set B Q.No. 7a

Evaluate: $\int\frac{dx}{3\sin x+4\cos x}$. **[4]**

➥ Pleaser refer to 2072 Supp. Q. No. 7a

73. 2074 Set B Q.No. 7a OR

Evaluate: $\int\sqrt{\frac{1+x}{1-x}}\,dx$. **[4]**

➥ Pleaser refer to 2061 Q. No. 3a

74. 2074 Supp Q.No. 7a

Evaluate: $\int\frac{dx}{2+3\cos x}$. **[4]**

➥ Please refer to 2069 Set B Q.N. 7a

75. 2075 Set A Q.No. 7a

Evaluate: $\int\frac{dx}{\sin x+\cos x}$ **[4]**

➥ Please refer to 2069 Set A (Old) Q.No. 11b

76. 2075 Set B Q.No. 7b

Evaluate: $\int(2x+3)\sqrt{x^2-2x-3}\,dx$. **[4]**

SOLUTION

Let

$I=\int(2x+3)\sqrt{x^2-2x-3}\,dx$

$=\int(2x-2+2+3)\sqrt{x^2-2x-3}\,dx$

$=\int\{(2x-2)+5\}\sqrt{x^2-2x-3}\,dx$

$=\int(2x-2)\sqrt{x^2-2x-3}\,dx+\int5\sqrt{x^2-2x-3}\,dx$

$=I_1+I_2$ (say)

$I_1=\int(2x-2)\sqrt{x^2-2x-3}\,dx$

Put $y=x^2-2x-3$

$dy=(2x-2)dx$

$\therefore\ I_1=\int\sqrt y\,dy$

$=\int y^{1/2}dy$

$=\frac{y^{3/2}}{3/2}+C_1$

$=\frac23y^{3/2}+C_1$

$=\frac23(x^2-2x-3)^{3/2}+C_1$

And,

$I_2=5\int\sqrt{x^2-2x-3}\,dx$

$=5\int\sqrt{x^2-2.x.1+1^2-1-3}\,dx$

$=5\int\sqrt{(x-1)^2-4}\,dx$

$=5\int\sqrt{(x-1)^2-2^2}\,dx$

$=5\left[\frac12(x-1)\sqrt{(x-1)^2-4}-\frac12 2^2\log\left(x-1+\sqrt{(x-1)^2-4}\right)\right]+C_2$

$=\frac52(x-1)\sqrt{x^2-2x-3}-$

$10\log\left(x-1+\sqrt{x^2-2x-3}\right)+C_2$

$\therefore\ I=I_1+I_2$

$= \frac{2}{3}(x^2 - 2x - 3)^{3/2} + C_1 + \frac{5}{2}(x-1)\sqrt{x^2-2x-3} - 10\log(x - 1 + \sqrt{x^2-2x-3}) + C_2$

$= \frac{2}{3}(x^2 - 2x - 3)^{3/2} + \frac{5}{2}(x-1)\sqrt{x^2-2x-3} - 10\log(x - 1 + \sqrt{x^2-2x-3}) + C$, where $C = C_1 + C_2$.

77. 2075 Set C Q.No. 7a

Evaluate: $\int \frac{1}{2\sin t + 3\cos t} dt.$ **[4]**

SOLUTION

Let,

$I = \int \frac{1}{2\sin t + 3\cos t} dt$

$= \int \frac{dt}{2.2\sin\frac{t}{2}\cos\frac{t}{2} + 3\left(\cos^2\frac{t}{2} - \sin^2\frac{t}{2}\right)}$

$= \int \frac{dt}{4.\sin\frac{t}{2}\cos\frac{t}{2} + 3\cos^2\frac{t}{2} - 3\sin^2\frac{t}{2}} \times \frac{\sec^2\frac{t}{2}}{\sec^2\frac{t}{2}}$

$= \int \frac{\sec^2\frac{t}{2}dt}{4\tan\frac{t}{2} + 3 - 3\tan^2\frac{t}{2}}$

$= \frac{1}{3}\int \frac{\sec^2\frac{t}{2}dt}{1 + \frac{4}{3}\tan\frac{t}{2} - \tan^2\frac{t}{2}}$

Put $\tan\frac{t}{2} = y$

Then, $\frac{1}{2}\sec^2\frac{t}{2} dt = dy$

$\therefore \sec^2\frac{t}{2} dt = 2dy$

Now,

$I = \frac{2}{3}\int \frac{dy}{1 + \frac{4}{3}y - y^2}$

$= \frac{2}{3}\int \frac{dy}{\frac{13}{9} - \left(y - \frac{2}{3}\right)^2}$

$= \frac{2}{3}\int \frac{dy}{\left(\frac{\sqrt{13}}{3}\right)^2 - \left(y - \frac{2}{3}\right)^2}$

$= \frac{2}{3}.\frac{1}{\frac{2\sqrt{13}}{3}}\log\left\{\frac{\frac{\sqrt{13}}{3} - \left(y - \frac{2}{3}\right)}{\frac{\sqrt{13}}{3} + y - \frac{2}{3}}\right\} + C$

$= \frac{1}{\sqrt{13}}\log\frac{\sqrt{13} + 2 - y}{\sqrt{13} - 2 + y} + C$

$= \frac{1}{\sqrt{13}}\log\left(\frac{\sqrt{13} + 2 - \tan\frac{x}{2}}{\sqrt{13} - 2 + \tan\frac{x}{2}}\right) + C$

Alternatively,

Let, $I = \int \frac{dt}{3\sin t + 4\cos t}$

Put $2 = r\cos\theta$ and $3 = r\sin\theta$.

Then, $r^2 = 2^2 + 3^2 = 13$

Also, $\tan\theta = \frac{3}{2}$

$\therefore \theta = \tan^{-1}\left(\frac{3}{2}\right)$

Now,

$I = \int \frac{dt}{r\cos\theta\sin t + r\sin\theta\cos t}$

$= \frac{1}{r}\int \frac{dt}{\sin(t+\theta)}$

$= \frac{1}{r}\int \text{cosec}(t+\theta)\text{ at}$

$= \frac{1}{r}\log\tan\frac{1}{2}(t+\theta) + C$

$= \frac{1}{\sqrt{13}}\log\tan\frac{1}{2}\left(t + \tan^{-1}\frac{3}{2}\right) + C$

$= \frac{1}{\sqrt{13}}\log\tan\left(\frac{t}{2} + \frac{1}{2}\tan^{-1}\frac{3}{2}\right) + C$

UNIT 9

DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

2 MARKS QUESTIONS

1. 2057 Q.No. 6 c

Solve: $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$ **[2]**

SOLUTION

Here, $\frac{dy}{dx} = \frac{x^2+x+1}{y^2+y+1}$

or, $(y^2+y+1)\,dy = (x^2+x+1)\,dx$

Integrating,

$\frac{y^3}{3} + \frac{y^2}{2} + y = \frac{x^3}{3} + \frac{x^2}{2} + x + c$

2. 2058 Q.No. 6 c

Solve: $\frac{dy}{dx} = \frac{x^3+1}{y^3+1}$. **[2]**

SOLUTION

Given, $\frac{dy}{dx} = \frac{x^3+1}{y^3+1}$

or, $(y^3+1)\,dy = (x^3+1)\,dx$

Integrating, we get

$\frac{y^4}{4} + y = \frac{x^4}{4} + x + c$

3. 2059 Q.No. 2 b

Solve: $x\,dy - y\,dx = 0$ **[2]**

SOLUTION

Given, $x\,dy - y\,dx = 0$

or, $x\,dy = y\,dx$

or, $\frac{dy}{y} = \frac{dx}{x}$

Integrating, we get

$\int\frac{dy}{y} = \int\frac{dx}{x} + \log c$

$\log y = \log x + \log c$

or, $\log y = \log c\,x$

$\therefore y = c\,x$

4. 2060 Q.No. 6 c

Solve: $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0.$ **[2]**

SOLUTION

Given, $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}dx = 0$

or, $\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$

Integrating, we have

$\sin^{-1}x + \sin^{-1}y = \sin^{-1}c$

or, $\sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}) = \sin^{-1}c$

$\therefore x\sqrt{1-y^2} + y\sqrt{1-x^2} = c$

5. 2061 Q.No. 6 c

Solve: $x^2dy - y^2dx = 0$ **[2]**

SOLUTION

Here, $x^2dy - y^2dx = 0$

or, $x^2dy = y^2dx$

or, $\frac{1}{y^2}dy = \frac{1}{x^2}dx$

or, $\frac{1}{x^2}dx - \frac{1}{y^2}dy = 0$

Integrating, we have

$\int x^{-2}dx - \int y^{-2}dy = c$

or, $\frac{x^{-2+1}}{-2+1} - \frac{y^{-2+1}}{-2+1} = c$

or, $\frac{-1}{x} + \frac{1}{y} = c$

or, $\frac{x-y}{xy} = c$

$\therefore x - y = cxy$

6. 2062 Q.No. 6 c

Solve: $x^2dy - y^2dx = 0$ **[2]**

➥ Please refer to 2061 Q.No. 6c

7. 2063 Q.No. 6 c

Solve: $e^{x-y}\,dx + e^{y-x}\,dy = 0$ **[2]**

SOLUTION

Here, $e^{x-y}.\,dx + e^{y-x}\,dy = 0$

or, $\frac{e^x}{e^y}dx + \frac{e^y}{e^x}dy = 0$

or, $e^{2x}\,dx + e^{2y}\,dy = 0$

Integrating we have

$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{c}{2}$

$\therefore e^{2x} + e^{2y} = c$

8. 2064 Q.No. 6 c

Solve: $x\,dy + (x+y)\,dx = 0$ **[2]**

SOLUTION

Given, $x\,dy + (x+y)\,dx = 0$

or, $x\,dy + x\,dx + y\,dx = 0$

or, $(x\,dy + y\,dx) + x\,dx = 0$

or, $d(xy) + d\left(\frac{x^2}{2}\right) = 0$

or, $d\left(xy + \frac{x^2}{2}\right) = 0$

Integrating, we get

$xy + \frac{x^2}{2} = c$

9. 2065 Q.No 6 c

Solve: $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0$ [2]

SOLUTION

Here, $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0$

or, $x(y^2 + 1)\, dx + y(x^2 + 1)\, dy = 0$

Dividing both sides by $(x^2 + 1)(y^2 + 1)$, we get

or, $\frac{x}{x^2+1}dx + \frac{y}{y^2+1}dy = 0$

or, $\frac{2x\,dx}{x^2+1} + \frac{2y\,dy}{y^2+1} = 0$

Integrating, we get

$\log(x^2 + 1) + \log(y^2 + 1) = \log c$

or, $\log(x^2 + 1)(y^2 + 1) = \log c$

$\therefore (x^2 + 1)(y^2 + 1) = c$

10. 2066 C Q.No. 6 c

Solve: $\sqrt{1 - x^2}\, dy + \sqrt{1 - y^2}dx = 0$ [2]

➥ Please refer to 2060 Q.No. 6c

11. 2066 Q.No. 6 c

Solve the differential equation

$(x + 2y - 3)\, dy - (2x - y + 1)\, dx = 0$ [2]

SOLUTION

Here, $(x + 2y - 3)\, dy - (2x - y + 1)\, dx = 0$

or, $xdy + 2ydy - 3dy - 2xdx + ydx - dx = 0$

or, $(xdy + ydx) + d(y^2) - d(x^2) - d(3y) - d(x) = 0$

or, $d(xy) + d(y^2) - d(x^2) - d(3y) - d(x) = 0$

or, $d(xy + y^2 - x^2 - 3y - x) = 0$

Integrating, we get

$xy + y^2 - x^2 - 3y - x = c$

12. 2068 Q.No. 6c

Solve: $x^2dy - y^2dx = 0$ [2]

➥ Please refer to 2061 Q.No. 6c

13. 2069 (Set A) Q.No. 4a

Solve $\frac{dy}{dx} = e^{x-y} + x^3.e^{-y}$ [2]

SOLUTION

Here, $\frac{dy}{dx} = e^{x-y} + x^3.e^{-y}$

or, $\frac{dy}{dx} = e^x \cdot e^{-y} + x^3 e^{-y}$

or, $\frac{dy}{dx} = e^{-y}(e^x + x^3)$

or, $\frac{dy}{e^{-y}} = (e^x + x^3)\, dx$

or, $e^y\, dy = (e^x + x^3)\, dx$

Integrating, we get

$\int e^y\, dy = \int (e^x + x^3)\, dx$

$\therefore e^y = e^x + \frac{x^4}{4} + c$

14. 2069 (Set B) Q.No. 4a

Solve: $e^{x-y}\, dx + e^{y-x}.\, dy = 0$ [2]

➥ Please refer to 2063 Q.No. 6c

15. 2069 (Set A) Old Q.No. 6c

Solve: $\frac{dy}{dx} = \frac{e^x + 1}{y}$. [2]

SOLUTION

Here, $\frac{dy}{dx} = \frac{e^x + 1}{y}$

or, $y\, dy = (e^x + 1)\, dx$

Integrating, we have

$\frac{y^2}{2} = e^x + x + c$

or, $y^2 = 2e^x + 2x + 2c_1$

$\therefore y^2 = 2e^x + 2x + c$, where $C = 2c_1$

16. 2069 Old (Set B) Q.No. 2b

Solve : $2xydx - x^2dy = 0$ [2]

SOLUTION

$2xydx - x^2dy = 0$

or, $y.d(x^2) - x^2d(y) = 0$

Dividing both sides by y^2

$\frac{yd(x) - x^2d(y)}{y^2} = 0$

or, $d\left(\frac{x^2}{y}\right) = 0$

Integrating, we get $\frac{x^2}{y} = c$

$\therefore x^2 = cy$

17. 2070 Set C Q.No. 4 a

Solve : $\frac{dy}{dx} = \frac{e^x + 1}{y}$ [2]

➥ Please refer to 2069 (Set A) Old Q.No. 6c

18. 2070 Set D Q.No. 4 a

Solve : $x^2dy - y^2dx = 0$. [2]

➥ Please refer to 2061 Q.No. 6c

19. 2070 (Old) Q.No. 5 c

Solve: $(1 + x^2)\frac{dy}{dx} = 1$. [2]

SOLUTION

Given, $(1 + x^2)\frac{dy}{dx} = 1$

or, $\frac{dy}{dx} = \frac{1}{1 + x^2}$

or, $dy = \frac{1}{1 + x^2}dx$

Integrating, we have

$\int dy = \int \frac{1}{1 + x^2}dx + c$

$\therefore y = \tan^{-1} x + c$

20. 2070 Supp. Q.No. 4 a

Solve the differential equation: $\frac{dy}{dx} = \frac{2x + 1}{5y^4 + 1}$ [2]

SOLUTION

Here, $\frac{dy}{dx} = \frac{2x + 1}{5y^4 + 1}$

$(2x + 1)dx = (5y^4 + 1)\, dy$

Integrating, we have

$\int(2x + 1)\, dx = \int(5y^4 + 1)\, dy$

or, $\frac{2x^2}{2} + x = \frac{5y^5}{5} + y + c$

or, $x^2 + x = y^5 + y + c$

$\therefore x^2 + x - y^5 - y = c$

21. 2071 Set C Q.No. 4 a

Solve: $e^{x-y}.\, dx + e^{y-x}\, dy = 0$ [2]

➥ Please refer to 2063 Q.No. 6c

22. 2071 Set D Q.No. 4 a

Solve: $\frac{dy}{dx} + 4x = 2\, e^{2x}$ [2]

SOLUTION

$\frac{dy}{dx} + 4x = 2\, e^{2x}$

or, $\frac{dy}{dx} = 2e^{2x} - 4x$

or, $dy = (2e^{2x} - 4x)\, dx$

Integrating, we have

$y = \frac{2e^{2x}}{2} - \frac{4x^2}{2} + c$

$\therefore y = e^{2x} - 2x^2 + c$

23. 2071 Old Q.No. 6 a

Solve: $2xydy - y^2dx = 0$ [2]

SOLUTION

Here, $2xydy - y^2dx = 0$

or, $x\, d(y^2) - y^2\, dx = 0$

or, $\frac{x\, d(y^2) - y^2dx}{x^2} = 0$

or, $d\left(\frac{y^2}{x}\right) = 0$

Integrating, we have

$\frac{y^2}{x} = c$

$\therefore y^2 = cx$

24. 2071 Supp. Q.No. 3c

Solve $\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0.$ [2]

SOLUTION

Given equation is

$\frac{dy}{dx} + \sqrt{\frac{1 - y^2}{1 - x^2}} = 0$

or, $\frac{dy}{dx} = -\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}}$

or, $\sqrt{1 - x^2}\, dy = -\sqrt{1 - y^2}\, dx$

or, $\sqrt{1 - x^2}\, dy + \sqrt{1 - y^2}\, dx = 0$

or, $\frac{dy}{\sqrt{1 - y^2}} + \frac{dx}{\sqrt{1 - x^2}} = 0$

Integrating, we have

$\sin^{-1} x + \sin^{-1} y = \sin^{-1} c$

or, $\sin^{-1}\left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right) = \sin^{-1} c$

$\therefore x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = c$

25. 2072 Set C Q.No. 4a

Solve: $\frac{dy}{dx} + \frac{y}{x} = 1.$ [2]

SOLUTION

Here, $\frac{dy}{dx} + \frac{y}{x} = 1$ ...(i)

Comparing it with $\frac{dy}{dx} + Py = Q$, we get,

$P = \frac{1}{x}, Q = 1$

$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying both sides of (i) by x, we get

$x\frac{dy}{dx} + y = x$

or $d(y \cdot x) = x\, dx$

Integrating, we have

$xy = \frac{x^2}{2} + c$

26. 2072 Set D Q.No. 4a

Solve: $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2y} = 0.$ [2]

SOLUTION

Given differential equation is

$\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2y} = 0$

or, $\frac{dy}{dx} = -\frac{1 + \cos 2y}{1 - \cos 2y}$

or, $\frac{dy}{dx} = -\frac{1 + 2\cos^2 y - 1}{1 - 1 + 2\sin^2 y}$

or, $\frac{dy}{dx} = -\frac{\cos^2 y}{\sin^2 y}$

or, $\frac{dy}{dx} = -\cot^2 y$

or, $\frac{dy}{\cot^2 y} = -dx$

or, $dx + \tan^2 y\, dy = 0$

or, $dx + (\sec^2 y - 1)dy = 0$

Integrating, we get

$x + \tan y - y = c$

27. 2072 Set E Q.No. 4a

Solve: $x\frac{dy}{dx} + y - 1 = 0$. [2]

➥ Please see Model Set II, Q.No.4a

28. 2072 Supp Q.No. 4a

Solve: $\frac{dy}{dx} + 4x = 2e^{2x}$. [2]

➥ Please refer to 2071 Set D Q.No. 4a

29. 2073 Set C Q.No. 4a

Solve: $\frac{dx}{1 + x^2} + \frac{dy}{1 + y^2} = 0$ [2]

SOLUTION

Integrating both sides, we get

$$\int\frac{dx}{1+x^2}+\int\frac{dy}{1+y^2}=\tan^{-1}c$$

or, $\tan^{-1}x+\tan^{-1}y=\tan^{-1}c$

or, $\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1}c$

or, $\frac{x+y}{1-xy}=C$

$\therefore\ x+y=C(1-xy)$

30. 2073 Set D Q.No. 4a [2]

Solve: $ydx - xdy = xy\,dy$.

SOLUTION

Here, $ydx - xdy = xy\,dy$

Dividing both sides by y^2, we get,

$$\frac{ydx-xdy}{y^2}=\frac{x}{y}dy$$

or, $d\left(\frac{x}{y}\right)=\frac{x}{y}dy$

or, $\frac{d\left(\frac{x}{y}\right)}{\frac{x}{y}}=dy$

Integrating, we get

$$\log\left(\frac{x}{y}\right)=y+c.$$

31. 2073 Supp Q.No. 4a [2]

Solve: $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$.

SOLUTION

$$\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

or, $(1+x^2)\,dy=(1+y^2)dx$

or, $\frac{dx}{1+x^2}=\frac{dy}{1+y^2}$

or, $\frac{dx}{1+x^2}-\frac{dy}{1+y^2}=0$

Integrating,

$\tan^{-1}x-\tan^{-1}y=\tan^{-1}c$

or, $\tan^{-1}\frac{x-y}{1+xy}=\tan^{-1}c$

or, $\frac{x-y}{1+xy}=c$

$\therefore\ x-y=c(1+xy)$

32. 2074 Set A Q.No. 3c [2]

Solve $\frac{dy}{dx}=\frac{e^x+1}{y}$

➥ Please refer to 2069 Set A Old Q.No. 6c

33. 2074 Set B Q.No. 4a [2]

Solve: $(1+x^2)\frac{dy}{dx}=1$.

➥ Pleaser refer to 2070 (old) Q. No. 5c

34. 2074 Supp Q.No. 4a [2]

Solve $\frac{dy}{dx}=e^{x-y}+x^3.e^{-y}$.

➥ Please refer to 2069 Set A Q.N. 4a

35. 2075 Set A Q.No. 4a [2]

Solve: $xdy+(x+y)dx=0$.

➥ Please refer to 2064 Q.No. 6c

36. 2075 Set B Q.No. 3c [2]

Solve: $\sec^2x\tan x\,dx+\sec^2y\tan x\,dy=0$.

SOLUTION

Given,

$\sec^2x\tan x\,dx+\sec^2y\tan x\,dy=0$

or, $\tan x(\sec^2x\,dx+\sec^2y\,dy)=0$

or, $\sec^2x\,dx+\sec^2y\,dy=0$

Integrating both sides, we get,

$\int\sec^2x\,dx+\int\sec^2y\,dy=c$

$\therefore\ \tan x+\tan y=c.$ $\quad[\because\int\sec^2t=\tan t+C]$

37. 2075 Set C Q.No. 4a [2]

Solve: $\frac{dy}{dx}=\frac{e^x}{e^y}+\frac{x^3}{e^y}$.

➥ Please refer to 2069 (Set A) Q.No. 4a

4 MARKS QUESTIONS

38. 2057 Q.No. 11 b OR [4]

Solve: $\tan x\frac{dy}{dx}+y=\sec x$.

SOLUTION

Here, $\tan x\frac{dy}{dx}+y=\sec x$

Dividing both sides by $\tan x$, we have

$$\frac{dy}{dx}+\cot x.\,y=\operatorname{cosec}x \qquad \ldots(i)$$

Comparing (i) with $\frac{dy}{dx}+Py=Q$, we get

$P=\cot x \qquad Q=\operatorname{cosec}x$

$I.F=e^{\int Pdx}=e^{\int\frac{\cos x}{\sin x}dx}=e^{\log\sin x}=\sin x$

Multiplying both sides of equation (i) by I.F, we get

$\sin x\frac{dy}{dx}+\cot x.\sin x.\,y=\operatorname{cosec}x.\sin x$

or, $\sin x\frac{dy}{dx}+\cos x.\,y=dx$

or, $d(y.\sin x)=dx$

Integrating, we have

$y\sin x=x+c$

39. 2058 Q.No. 11 b OR [4]

Solve: $\cos^2x\frac{dy}{dx}+y=1$

SOLUTION

Here, $\cos^2x\frac{dy}{dx}+y=1$

Dividing both sides by $\cos^2x$, we have

$$\frac{dy}{dx}+\sec^2x.\,y=\sec^2x \qquad \ldots(i)$$

Comparing equation (i) with $\frac{dy}{dx}+Py=Q$,

we get

$P=\sec^2x, \qquad Q=\sec^2x$

$I.F=e^{\int Pdx}=e^{\int\sec^2x\,dx}=e^{\tan x}$

Multiplying equation (i) both sides by $e^{\tan x}$, we get

$e^{\tan x}.\frac{dy}{dx}+e^{\tan x}.\sec^2x.\,y=e^{\tan x}.\sec^2x$

$d(y.\,e^{\tan x})=e^{\tan x}.\sec^2x\,dx$

Integrating, we have

$$y\cdot e^{\tan x}=\int d(e^{\tan x})+c$$

or, $y\cdot e^{\tan x}=e^{\tan x}+c$

$\therefore\ y=1+c\,e^{-\tan x}$

40. 2059 Q.No. 11 b OR [4]

Solve: $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$

SOLUTION

Here, $\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$

Put $y=vx$, then

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Now, the given equation can be written as

$v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2x.\,vx}$

or, $v+x\frac{dv}{dx}=\frac{v^2-1}{2v}$

or, $x\frac{dv}{dx}=\frac{v^2-1}{2v}-v$

or, $x\frac{dv}{dx}=\frac{v^2-1-2v^2}{2v}$

or, $x\frac{dv}{dx}=\frac{-1-v^2}{2v}$

or, $\frac{2v\,dv}{v^2+1}=\frac{-dx}{x}$

or, $\frac{2v\,dv}{v^2+1}+\frac{dx}{x}=0$

Integrating, we get

$\log(v^2+1)+\log x=\log c$

or, $\log\{(v^2+1)\,x\}=\log c$

or, $(v^2+1)\,x=c$

or, $\left(\frac{y^2}{x^2}+1\right)x=c$

or, $\frac{x^2+y^2}{x}=c$

$\therefore\ x^2+y^2=cx$

41. 2060 Q.No. 11 b OR [4]

Solve: $\tan x\frac{dy}{dx}+y=\sec x$

➥ Please refer to 2057 Q.No. 11b OR

42. 2061 Q.No. 11 b OR [4]

Solve: $(x^2-y^2)\frac{dy}{dx}=xy$

SOLUTION

Given, $(x^2-y^2)\frac{dy}{dx}=xy$

or, $\frac{dy}{dx}=\frac{xy}{x^2-y^2} \qquad (i)$

Put $y=vx$, then

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Now, equation (i) can be written as

$v+x\frac{dv}{dx}=\frac{x.\,vx}{x^2-v^2x^2}$

or, $x\frac{dv}{dx}=\frac{v}{1-v^2}-v$

or, $x\frac{dv}{dx}=\frac{v-v+v^3}{1-v^2}$

or, $x\frac{dv}{dx}=\frac{v^3}{1-v^2}$

or, $\frac{1-v^2}{v^3}dv=\frac{dx}{x}$

or, $\left(\frac{1}{v^3}-\frac{1}{v}\right)dv=\frac{dx}{x}$

or, $\frac{1}{v^3}dv=\frac{1}{v}dv+\frac{1}{x}dx$

Integrating, we get

$$\int v^{-3}dv=\int\frac{1}{v}dv+\int\frac{1}{x}dx+\log c$$

or, $-\frac{1}{2v^2}=\log v+\log x+\log c$

or, $\frac{-1}{2\left(\frac{y}{x}\right)^2}=\log(vxc)$

or, $\frac{-x^2}{2y^2}=\log\left(\frac{y}{x}\cdot x\cdot c\right)$

or, $\frac{-x^2}{2y^2}=\log(cy)$

or, $-x^2=2y^2\log(cy)$

$\therefore\ x^2+2y^2\log(cy)=0$

43. 2062 Q.No. 11 b OR [4]

Solve: $\frac{dy}{dx}+\frac{1}{x}y=x^2$

SOLUTION

Given, $\frac{dy}{dx}+\frac{1}{x}y=x^2 \qquad \ldots(i)$

Comparing equation (i) with $\frac{dy}{dx}+Py=Q$,

we get

$P=\frac{1}{x},\ Q=x^2$

$I.F=e^{\int Pdx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$

Multiplying both sides of equation (i) by x, we get,

$x\frac{dy}{dx}+y=x^3$

or, $d(xy) = x^3\,dx$

Integrating, we get,

$xy = \frac{x^4}{4} + c$

44. 2063 Q.No. 11 b OR

Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ [4]

SOLUTION

Here, $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ ... (i)

Put $y = vx$, then

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{vx}{x} + \tan\frac{vx}{x}$

or, $x\frac{dv}{dx} = \tan v$

or $\frac{dv}{\tan v} = \frac{dx}{x}$

or, $\frac{\cos v}{\sin v}dv = \frac{dx}{x}$

Integrating, we get

$\log\sin v = \log x + \log c$

or, $\log\sin v = \log cx$

or, $\sin v = cx$

$\therefore \sin\left(\frac{y}{x}\right) = cx$

45. 2064 Q.No. 11 b OR

Solve: $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$ [4]

SOLUTION

Given, $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

Put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, given equation can be written as

$2\left(v + x\frac{dv}{dx}\right) = \frac{vx}{x} + \frac{v^2x^2}{x^2}$

or, $2v + 2x\frac{dv}{dx} = v + v^2$

or, $2x\frac{dv}{dx} = v^2 - v$

or, $\frac{2}{v(v-1)}dv = \frac{dx}{x}$

or, $2\left(\frac{1}{v-1} - \frac{1}{v}\right)dv = \frac{dx}{x}$

Integrating, we get

$2\left[\int\frac{1}{v-1}dv - \int\frac{1}{v}dv\right] = \int\frac{dx}{x} + \log c$

or, $2\{\log(v-1) - \log v\} = \log x + \log c$

or, $2\log\left(\frac{v-1}{v}\right) = \log cx$

or, $\log\left(\frac{v-1}{v}\right)^2 = \log cx$

or, $\left(\frac{v-1}{v}\right)^2 = cx$

or, $\left(\frac{\frac{y}{x}-1}{\frac{y}{x}}\right)^2 = cx$

or, $\left(\frac{y-x}{y}\right)^2 = cx$

$\therefore (y-x)^2 = cxy^2$

46. 2065 Q.No 11 b OR

Solve: $\frac{dy}{dx} + y\cot x = x$ [4]

SOLUTION

Here, $\frac{dy}{dx} + y\cot x = x$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$

we get

$P = \cot x,\ Q = x$

$I.F = e^{\int Pdx} = e^{\int\cot x\,dx} = e^{\int\frac{\cos x}{\sin x}dx} = e^{\log\sin x} = \sin x$

Multiplying both sides of equation (i) by $\sin x$, we get

$\sin x\frac{dy}{dx} + y\cdot\sin x\cot x = x\sin x$

$d(y\sin x) = x\sin x\,dx$

Integrating we get,

$y\sin x = \int x\sin x\,dx + c$

or, $y\sin x = x\int\sin x\,dx - \int\left[\frac{d(x)}{dx}\int\sin x\,dx\right]dx + c$

or, $y\sin x = -x\cos x - \int 1\cdot(-\cos x)\,dx + c$

or, $y\sin x = -x\cos x + \int\cos x\,dx + c$

$\therefore y\sin x = -x\cos x + \sin x + c$

47. 2066 C Q.No. 11 b OR

Solve: $(x^2 - y^2)\frac{dy}{dx} = xy$ [4]

➥ Please refer to 2061 Q.No. 11b OR

48. 2066 Q.No.11 b OR

Solve the differential equation: [4]

$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

SOLUTION

Given equation is: $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$

or, $\frac{dy}{dx} + \frac{1}{1+x^2}\cdot y = \frac{e^{\tan^{-1}x}}{1+x^2}$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{1}{1+x^2},\ Q = \frac{e^{\tan^{-1}x}}{1+x^2}$

$I.F = e^{\int Pdx} = e^{\int\frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

Multiplying both sides of (i) by $e^{\tan^{-1}x}$, we have

$e^{\tan^{-1}x}\cdot\frac{dy}{dx} + e^{\tan^{-1}x}\frac{1}{1+x^2}\cdot y = \frac{(e^{\tan^{-1}x})^2}{1+x^2}$

or, $d(y.\,e^{\tan^{-1}x}) = \frac{(e^{\tan^{-1}x})^2}{1+x^2}dx$

Integrating, we get

$y\,e^{\tan^{-1}x} = \int\frac{(e^{\tan^{-1}x})^2}{1+x^2}dx$

Put $\tan^{-1}x = z$

Then, $\frac{1}{1+x^2}dx = dz$

$\int\frac{(e^{\tan^{-1}x})^2}{1+x^2}dx = \int e^{2z}dz = \frac{1}{2}e^{2z} + c = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$

$\therefore y\,e^{\tan^{-1}x} = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$

$y = \frac{1}{2}e^{\tan^{-1}x} + c\,e^{-\tan^{-1}x}$

49. 2067 Q.No. 6c

Solve: $xdy + (x + y)dx = 0$ [4]

SOLUTION

Given, $xdy + (x+y)dx = 0$

or, $xdy + xdx + ydx = 0$

or, $xdy + ydx + xdx = 0$

or, $d(xy) + d\left(\frac{x^2}{2}\right) = 0$

or, $d\left(xy + \frac{x^2}{2}\right) = 0$

Integrating, we get

$xy + \frac{x^2}{2} = c$

50. 2067 Q.No. 10b OR

Solve: $(1 - x^2)\frac{dy}{dx} = 1 + xy$ [4]

➥ Please see Model Set II, Q.No. 7b or

51. 2068 Q.No. 11 b OR

Solve: $\sin x\frac{dy}{dx} + \cos x.y = x\sin x$ [4]

SOLUTION

Given, $\sin x\frac{dy}{dx} + \cos x.y = x\sin x$

Dividing both sides by $\sin x$,

$\frac{dy}{dx} + \frac{\cos x}{\sin x}y = x$ ... (i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{\cos x}{\sin x},\quad Q = x$

$I.F = e^{\int Pdx} = e^{\int\frac{\cos x}{\sin x}dx} = e^{\log\sin x} = \sin x$

Multiplying both sides of equation (i) by $\sin x$

$\sin x\frac{dy}{dx} + \cos x.\,y = x\sin x$

or, $d(y.\sin x) = x\sin x\,dx$

Integrating,

$y\sin x = \int x\sin x\,dx + c$

or, $y\sin x = x\int\sin x\,dx - \int\left[\frac{dx}{dx}\int\sin x\,dx\right]dx + c$

or, $y\sin x = -x\cos x - \int 1(-\cos x)\,dx + c$

or, $y\sin x = -x\cos x + \sin x + c$

$\therefore y\sin x = \sin x - x\cos x + c$

52. 2069 (Set A) Q.No. 7b

Solve: $\tan x\frac{dy}{dx} + y = \sec x$ [4]

➥ Please refer to 2060 Q.No. 11b OR

53. 2069 (Set A) Q.No. 7b or

Solve: $xy\frac{dy}{dx} = x^2 + y^2$ [4]

SOLUTION

Given differential equation is

$xy\frac{dy}{dx} = x^2 + y^2$

or, $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$ ...(i)

Put $y = vx$, then

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then, from (i)

$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x.\,vx}$

or, $v + x\frac{dv}{dx} = \frac{1+v^2}{v}$

or, $x\frac{dv}{dx} = \frac{1+v^2}{v} - v$

or, $x\frac{dv}{dx} = \frac{1+v^2-v^2}{v}$

or, $x\frac{dv}{dx} = \frac{1}{v}$

or, $vdv = \frac{dx}{x}$

Integrating, we have

$\frac{v^2}{2} = \log x + c$

or, $\left(\frac{y}{x}\right)^2 = 2(\log x + c)$

$\therefore y^2 = 2x^2(\log x + c)$

54. 2069 (Set B) Q.No. 7b

Solve: $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ [4]

SOLUTION

Here, $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

or, $\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2}$ ...(i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{2x}{1 + x^2}$ and $Q = \frac{4x^2}{1 + x^2}$

Now, I.F $= e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Multiplying both sides of (i) by $1 + x^2$, we get

$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

or, $\frac{d}{dx}\{y(1 + x^2)\} = 4x^2$

or, $d\{y(1 + x^2)\} = 4x^2\,dx$

Integrating, we get

$y(1 + x^2) = \int 4x^2\,dx + c$

or, $y(1 + x^2) = 4\cdot\frac{x^3}{3} + c$

$\therefore\ y(1 + x^2) = \frac{4x^3}{3} + c$

55. 2069 (Set B) Q.No. 7b Or

Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ [4]

➥ Please refer to 2063 Q.No. 11b OR

56. 2069 (Set A) Old Q.No. 11b or

Solve: $\cos^2 x\frac{dy}{dx} + y = 1.$ [4]

➥ Please refer to 2058 Q.No. 11b OR

57. 2069 Old (Set B) Q.No. 11b Or

Solve: $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$ [4]

SOLUTION

Here, $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2}$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we have

$P = \frac{1}{x^2}$ and $Q = \frac{1}{x^2}$

I.F $= e^{\int\frac{1}{x^2}dx} = e^{-\frac{1}{x}}$

Multiplying both side of (i) by $e^{-\frac{1}{x}}$

$e^{-\frac{1}{x}}\left(\frac{dy}{dx} + \frac{y}{x^2}\right) = e^{-\frac{1}{x}}\cdot\frac{1}{x^2}$

or, $d(y\cdot e^{-\frac{1}{x}}) = e^{-\frac{1}{x}}\cdot\frac{1}{x^2}\,dx$

Integrating,

$y\,e^{-\frac{1}{x}} = \int e^{-\frac{1}{x}}\cdot\frac{1}{x^2}\,dx + c$ ...(ii)

Let $I = \int e^{-\frac{1}{x}}\cdot\frac{1}{x^2}\,dx$

Put $z = \frac{-1}{x}$, $dz = \frac{1}{x^2}dx$

Then,

$I = \int e^z\,dz = e^z + c = e^{-\frac{1}{x}} + c$

Then from (ii),

$y\,e^{-\frac{1}{x}} = e^{-\frac{1}{x}} + c$

$\therefore\ y = 1 + ce^{\frac{1}{x}}$

58. 2070 Set C Q.No. 7 b

Solve: $\cos^2 x\frac{dy}{dx} + y = 1.$ [4]

➥ Please refer to 2058 Q.No. 11b OR

59. 2070 Set C Q.No. 7 b or

Solve: $\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$ [4]

SOLUTION

Given equation is, $\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, the given equation can be written as

$v + x\frac{dv}{dx} = v - \sin^2 v$

or, $x\frac{dv}{dx} = -\sin^2 v$

or, $\frac{-dv}{\sin^2 v} = \frac{dx}{x}$

Integrating, we have

$\int -\text{cosec}^2 v\,dv = \int\frac{dx}{x} + c$

or, $\cot v = \log x + c$

$\therefore\ \cot\left(\frac{y}{x}\right) = \log x + c$

60. 2070 Set D Q.No. 7 b

Solve $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$ [4]

➥ Please refer to 2069 (Set B) Q.No. 7b

61. 2070 Set D Q.No. 7 b Or

Solve: $(x^2 + y^2)\,dy = xy\,dx$ [4]

SOLUTION

Here, $(x^2 + y^2)\,dy = xy\,dx$

or, $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ ...(i)

Put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} = \frac{x.\,vx}{x^2 + v^2x^2}$

or, $x\frac{dv}{dx} = \frac{v}{1 + v^2} - v$

or, $x\frac{dv}{dx} = \frac{v - v - v^3}{1 + v^2}$

or, $x\frac{dv}{dx} = \frac{-v^3}{1 + v^2}$

or, $\frac{1 + v^2}{v^3}dv = \frac{-dv}{x}$

or, $\frac{1}{v^3}dv + \frac{1}{v}dv + \frac{dx}{x} = 0$

or, $\frac{1}{v}dv + \frac{dx}{x} = -\frac{1}{v^3}dv$

Integrating, we get

$\log v + \log x + \log c = \frac{-v^{-3+1}}{(-3+1)}$

or, $\log(v.\,x.\,c) = \frac{1}{2v^2}$

or, $\log\left(\frac{y}{x}.x.\,c\right) = \frac{1}{2\left(\frac{y}{x}\right)^2}$

or, $\log(cy) = \frac{x^2}{2y^2}$

$\therefore\ x^2 = 2y^2\log cy$

62. 2070 (Old) Q.No. 11 b Or

Solve: $(1 + x)\frac{dy}{dx} = xy - x + 1$ [4]

SOLUTION

Here, $(1 + x)\frac{dy}{dx} = xy - x + 1$

or, $(1 + x)\frac{dy}{dx} - xy = 1 - x$

Dividing both sides by $(1 + x)$, we get

$\frac{dy}{dx} - \frac{x}{1 + x}\cdot y = \frac{1 - x}{1 + x}$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{-x}{1 + x}$, $Q = \frac{1 - x}{1 + x}$

I.F $= e^{\int Pdx} = e^{\int\frac{-x}{1+x}dx} = e^{\int\frac{1-(1+x)}{1+x}dx} = e^{\int\left(\frac{1}{1+x} - 1\right)dx}$

$= e^{\log(1+x) - x} = e^{\log(x+1)}.\,e^{-x} = (1 + x)\,e^{-x}$

Multiplying both sides of equation (i) by $(1 + x)\,e^{-x}$, we have

$(1 + x)\,e^{-x}\left(\frac{dy}{dx} - \frac{x}{1 + x}y\right) = e^{-x}(1 - x)$

or, $d\{e^{-x}(1 + x)\,y\} = e^{-x}(1 - x)\,dx$

Integrating, we get

$y(1 + x)\,e^{-x} = \int(1 - x)\,e^{-x}\,dx + c$

or, $y(1 + x)\,e^{-x} = \int e^{-x}\,dx - \int x\,e^{-x}\,dx + c$

$= \frac{e^{-x}}{(-1)} - \left[x\int e^{-x}\,dx - \int\left\{\left(\frac{dx}{dx}\right)\int e^{-x}\,dx\right\}dx\right] + c$

$= -e^{-x} - \left[\frac{x\,e^{-x}}{-1} - \int 1.\,\frac{e^{-x}}{-1}\,dx\right] + c$

$= -e^{-x} - \left[-xe^{-x} + \frac{e^{-x}}{(-1)}\right] + c$

$= -e^{-x} + xe^{-x} + e^{-x} + c = xe^{-x} + c$

Dividing both sides by e^{-x}, we get

$\therefore\ y(1 + x) = x + ce^x$

63. 2070 Supp. Q.No. 7 b

Solve: $x^2dy + y(x + y)dx = 0.$ [4]

SOLUTION

Here, $x^2dy + y(x + y)dx = 0$

or, $\frac{dy}{dx} = \frac{-y(x + y)}{x^2}$ ...(i)

Put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) becomes,

$v + x\frac{dv}{dx} = -\frac{vx(x + vx)}{x^2}$

or, $v + x\frac{dv}{dx} = -v - v^2$

or, $x\frac{dv}{dx} = -2v - v^2$

or, $\frac{dv}{-2v - v^2} = \frac{dx}{x}$

or, $\frac{dv}{-2v(v + 1)} = \frac{dx}{x}$

or, $\frac{dv}{v(v + 1)} + \frac{2\,dx}{x} = 0$

or, $\left(\frac{1}{v} - \frac{1}{v + 1}\right)dv + \frac{2\,dx}{x} = 0$

Integrating, we get

$\log v - \log(v + 1) + 2\log x = \log c$

or, $\log\frac{v}{v + 1} + \log x^2 = \log c$

or, $\log\left(\frac{\frac{y}{x}}{\frac{y}{x} + 1}\right) = \log c - \log x^2$

or, $\log\left(\frac{y}{x + y}\right) = \log\frac{c}{x^2}$

or, $\frac{y}{x + y} = \frac{c}{x^2}$

$\therefore\ x^2y = c(x + y)$

64. 2071 Set C Q.No. 7 b

Solve: $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$ [4]

SOLUTION

The given equation is

$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$

Put $y = vx$

then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, the given equation is

$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x^2}$

or $v + x\frac{dv}{dx} = \frac{1+v^2}{2}$

or, $x\frac{dv}{dx} = \frac{1+v^2}{2} - v$

or, $x\frac{dv}{dx} = \frac{v^2-2v+1}{2}$

or, $x\frac{dv}{dx} = \frac{(v-1)^2}{2}$

or, $\frac{2dv}{(v-1)^2} = \frac{dx}{x}$

Integrating, we have,

$\frac{-2}{v-1} = \log x + \log c$

or, $\frac{-2}{\frac{y}{x}-1} = \log cx$

or, $\frac{2x}{x-y} = \log cx$

∴ $2x = (x-y)\log cx$

65. 2071 Set C Q.No. 7 b OR

Solve: $\sin x\frac{dy}{dx} + (\cos x)y = \sin x\cos x$ **[4]**

SOLUTION

Here, $\sin x\frac{dy}{dx} + (\cos x)y = \sin x\cos x$

or, $\frac{dy}{dx} + \frac{\cos x}{\sin x}y = \cos x$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{\cos x}{\sin x}$, $Q = \cos x$

I.F $= e^{\int Pdx} = e^{\int\frac{\cos x}{\sin x}dx} = e^{\log\sin x} = \sin x$

Multiplying both sides of (i) by sin x, we have

$\sin x\frac{dy}{dx} + \cos x.\,y = \sin x.\cos x$

or, $d(y.\sin x) = \frac{1}{2}\sin 2x\,dx$

Integrating, we have

$y\sin x = \int\frac{1}{2}\sin 2x\,dx + c$

or, $y\sin x = \frac{1}{2}\left(\frac{-\cos 2x}{2}\right) + c$

∴ $y\sin x + \frac{1}{4}\cos 2x = c$

66. 2071 Set D Q.No. 7 b

Solve: $xy\frac{dy}{dx} = x^2 + y^2$ **[4]**

➥ Please refer to 2069 (Set A) Q.No. 7b OR

67. 2071 Set D Q.No. 7 b OR

Solve: $\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{1}{(1+x^2)^2}$ **[4]**

SOLUTION

Here, $\frac{dy}{dx} + \frac{2x}{1+x^2}\cdot y = \frac{1}{(1+x^2)^2}$ ...(i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{2x}{1+x^2}$, $Q = \frac{1}{(1+x^2)^2}$

I.F. $= e^{\int Pdx} = e^{\int\frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Multiplying both sides of equation (i) by $(1+x^2)$, we get

$(1+x^2)\frac{dy}{dx} + 2x.y = \frac{1}{1+x^2}$

or, $d(y.1+x^2) = \frac{1}{1+x^2}dx$

Integrating, we have

$y(1+x^2) = \int\frac{1}{1+x^2}dx + c$

$y(1+x^2) = \tan^{-1}x + c$

68. 2071 Old Q.No. 10 a OR

Solve: $\tan x\frac{dy}{dx} + y = \sec x$ **[4]**

➥ Please refer to 2057 Q.No. 11b OR

69. 2071 Supp. Q.No. 7b

Solve $x^2y\,dx = (x^3+y^3)\,dy$

SOLUTION

Here, $x^2y\,dx = (x^3+y^3)\,dy$

or, $\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$

This is homogeneous differential equation

Put $y = vx$

∴ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Hence equation (i) becomes,

$v + x\frac{dv}{dx} = \frac{x^2\cdot vx}{x^3+v^3x^3}$

or, $v + x\frac{dv}{dx} = \frac{v}{1+v^3}$

or, $x\frac{dv}{dx} = \frac{v}{1+v^3} - v$

or, $x\frac{dv}{dx} = \frac{v-v-v^4}{1+v^3}$

or, $x\frac{dv}{dx} = \frac{-v^4}{1+v^3}$

or, $\frac{1+v^3}{v^4}dv = \frac{-dx}{x}$

or, $\left(v^{-4}+\frac{1}{v}\right)dv = -\frac{dx}{x}$

Integrating, we get

$\frac{v^{-3}}{-3} + \log v = -\log x + \log c$

or, $\frac{-1}{3}\cdot\frac{x^3}{y^3} = -\left(\log\frac{y}{x} + \log x - \log c\right)$

or, $\frac{x^3}{3y^3} = \log\left(\frac{y}{c}\right)$

∴ $x^3 = 3y^3\log\left(\frac{y}{c}\right)$

70. 2071 Supp. Q.No. 7b OR

$\frac{dy}{dx} + \frac{y}{x} = \sin x^2$ **[4]**

SOLUTION

$\frac{dy}{dx} + \frac{y}{x} = \sin x^2$ ...(i)

Here, $P = \frac{1}{x}$, $Q = \sin x^2$

I.F $= e^{\int Pdx} = e^{\int\frac{1}{x}dx} = e^{\log x} = x$

Multiplying both sides of (i) by x,

$x\frac{dy}{dx} + y = x\sin x^2$

$d(xy) = x\sin x^2\,dx$

Integrating

$xy = \int x\sin x^2\,dx + c$

$xy = \frac{1}{2}\int 2x\sin x^2\,dx + c$

or, $xy = \frac{-1}{2}\int d(\cos x^2) + c$

or, $xy = \frac{-1}{2}\cos x^2 + c$

∴ $xy + \frac{1}{2}\cos x^2 = c$

71. 2072 Set C Q.No. 7b

Solve: $\frac{dy}{dx} = y\tan x - 2\sin x$ **[4]**

SOLUTION

Given equation is

$\frac{dy}{dx} = y\tan x - 2\sin x$

or, $\frac{dy}{dx} - \tan x\cdot y = -2\sin x$ ...(i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = -\tan x$, $Q = -2\sin x$

I.F $= e^{\int Pdx} = e^{\int -\tan x\,dx} = e^{\int\frac{-\sin x}{\cos x}dx} = e^{\log\cos x} = \cos x$

Multiplying both sides of (i) by cos x, we get,

$\cos x\frac{dy}{dx} - \cos x\cdot\tan x\cdot y = -2\sin x\cos x$

or, $d(y\cdot\cos x) = -\sin 2x\,dx$

Integrating

$y\cos x = \frac{\cos 2x}{2} + c$

∴ $y\cos x = \frac{1}{2}\cos 2x + c$

72. 2072 Set C Q.No. 7b OR

Solve: $xy\frac{dy}{dx} - y^2 = x^2$ **[4]**

➥ Please refer to 2069 (Set A) Q.No. 7b OR

73. 2072 Set D Q.No. 7b

Reduce the equation $\frac{dy}{dx} + \frac{y}{x} = y^2$ in linear form hence solve it. **[4]**

SOLUTION

Given equation is

$\frac{dy}{dx} + \frac{y}{x} = y^2$

Dividing both sides by y^2, we have

$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1$ ...(i)

Put $y^{-1} = z$ then

$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

Then equation (i) can be written as

$\frac{dz}{dx} - \frac{1}{x}z = -1$ ...(ii)

which is a linear equation.

Here, $P = \frac{-1}{x}$, $Q = -1$

I.F $= e^{\int Pdx} = e^{\int\frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Multiplying equation (i) by $\frac{1}{x}$, we get

$\frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = -\frac{1}{x}$

or, $d\left(\frac{1}{x}\cdot z\right) = -\frac{1}{x}dx$

Integrating

$\frac{z}{x} = -\log x + c$

or, $\left(\frac{1/y}{x}\right) = c - \log x$

∴ $\frac{1}{xy} = c - \log x$

74. 2072 Set D Q.No. 7b OR

Solve: $\frac{dy}{dx} = \frac{y+1}{x+y+1}$

SOLUTION

Given equation is $\frac{dy}{dx} = \frac{y+1}{x+y+1}$

This equation is not homogeneous.

Put $y + 1 = z$

Then, $\frac{dy}{dx} = \frac{dz}{dx}$.

Now, given equation can be written as

$\frac{dz}{dx} = \frac{z}{z+x}$ ...(i)

Put, $v = \frac{z}{x}$

⇒ $z = vx$

Then, $\frac{dz}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} = \frac{vx}{vx + x}$

or, $x\frac{dv}{dx} = \frac{v}{v+1} - v$

or, $x\frac{dv}{dx} = \frac{-v^2}{v+1}$

or, $\frac{v+1}{v^2} dv = \frac{-dx}{x}$

or, $\frac{dv}{v} + \frac{dv}{v^2} = \frac{-dx}{x}$

Integrating, we get

$\log v - v^{-1} = -\log x + \log c$

or, $\log v + \log x - \log c = \frac{1}{v}$

or, $\log \frac{vx}{c} = \frac{1}{v}$

or, $\log\left(\frac{\frac{z}{x}\cdot x}{c}\right) = \frac{1}{z/x}$

or, $\log\frac{z}{c} = \frac{x}{z}$

or, $\log\frac{y+1}{c} = \frac{x}{y+1}$

or, $\frac{y+1}{c} = e^{\frac{x}{y+1}}$

$\therefore\ y + 1 = c\,e^{\frac{x}{y+1}}$

75. 2072 Set E Q.No. 7b

Solve: $\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$. [4]

➥ Please refer to 2070 Set C Q.No. 7b OR

76. 2072 Set E Q.No. 7b OR

Solve: $\sin x\frac{dy}{dx} + \cos x\, y = x \sin x$.

➥ Please refer to 2068 Q.No. 11b OR

77. 2072 Supp Q.No. 7b

Solve: $(x + 1)\frac{dy}{dx} + 2y = \frac{e^x}{x+1}$. [4]

SOLUTION

Given equation is $(x + 1)\frac{dy}{dx} + 2y = \frac{e^x}{x+1}$

or, $\frac{dy}{dx} + \frac{2}{x+1}\cdot y = \frac{e^x}{(x+1)^2}$ ...(i)

Comparing (i) with $\frac{dy}{dx} + P\cdot y = Q$, we get

$P = \frac{2}{x+1}$, $Q = \frac{e^x}{(x+1)^2}$

$I.F = e^{\int Pdx} = e^{\int\frac{2}{x+1}dx} = e^{2\log(1+x)} = e^{\log(1+x)^2}$

$= (1 + x)^2$

Multiplying equation (i) both sides by $(1 + x)^2$, we get

$(1 + x)^2\frac{dy}{dx} + (1 + x)^2\cdot\frac{2}{(x+1)}\cdot y = (1 + x)^2\cdot\frac{e^x}{(x+1)^2}$

or, $(1 + x)^2\frac{dy}{dx} + 2(1 + x)y = e^x$

or, $d\{y\cdot(1 + x)^2\} = e^x\,dx$

Integrating, we get

$y(1 + x)^2 = e^x + c$

78. 2072 Supp Q.No. 7b OR

Solve: $\frac{dy}{dx} = \frac{y}{x} - \sin^2\frac{y}{x}$. [4]

➥ Please refer to 2070 Set C Q.No. 7b OR

79. 2073 Set C Q.No. 7b

Solve $x\frac{dy}{dx} + 2y = x^2 \log x$. [4]

SOLUTION

Given, $x\frac{dy}{dx} + 2y = x^2\log x$

Dividing both sides by x, we get

$\frac{dy}{dx} + \frac{2}{x}\cdot y = x\log x$ ...(i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{2}{x}$, $Q = x\log x$

$I.F = e^{\int Pdx} = e^{\int\frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$

Multiplying both sides of (i) by x^2, we get

$x^2\frac{dy}{dx} + x^2\cdot\frac{2}{x}\cdot y = x^2\cdot x\log x$

$d(x^2\cdot y) = x^3\log x\,dx$

Integrating, we get

$x^2y = \int\log x\cdot x^3\,dx$

or, $x^2y = \log x\int x^3\,dx - \int\left[\frac{d}{dx}(\log x)\int x^3\,dx\right]dx$

[Using integration by parts in right side]

or, $x^2y = \log x\cdot\frac{x^4}{4} - \int\frac{1}{x}\cdot\frac{x^4}{4}dx$

or, $x^2y = \frac{x^4}{4}\log x - \frac{1}{4}\int x^3dx$

or, $x^2y = \frac{x^4}{4}\log x - \frac{x^4}{16} + c$

$\therefore\ y = \frac{1}{4}x^2\log x - \frac{1}{16}x^2 + \frac{c}{x^2}$.

80. 2073 Set D Q.No. 7b

Solve: $\frac{dy}{dx} - 2xy = x$. [4]

SOLUTION

Given equation is $\frac{dy}{dx} - 2xy = x$ ...(i)

The given equation is in the form $\frac{dy}{dx} + Py = Q$

where $P = -2x$, $Q = x$

$I.F = e^{\int Pdx} = e^{\int -2x\,dx} = e^{-x^2}$

Multiplying both sides of (i) by $I.F = e^{-x^2}$, we get

$e^{-x^2}\frac{dy}{dx} - 2xy\cdot e^{-x^2} = x\,e^{-x^2}$

$d(y\cdot e^{-x^2}) = xe^{-x^2}\,dx$

$d(y\cdot e^{-x^2}) = d\left(-\frac{1}{2}e^{-x^2}\right)$

Integrating, we get

$ye^{-x^2} = -\frac{1}{2}e^{-x^2} + c$

$\therefore\ y = -\frac{1}{2} + ce^{x^2}$

81. 2073 Set D Q.No. 7b OR

Solve : $x^2\frac{dy}{dx} + y^2 = xy$

SOLUTION

Given equation is $x^2\frac{dy}{dx} + y^2 = xy$

or, $\frac{dy}{dx} + \frac{y^2}{x^2} = \frac{y}{x}$ ...(i)

This is homogeneous differential equation.

So, put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, equation (i) can be written as

$v + x\frac{dv}{dx} + \frac{v^2x^2}{x^2} = \frac{vx}{x}$

or, $v + x\frac{dv}{dx} + v^2 = v$

or, $x\frac{dv}{dx} = -v^2$

or, $-\frac{dv}{v^2} = \frac{dx}{x}$

or, $-v^{-2}\,dv = \frac{dx}{x}$

Integrating, we get

$-\left(\frac{v^{-1}}{-1}\right) = \log x + c$

or, $\frac{1}{v} = \log x + c$

or, $\frac{1}{y/x} = \log x + c$

or, $\frac{x}{y} = \log x + c$

$\therefore\ x = y(\log x + c)$

82. 2073 Supp Q.No. 7b

Define exact differential equation hence solve $\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$. [4]

SOLUTION

Exact Differential Equation: A differential equation of the form $Mdx + Ndy = 0$ where M and N are functions of x or y or both, is said to be exact if there exists a function f(x, y) such that: $Mdx + Ndy = d\,f(x,y)$.

i.e. when $Mdx + Ndy$ is a perfect differential.

Second part:

Given,

$\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$

or, $ydy - xdy + 5dy = ydx - xdx + dx$

or, $xdx + ydy - xdy - ydx - dx + 5dy = 0$

or, $xdx + ydy - (xdy + ydx) - dx + 5dy = 0$

or, $d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) - d(xy) - d(x) + d(5y) = 0$

or, $d\left(\frac{x^2}{2} + \frac{y^2}{2} - xy - x + 5y\right) = 0$

Integrating, we get,

$\frac{x^2}{2} + \frac{y^2}{2} - xy - x + 5y = \frac{C}{2}$

$\therefore\ x^2 + y^2 - 2xy - 2x + 10y = C$

83. 2073 Supp Q.No. 7b or

$\frac{dy}{dx} + 2y\tan x = \sin x$. [4]

SOLUTION

Given equation is $\frac{dy}{dx} + 2y\tan x = \sin x$ ... (i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$P = 2\tan x$, $Q = \sin x$.

Now, $I.F = e^{\int Pdx} = e^{\int 2\tan x\,dx}$

$= e^{2\int\tan x\,dx}$

$= e^{2\log\sec x}$

$= e^{\log\sec^2 x}$

$= \sec^2 x$

Multiplying both sides of (i) by $\sec^2 x$

$\sec^2 x.\frac{dy}{dx} + \sec^2 x.\,2y.\,\tan x = \sec^2 x.\sin x$

or, $d(y.\sec^2 x) = \sec x\tan x\,dx$

Integrating, we get

$y\sec^2 x = \int\sec x\tan x\,dx + c$

$\therefore\ y\sec^2 x = \sec x + c$.

84. 2074 Set A Q.No. 7b

Solve $(xy - x^2)dy = y^2dx$ [4]

SOLUTION

Given, $(xy - x^2)dy = y^2\,dx$

or, $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ ... (i)

The equation (i) is homogeneous, so put $y = vx$

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now, from (i), we can write

$v + x\frac{dv}{dx} = \frac{v^2x^2}{x.vx - x^2}$

or, $v + x\frac{dv}{dx} = \frac{v^2}{v - 1}$

or, $x\frac{dv}{dx} = \frac{v^2}{v-1} - v$

or, $x\frac{dv}{dx} = \frac{v^2 - v(v-1)}{v-1}$

or, $x\frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$

or, $x\frac{dv}{dx} = \frac{v}{v-1}$

or, $\left(\frac{v-1}{v}\right) dv = \frac{dx}{x}$

or, $\left(1 - \frac{1}{v}\right) dv = \frac{dx}{x}$

or, $\frac{dx}{x} = \left(1 - \frac{1}{v}\right) dv$

Integrating,

$\log x = v - \log v + \log c$

or, $\log x + \log v - \log c = v$

or, $\log \frac{vx}{c} = v$

or, $\frac{vx}{c} = e^v$

or, $\frac{y}{x}.x = ce^{y/x}$ [$\because v = y/x$]

$\therefore$ $y = ce^{y/x}$

85. 2074 Set A Q.No. 7b or

$x \ln x \frac{dy}{dx} + y = 2 \ln x.$ **[4]**

SOLUTION

Given, $x \ln x \frac{dy}{dx} + y = 2 \ln x$

Dividing both sides by xlnx,

$\frac{dy}{dx} + \frac{1}{x\ln x}.y = \frac{2}{x}$ ...(i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$P = \frac{1}{x\ln x}, Q = \frac{2}{x}$

Now, I.F $= e^{\int P dx} = e^{\int \frac{1}{x\ln x}dx} = e^{\int \frac{1}{z}dz}$ [put $z = \ln x$]

$= e^{\log z} = z = \ln x.$

Multiplying both sides of (i) by I.F, we get

$\ln x \frac{dy}{dx} + \frac{1}{x}.y = \frac{2}{x}.\ln x$

or, $d(y.\ln x) = \frac{2}{x}\ln x\, dx$

Integrating,

$y.\ln x = \int \frac{2}{x} \ln x dx + c_1$

$= 2\int \frac{\ln x}{x} dx + c_1$ ... (ii)

Let $I = \int \frac{\ln x}{x} dx$

Put $t = \ln x$

$\frac{dt}{dx} = \frac{1}{x}$

$dt = \frac{1}{x} dx$

$\therefore$ $I = \int t dt = \frac{t^2}{2} + c_2 = \frac{(\ln x)^2}{2} + c_2$

Now, from (ii)

$y. \ln x = 2.\frac{(\ln x)^2}{2} + c_1 + c_2$

$y.\ln x = (\ln x)^2 + c$, where $c = c_1 + c_2$

86. 2074 Set B Q.No. 7b

Solve: $(1 + x)\frac{dy}{dx} - xy = 1 - x.$ [4]

➥ Please refer to 2070 (Old) Q. No. 11 b OR

87. 2074 Supp Q.No. 7b

Solve: $x^2\frac{dy}{dx} + y^2 = xy.$ [4]

➥ Please refer to 2073 Set D Q.N. 6b OR

88. 2074 Supp Q.No. 7b OR

Solve: $\cos^2 x\frac{dy}{dx} + y = 1.$ [4]

➥ Please refer to 2058 Q.N. 11b OR

89. 2075 Set A Q.No. 7b

Solve: $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}.$ [4]

➥ Please refer to 2063 Q.No. 11b OR

90. 2075 Set A Q.No. 7b OR

Solve: $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}.$ [4]

SOLUTION

Here,

$\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}$ ... (1)

Comparing equation (1) with $\frac{dy}{dx} + Py = Q$, we get,

$P = \frac{1}{x^2}, Q = \frac{1}{x^3}$

I.F. $= e^{\int P dx} = e^{\int \frac{1}{x^2}dx} = e^{\int x^{-2}dx} = e^{\frac{x^{-1}}{-1}} = e^{-\frac{1}{x}}$

Multiplying both sides of (1) by $e^{-\frac{1}{x}}$, we get

$e^{-\frac{1}{x}}\frac{dy}{dx} + e^{-\frac{1}{x}}.\frac{y}{x^2} = e^{-\frac{1}{x}}.\frac{1}{x^3}$

or, $d\left(e^{-\frac{1}{x}}.y\right) = e^{-\frac{1}{x}}.\frac{1}{x^3}dx$

Integrating, we get,

$y. e^{-\frac{1}{x}} = \int \frac{1}{x^3} e^{-\frac{1}{x}} dx + c$ ... (2)

Let $I = \int \frac{1}{x^3} e^{-\frac{1}{x}} dx$

Put $\frac{1}{x} = t$

$-\frac{1}{x^2} dx = dt$

or, $\frac{1}{x^2} dx = -dt$

Then,

$I = \int t. e^{-t} (-dt)$

$= -\int t.e^{-t} dt$

$= -\left[t\int e^{-t} dt - \int\left(\frac{d(t)}{dt}\int e^{-t} dt\right)dt\right]$

$= -\left[t.\frac{e^{-t}}{-1} - \int 1.\frac{e^{-t}}{-1}dt\right]$

$= -[-te^{-t} + \int e^{-t}dt]$

$= -\left[-te^{-t} + \frac{e^{-t}}{-1}\right]$

$= -[-te^{-t} - e^{-t}]$

$= te^{-t} + e^{-t}$

$= \frac{1}{x}e^{-\frac{1}{x}} + e^{-\frac{1}{x}}$

$= e^{-\frac{1}{x}}\left(\frac{1}{x} + 1\right)$

Then from (2)

$y.e^{-\frac{1}{x}} = e^{-\frac{1}{x}}\left(\frac{1}{x} + 1\right) + c$

$y = \frac{1}{x} + 1 + c\,e^{\frac{1}{x}}$

$xy = 1 + x + c\,xe^{\frac{1}{x}}$

91. 2075 Set B Q.No. 8a

Solve: $\frac{dy}{dx} + \frac{x^2 - y^2}{3xy} = 0$ [4]

SOLUTION

Given,

$\frac{dy}{dx} + \frac{x^2 - y^2}{3xy} = 0$

or, $\frac{dy}{dx} = -\frac{x^2 - y^2}{3xy} = \frac{y^2 - x^2}{3xy}$ ... (1)

This is the homogenous equation. So, put $y = vx$.

Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then (1) becomes,

$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{3x.vx}$

or, $v + x\frac{dv}{dx} = \frac{v^2 - 1}{3v}$

or, $x\frac{dv}{dx} = \frac{v^2 - 1}{3v} - v$

or, $x\frac{dv}{dx} = \frac{v^2 - 1 - 3v^2}{3v}$

or, $x\frac{dv}{dx} = \frac{-1 - 2v^2}{3v}$

or, $\frac{3v}{-1-2v^2}dv = \frac{dx}{x}$

or, $\frac{dx}{x} + \frac{3v}{1 + 2v^2}dv = 0$

or, $\frac{dx}{x} + \frac{3}{4}.\frac{4v}{2v^2 + 1}dv = 0$

Integrating, we get

$\log x + \frac{3}{4}\log(2v^2 + 1) = \log c$

or, $\log x + \log(2v^2 + 1)^{3/4} = \log c$

or, $\log x(2v^2 + 1)^{3/4} = c$

or, $x\left\{2.\left(\frac{y}{x}\right)^2 + 1\right\}^{3/4} = c$

or, $x\left(\frac{x^2 + 2y^2}{x^2}\right)^{3/4} = c$

$\therefore$ $(x^2 + 2y^2)^{3/4} = c\sqrt{x}$

92. 2075 Set B Q.No. 8a OR

$\frac{dy}{dx} + 2y\tan x = \sin x.$ [4]

➥ Please refer to 2073 Supp. Q.No. 7b OR

93. 2075 Set C Q.No. 7b

Solve: $\frac{dy}{dx} = \frac{(y - x)(y + x)}{2xy}.$ [4]

➥ Please refer to 2059 Q.No. 11b OR

94. 2075 Set C Q.No. 7b OR

Solve: $y + (x^2 + 1)\frac{dy}{dx} = e^{\text{arc}(\tan x)}.$ [4]

➥ Please refer to 2066 Q.No. 11 b OR

UNIT 10

DISPERSION, CORRELATION AND REGRESSION

A. DISPERSION

2 MARKS QUESTIONS

1. 2058 Q.No. 4 b

Find the standard deviation of the following data: 100, 150, 200, 250, 300 [2]

SOLUTION

Calculation of S.D

X	$d' = \frac{x - a(200)}{h(50)}$	d'^2
100	−2	4
150	−1	1
200	0	0
250	1	1
300	2	4
	$\Sigma d' = 0$	$\Sigma d'^2 = 10$

Here, n = 5

S.D (σ) = ?

We have, $S.D\ (\sigma) = h \times \sqrt{\frac{\Sigma d'^2}{n} - \left(\frac{\Sigma d'}{n}\right)^2} = 50 \times \sqrt{\frac{10}{5} - \left(\frac{0}{5}\right)^2} = 50 \times \sqrt{2} = 70.71$

2. 2059 Q.No. 4 c

The coefficient of variation and mean of a certain frequency distribution are 50.2% and 22.8 respectively. Find the s.d. [2]

SOLUTION

Here, C.V = 50.2%

Mean $(\bar{x})$ = 22.8

S.D (σ) = ?

We have,

$C.V = \frac{\sigma}{\bar{x}} \times 100$

or, $50.2 = \frac{\sigma}{22.8} \times 100$

or, $\sigma = \frac{50.2 \times 22.8}{100} = 11.45$

∴ σ = 11.45

3. 2060 Q.No. 4 b

Find the mean deviation from median of the numbers 5, 7, 10, 12 and 6. [2]

SOLUTION

Arranging the given data in ascending order

5, 6, 7, 10, 12

∴ n = 5

Median (M_d) = Value of $\left(\frac{n+1}{2}\right)^{th}$ item = Value of $\left(\frac{5+1}{2}\right)^{th}$ item = Value of 3rd item

∴ Median (M_d) = 7

Calculation of M.D from Median

x	$\lvert x - M_d \rvert = \lvert x - 7 \rvert$
5	2
6	1
7	0
10	3
12	5
	$\Sigma \lvert x - M_d \rvert = 11$

M.D from Median $= \frac{\Sigma \lvert x - M_d \rvert}{n} = \frac{11}{5} = 2.2$

4. 2061 Q.No. 4 b

Find the mean deviation from mean of the following data: 6, 8, 10, 13 and 5. [2]

SOLUTION

Given data is: 6, 8, 10, 13 and 5

$\Sigma x = 6 + 8 + 10 + 13 + 5 = 42$

n = 5

Mean $(\bar{x}) = \frac{\Sigma x}{n} = \frac{42}{5} = 8.4$

Calculation of M.D from Mean

x	$\lvert x - \bar{x} \rvert = \lvert x - 8.4 \rvert$
6	2.4
8	0.4
10	1.6
13	4.6
5	3.4
	$\Sigma \lvert x - \bar{x} \rvert = 12.4$

Mean deviation from mean $= \frac{\Sigma \lvert x - \bar{x} \rvert}{n} = \frac{12.4}{5} = 2.48$

5. 2061 Q.No. 12 a

Find out the mean and Standard Deviation from the following data: [2]

Variable	5–10	10–15	15–20	20–25	25–30	30–35
Frequency	2	9	29	54	11	5

SOLUTION

Calculation of Mean and S.D

Profit (in Rs.)	Mid value (x)	f	$d' = \frac{x - a(17.5)}{h(5)}$	fd'	fd'^2
5 - 10	7.5	2	-2	-4	8
10 - 15	12.5	9	-1	-9	9
15 - 20	17.5	29	0	0	0
20 - 25	22.5	54	1	54	54
25 - 30	27.5	11	2	22	44
30 - 35	32.5	5	3	15	45
		N = 110		$\Sigma fd' = 78$	$\Sigma fd'^2 = 160$

Mean $(\bar{x}) = a + \frac{\Sigma fd'}{N} \times h = 17.5 + \frac{78}{110} \times 5 = 21.05$

S.D (σ) $= h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2}$

$= 5 \times \sqrt{\frac{160}{110} - \left(\frac{78}{110}\right)^2}$

$= 5 \times \sqrt{1.45 - 0.5028}$

$= 5 \times \sqrt{0.9472}$

$= 4.87$

6. 2062 Q.No. 4 b

Find the standard deviation of the following data: 10, 15, 20, 25, 30, 35, 40 [2]

SOLUTION

Calculation of S.D

x	x^2
10	100
15	225
20	400
25	625
30	900
35	1225
40	1600
$\Sigma x = 175$	$\Sigma x^2 = 5075$

Here, n = 7

S.D (σ) = ?

We have, S.D(σ) $= \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{5075}{7} - \left(\frac{175}{7}\right)^2} = \sqrt{725 - 625} = \sqrt{100} = 10$

7. 2066 C Q.No. 4 b

The coefficient of variation and mean of a certain frequency distribution are 50.2% and 22.8 respectively. Find the standard deviation. [2]

➥ Please refer to 2059 Q.No. 4c

8. 2066 Q.No. 4 b

Find the standard deviation from the following data: [2]

x	10	11	12	13	14
f	3	12	18	12	2

SOLUTION

Calculation of S.D.

x	f	fx	fx²
10	3	30	300
11	12	132	1,452
12	18	216	2,592
13	12	156	2,028
14	2	28	392
	N = 47	Σfx = 562	Σfx² = 6764

Mean $(\bar{x}) = \frac{\Sigma fx}{N} = \frac{562}{47} = 11.96$

S.D (σ) $= \sqrt{\frac{\Sigma fx^2}{N} - (\bar{x})^2} = \sqrt{\frac{6764}{47} - (11.96)^2} = \sqrt{143.91 - 143.04} = \sqrt{0.87} = 0.93$

9. 2064 Q.No. 4 b

The information about the wages distribution of the firms A and B are given below:

	Firm A	Firm B
No. of workers	500	600
Average monthly wages	Rs. 586	Rs. 575
Variance of wages distribution	81	100

In which firm is the wages distribution uniform? [2]

SOLUTION

For Firm A

n = 500

$\bar{x}$ = Rs.586

$\sigma^2 = 81$

∴ σ = 9

C.V (A) $= \frac{\sigma}{\bar{x}} \times 100 = \frac{9}{586} \times 100 = 1.54\%$

For Firm B

n = 600

$\bar{x}$ = Rs.575

$\sigma^2 = 100$

∴ σ = 10

C.V (B) $= \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{575} \times 100 = 1.74\%$

Since, C.V (A) < C.V (B), the wages distribution is uniform in firm A.

10. 2070 (Old) Q.No. 3 c

Find the coefficient of mean deviation from median of the data 5, 4, 2, 8 and 6. [2]

SOLUTION

Arranging the given data in ascending order

2, 4, 5, 6, 8

Here, n = 5

Median (M_d) = Value of $\left(\frac{n+1}{2}\right)^{th}$ item

= Value of $\left(\frac{5+1}{2}\right)^{th}$ item

= Value of 3rd item

∴ $M_d = 5$

Calculation of Coefficient of M.D from Median

x	$\lvert x - M_d \rvert = \lvert x - 5 \rvert$
2	3
4	1
5	0
6	1
8	3
	$\Sigma \lvert x - M_d \rvert = 8$

M.D from median $= \frac{\Sigma \lvert x - M_d \rvert}{n} = \frac{8}{5} = 1.6$

Coefficient of M.D. from median $= \frac{\text{M.D from median}}{\text{Median}} = \frac{1.6}{5} = 0.32$

11. 2070 Supp. Q.No. 4 b

Calculate the quartile deviation from the data: [2]

15, 7, 25, 12, 4, 22, 19, 10

SOLUTION

Arranging the given data in ascending order

4, 7, 10, 12, 15, 19, 22, 25

Here, n = 8

$Q_1 = \left(\frac{n+1}{4}\right)^{th}$ item $= \left(\frac{8+1}{4}\right)^{th}$ item $= (2.25)^{th}$ item

∴ Q_1 = 2nd item + 0.25 (3rd item − 2nd item)

= 7 + 0.25 (10 − 7) = 7.75

$Q_3 = \left[3\left(\frac{n+1}{4}\right)\right]^{th}$ item $= \left[3\left(\frac{8+1}{4}\right)\right]^{th} = (6.75)^{th}$ item

∴ Q_3 = 6th item + 0.75 (7th item − 6th item)

= 19 + 0.75 (22 − 19) = 21.25

Quartile deviation (Q.D.) $= \frac{Q_3 - Q_1}{2} = \frac{21.25 - 7.75}{2} = 6.75$

12. 2071 Set D Q.No. 4 b

If $n = 10$, $\Sigma x = 120$, $\Sigma x^2 = 1530$, find the standard deviation and the coefficient of variation. [2]

SOLUTION

Given, $n = 10$, $\Sigma x = 120$,

$\Sigma x^2 = 1530$ S.D (σ) = ?

Coefficient of variation (C.V.) = ?

We have, $S.D\ (\sigma) = \sqrt{\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2} = \sqrt{\frac{1530}{10} - \left(\frac{120}{10}\right)^2} = \sqrt{153 - 144} = 3$

$A.M\ (\bar{x}) = \frac{\Sigma x}{n} = \frac{120}{10} = 12$

$C.V. = \frac{\sigma}{\bar{x}} \times 100 = \frac{3}{12} \times 100 = 25\%$

13. 2071 Old Q.No. 4 b

Find the mean deviation of the data 10, 5, 6, 12, 7 from median. [2]

➡ Please refer to 2060 Q.No. 4b

14. 2071 Supp. Q.No. 4a

The mean of two samples of size 50 and 100 are 54.1 and 50.3 respectively and the standard deviations are 8 and 7 respectively. Obtain the mean and the standard deviation of the sample of size 150 obtained by combining the two samples. [2]

SOLUTION

Given, $n_1 = 50$, $n_2 = 100$

$\bar{x}_1 = 54.1$, $\bar{x}_2 = 50.3$

$\sigma_1 = 8$, $\sigma_2 = 7$

Combined mean $(\bar{x}) = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{50 \times 54.1 + 100 \times 50.3}{50 + 100} = 51.57$

Again,

We have, Combined S.D $(\sigma) = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$

Where, $d_1 = 54.1 - 51.57 = 2.53$

$d_2 = 50.3 - 51.57 = -1.27$

$\therefore \sigma = \sqrt{\frac{50\{8^2 + (2.53)^2\} + 100\{7^2 + (-1.27)^2\}}{50 + 100}}$

$= \sqrt{\frac{3520.05 + 5{,}061.29}{150}} = \sqrt{57.21} = 7.56$

15. 2072 Set C Q.No. 4b

The information about the daily temperature of two cities X and Y are as follows: [2]

	X	Y
Average temp. (°F)	84	92
Variance of temp.	16	25

Determine which city has greater consistency in climate.

SOLUTION

City X

Average $(\bar{X}) = 84$

Variance $(\sigma^2) = 16$

$\therefore \sigma = 4$

$C.V(X) = \frac{\sigma}{\bar{X}} \times 100 = \frac{4}{84} \times 100 = 4.76\%$

City Y

Average $(\bar{Y}) = 92$

Variance $(\sigma^2) = 25$

$\therefore \sigma = 5$

$C.V(Y) = \frac{\sigma}{\bar{Y}} \times 100 = \frac{5}{92} \times 100 = 5.43\%$

Since C.V (X) < C.V (Y), the city X has greater consistency in climate.

16. 2072 Set D Q.No. 4b

In the distribution of data 20, 25, 30, 36, 32, 43; find standard deviation. [2]

SOLUTION

Calculation of Standard deviation

X	X²
20	400
25	625
30	900
36	1296
32	1024
43	1849
ΣX = 186	ΣX² = 6094

Here, $n = 6$

S.D (σ) = ?

We have,

$S.D\ (\sigma) = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2} = \sqrt{\frac{6094}{6} - \left(\frac{186}{6}\right)^2} = \sqrt{1015.67 - 961} = \sqrt{54.67} = 7.39$

17. 2073 Supp Q.No. 4b

In the distribution of two sets of data, which of the distribution is consistent? [2]

	Distribution X	Distribution Y
AM	100	90
S.D.	10	18

SOLUTION

Distribution X

A.M = 100

S.D = 10

$C.V\ (X) = \frac{S.D}{A.M} \times 100 = \frac{10}{100} \times 100 = 10$

Distribution Y

A.M = 90

S.D = 18

$C.V(Y) = \frac{S.D}{A.M} \times 100 = \frac{18}{90} \times 100 = 20$

Since C.V (X) < C.V (Y), distribution X is consistent.

18. 2074 Supp Q.No. 4b

Following are the information about the marks of two students A and B.

	A	B
Average marks	84	92
Variance of marks	16	25

Examine who has got the uniform mark. [2]

SOLUTION

City A

Average $(\bar{X}) = 84$

Variance $(\sigma^2) = 16$

$\therefore \sigma = 4$

$C.V.(A) = \frac{\sigma}{\bar{X}} \times 100 = \frac{4}{84} \times 100 = 4.76\%$

City B

Average $(\bar{Y}) = 92$

Variance $(\sigma^2) = 25$

$\therefore \sigma = 5$

$C.V.(B) = \frac{\sigma}{\bar{Y}} \times 100 = \frac{5}{92} \times 100 = 5.43\%$

Since C.V. (A) < C.V. (B), the city A has greater uniform mark.

19. 2075 Set B Q.No. 4a

Find the mean deviation from mean of the data: 6, 8, 10, 13, 5. [2]

➥ Please refer 2061 Q.No. 4b

20. 2075 Set C Q.No. 4b

If total items (n) = 10, sum of items $(\Sigma x) = 120$ and the sum of square of items $(\Sigma x^2) = 1530$, find the standard deviation and coefficient of variation. [2]

➥ Please refer to 2071 Set D Q.No. 4b

4 MARKS QUESTIONS

21. 2060 Q.No. 12 a

Find the mean and S.D. from the following table: [4]

Wages (Rs.)	10–20	10–30	10–40	10–50	10–60
No. of workers	15	33	63	83	100

SOLUTION

Calculation of Mean and S.D

Wages (Rs.)	Mid value (x)	No. of workers(f)	$d' = \frac{x - a(35)}{h(10)}$	fd'	fd'²
10 - 20	15	15	-2	-30	60
20 - 30	25	33 – 15 = 18	-1	-18	18
30 - 40	35	63 – 33 = 30	0	0	0
40 - 50	45	83 – 63 = 20	1	20	20
50 - 60	55	100– 83 = 17	2	34	68
		N = 100		Σfd' = 6	Σfd'² = 166

$$\text{Mean } (\bar{x}) = a + \frac{\Sigma fd'}{N} \times h = 35 + \frac{6}{100} \times 10 = 35.6$$

$$\text{S.D } (\sigma) = h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} = 10 \times \sqrt{\frac{166}{100} - \left(\frac{6}{100}\right)^2}$$

$$= 10 \times \sqrt{1.66 - 0.0036} = 10 \times \sqrt{1.6564} = 12.87$$

22. 2067 Q.No. 12b

Define standard deviation. Also prove that the root mean square deviation is not less than the standard deviation. [4]

SOLUTION

Standard deviation

Standard deviation is defined as the positive square root of the mean of the square of the deviations taken from the arithmetic mean. It is denoted by Greek letter σ (sigma), it is given by

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})}{n}} \text{ for individual series}$$

$$\text{and } \sigma = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{n}} \text{ for continuous and discrete series,}$$

Next part

We know that root mean square deviation is defined by:

$$s = \sqrt{\frac{\Sigma f(x - a)^2}{n}}, \text{ where a is an arbitrary number.}$$

Now,

$$s^2 = \frac{1}{N}\Sigma f(x - a)^2 = \frac{1}{N}\Sigma f[(x - \bar{x}) + (\bar{x} - a)]^2 = \frac{1}{N}\Sigma f(x - \bar{x})^2 + 2(\bar{x} - a)\frac{1}{N}\Sigma f(x - \bar{x}) + (\bar{x} - a)^2\frac{1}{N}\Sigma f$$

$$= \sigma^2 + 2(\bar{x} - a)\left[\frac{1}{N}\Sigma fx - \bar{x}\frac{1}{N}\Sigma f\right] + (\bar{x} - a)^2 \cdot \frac{1}{N} \cdot N = \sigma^2 + 2(\bar{x} - a)(\bar{x} - \bar{x}) + (\bar{x} - a)^2$$

$$= \sigma^2 + (\bar{x} - a)^2$$

Since, $(\bar{x} - a)^2 \geq 0$, we have

$s^2 \geq \sigma^2$

$\Rightarrow s \geq \sigma$

Hence, the root mean square deviation is not less than the standard deviation.

23. 2068 Q.No. 12 a

Weights of a group of individuals are given below. Find out the mean and the standard deviation. [4]

Weight (in kg)	0-10	10-20	20-30	30-40	40-50
Frequency	12	33	30	15	10

SOLUTION

Calculation of Mean and S.D

Weight (in kg)	Mid value (x)	Frequency (f)	$d' = \frac{x - a(25)}{h(10)}$	fd'	fd'²
0 - 10	5	12	-2	-24	48
10 - 20	15	33	-1	-33	33
20 - 30	25	30	0	0	0
30 - 40	35	15	1	15	15
40 - 50	45	10	2	20	40
		N = 100		Σfd' = –22	Σfd'² = 136

$$\text{Mean } (\bar{x}) = a + \frac{\Sigma fd'}{N} \times h = 25 + \frac{(-22)}{100} \times 10 = 25 - 2.2 = 22.8$$

$$\text{S.D } (\sigma) = h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} = 10 \times \sqrt{\frac{136}{100} - \left(\frac{-22}{100}\right)^2} = 10 \times \sqrt{1.36 - 0.0484} = 10 \times \sqrt{1.3116} = 11.45$$

24. 2069 (Set A) Q.No. 8a

Determine the standard deviation and the coefficient of variation from the following distribution. [4]

Profit (in Rs.)	0–10	10–20	20–30	30–40	40–50
No. of shops	8	13	16	8	5

SOLUTION

Calculation of S.D and C.V

Profit (in Rs.)	Mid value (x)	No. of shops (f)	$d' = \frac{x - a(25)}{h(10)}$	fd'	fd'²
0 - 10	5	8	-2	-16	32
10 - 20	15	13	-1	-13	13
20 - 30	25	16	0	0	0
30 - 40	35	8	1	8	8
40 - 50	45	5	2	10	20
		N = 50		Σfd' = -11	Σfd'² = 73

$$\text{A.M } (\bar{x}) = a + \frac{\Sigma fd'}{N} \times h = 25 + \frac{(-11)}{50} \times 10 = 25 - 2.2 = \text{Rs.}22.8$$

$$\text{S.D } (\sigma) = h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} = 10 \times \sqrt{\frac{73}{50} - \left(\frac{-11}{50}\right)^2}$$

$$= 10 \times \sqrt{1.46 - 0.0484} = 10 \times \sqrt{1.4116} = \text{Rs.}11.88$$

$$\text{C.V} = \frac{\sigma}{\bar{x}} \times 100 = \frac{\text{Rs.}11.88}{\text{Rs.}22.8} \times 100 = 52.11\%$$

25. 2069 (Set A) Old Q.No. 12a

Find the mean and the standard deviation from the following data. [4]

Marks	0–10	10–20	20–30	30–40	40–50
No. of students	5	8	15	16	6

SOLUTION

Calculation of Mean and S.D

Marks	Mid value (x)	No. of students(f)	$d' = \frac{x - a(25)}{h(10)}$	fd'	fd'²
0 - 10	5	5	-2	-10	20
10 - 20	15	8	-1	-8	8
20 - 30	25	15	0	0	0
30 - 40	35	16	1	16	16
40 - 50	45	6	2	12	24
		N = 50		Σfd' = -10	Σfd'² = 68

$$\text{Mean } (\bar{x}) = a + \frac{\Sigma fd'}{N} \times h = 25 + \frac{10}{50} \times 10$$

$$\text{S.D } (\sigma) = h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} = 10 \times \sqrt{\frac{68}{50} - \left(\frac{10}{50}\right)^2} = 10 \times \sqrt{1.36 - 0.04} = 10 \times \sqrt{1.32} = 11.49$$

26. 2069 Old (Set B) Q.No. 12b

Find out the mean and standard deviation from the following data: [4]

x	10	11	12	13	14
f	3	12	18	12	2

SOLUTION

Calculation of Mean and S.D.

x	f	fx	fx²
10	3	30	300
11	12	132	1,452
12	18	216	2,592
13	12	156	2,028
14	2	28	392
	N = 47	Σfx = 562	Σfx² = 6764

$$\text{Mean } (\bar{x}) = \frac{\Sigma fx}{N} = \frac{562}{47} = 11.96$$

$$\text{S.D } (\sigma) = \sqrt{\frac{\Sigma fx^2}{N} - (\bar{x})^2} = \sqrt{\frac{6764}{47} - (11.96)^2} = \sqrt{143.91 - 143.04} = \sqrt{0.87} = 0.93$$

27. 2074 Set B Q.No. 8a

Following are the marks obtained by the two students in 6 tests.

A	56	72	48	69	64	81
B	63	74	45	57	82	63

Which of the student will get performance award for the consistency in tests? [4]

SOLUTION

Student A		Student B	
Marks (X)	X²	Marks (Y)	Y²
56	3136	63	3969
72	5184	74	5476
48	2304	45	2025
69	4761	57	3249
64	4096	82	6724
81	6561	63	3969
ΣX = 390	ΣX² = 26,042	ΣY = 384	ΣY² = 25,412

Here, n = 6

For Student A

$$\text{A. M } (\overline{X}) = \frac{\Sigma X}{n} = \frac{390}{6} = 65$$

$$\text{S.D} = \sqrt{\frac{\Sigma X^2}{n} - (\overline{X})^2} = \sqrt{\frac{26042}{6} - (65)^2} = \sqrt{4340.33 - 4225} = \sqrt{115.33} = 10.74$$

$$\text{C.V (A)} = \frac{\text{S.D}}{\text{Mean}} \times 100 = \frac{10.74}{65} \times 100 = 16.52\ \%$$

For Student B

$$\text{A.M } (\overline{Y}) = \frac{\Sigma Y}{n} = \frac{384}{6} = 64$$

$$\text{S.D} = \sqrt{\frac{\Sigma Y^2}{n} - (\overline{Y})^2} = \sqrt{\frac{25412}{6} - (64)^2} = \sqrt{4235.33 - 4096} = \sqrt{139.33} = 11.8$$

$$\text{C.V(B)} = \frac{\text{S.D}}{\text{Mean}} \times 100 = \frac{11.8}{64} \times 100 = 18.44\%$$

Since, C.V (A) < C.V (B), A will get performance award for the consistency tests.

B. SKEWNESS

2 MARKS QUESTIONS

28. 2063 Q.No. 4 b

Consider the following distribution.

	Distribution a	Distribution b
Arithmetic mean:	100	90
Median:	90	80
Standard deviation	10	10

Is the distribution A same as the distribution B regarding the skewness? [2]

SOLUTION

For distribution A

$\bar{x} = 100$, $M_d = 90$, $\sigma = 10$

We have, $M_0 = 3M_d - 2\bar{x} = 3 \times 90 - 2 \times 100 = 70$

$$\text{C.V(A)} = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{100} \times 100 = 10\%$$

$$S_k(A) = \frac{\bar{x} - M_0}{\sigma} = \frac{100 - 70}{10} = 3$$

For distribution B

$\bar{x} = 90$, $M_d = 80$, $\sigma = 10$

We have, $M_0 = 3M_d - 2\bar{x} = 3 \times 80 - 2 \times 90 = 240 - 180 = 60$

$$\text{C.V(B)} = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{90} \times 100 = 11.11\%$$

$$S_k(B) = \frac{\bar{x} - M_0}{\sigma} = \frac{90 - 60}{10} = 3$$

Since C.V(B) > C.V(A), the degree of variation in distribution B is greater that that of A but $S_k(A) = S_k(B)$ i.e. distribution A is same as the distribution B regarding to the skewness.

29. 2067 Q.No. 4b

In a frequency distribution of a set of data C.V. = 5%, σ = 2 and karl Pearson coefficient of skewness = 0.5; find the mean of the data. [2]

SOLUTION

Here, C.V = 5%

σ = 2

Karl Pearson coefficient of skewness $S_k(P) = 0.5$

Mean $(\bar{x})$ = ?

We have,

$$\text{C.V} = \frac{\sigma}{\bar{x}} \times 100$$

or, $5 = \frac{2}{\bar{x}} \times 100$

or, $5\bar{x} = 200$

∴ $\bar{x} = 40$

30. 2071 Supp. Q.No. 4c

In a distribution, the difference of the two quartiles is 20 and their sum is 70 and the median is 36. Find the coefficient of skewness. [2]

SOLUTION

Given, $Q_3 - Q_1 = 20$

$Q_3 + Q_1 = 70$

$M_d = 36$

Coefficient of skewness S_k (B) = ?

We have,

$$S_k(B) = \frac{Q_3 + Q_1 - 2M_d}{Q_3 - Q_1} = \frac{70 - 2 \times 36}{20} = \frac{-2}{20} = -0.1$$

31. 2072 Set E Q.No. 4b

For a group of 50 items; circle $\Sigma x^2 = 600$, $\Sigma x = 150$ and $M_0 = 1.75$, find the Pearsonian coefficient of skewness.

SOLUTION

Given, $n = 50$, $\Sigma x = 150$, $\Sigma x^2 = 600$, $M_0 = 1.75$

We know that $(\bar{x}) = \frac{\Sigma x}{n} = \frac{150}{50} = 3$

S.D $(\sigma) = \sqrt{\frac{\Sigma x^2}{n} - (\bar{x})^2} = \sqrt{\frac{600}{50} - 3^2} = \sqrt{12-9} = 1.73$

Pearsonian coefficient of skewness

$S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{3 - 1.75}{1.73} = 0.72$

32. 2073 Set C Q.No. 4b

A frequency distribution gives the following results. C.V. = 5%, Mean = 40 and Mode = 39. Calculate Karl Pearson's coefficient of skewness of the distribution.

SOLUTION

Mean $(\bar{x}) = 40$

Mode $(M_0) = 39$

C.V = 5%

Karl Pearson's coefficient of skewness $(S_k(P)) = ?$

We have,

C.V $= \frac{S.D}{Mean} \times 100$

or, $5 = \frac{S.D}{40} \times 100$

or, S.D $= \frac{5 \times 40}{100} = 2$

$\therefore$ S.D $(\sigma) = 2$

Again, we have, $S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{40-39}{2} = 0.5$

33. 2074 Set A Q.No. 4b

Find Skewness and C.V. if mean, median and S.D. are respectively 56.80, 59.50 and 12.40.

SOLUTION

Given, mean $(\bar{x}) = 56.80$

Median $(M_d) = 59.50$

S.D $(\sigma) = 12.40$

We have, coefficient of skewness $(S_k) = \frac{3(\bar{x} - M_d)}{\sigma} = \frac{3(56.80 - 59.50)}{12.40} = -0.653$

Again, we have, C.V $= \frac{S.D}{Mean} \times 100 = \frac{12.4}{56.8} \times 100 = 21.83\%$

34. 2074 Set B Q.No. 4b

The C.V, S.D. and mode of a distribution are 5%, 2 and 39 respectively. Calculate the Pearson's coefficient of Skewness of the distribution.

SOLUTION

Given

C.V. = 5%

S.D = 2

Mode = 39

We have, C.V $= \frac{S.D}{Mean} \times 100$

or, $5 = \frac{2}{Mean} \times 100$

or, 5 Mean = 200

$\therefore$ Mean $= \frac{200}{5} = 40$

Coefficient of Skewness $(S_K) = \frac{Mean - Mode}{S.D} = \frac{40-39}{2} = 0.5$

35. 2075 Set A Q.No. 4b

For a group of 50 items, $\Sigma x^2 = 600$, $\Sigma x = 150$ and $M_o = 1.75$, find the pearsonian coefficient of skewness. [2]

- Please refer to 2072 Set E Q.No. 4b

36. 2075 Set B Q.No. 4b

Find the Pearson's coefficient of skewness when $\Sigma x = 735$, $\Sigma x^2 = 28730$, mode = 35.25, n = 20. [2]

SOLUTION

Given,

$\Sigma x = 735$

$\Sigma x^2 = 28730$

Mode $(M_0) = 35.25$

n = 20

Mean $(\bar{x}) = \frac{\Sigma x}{n} = \frac{735}{20} = 36.75$

S.D $(\sigma) = \sqrt{\frac{\Sigma x^2}{n} - (\bar{x})^2} = \sqrt{\frac{28730}{20} - (36.75)^2} = \sqrt{1436.5 - 1050.5625} = \sqrt{85.9375} = 9.27$

Pearson's coefficient of skewness,

$S_k(P) = \frac{\bar{X} - M_0}{\sigma} = \frac{36.75 - 35.25}{9.27} = 0.16$

4 MARKS QUESTIONS

37. 2061 Q.No. 12 b

The median, mode and coefficient of skewness for a certain distribution are respectively 17.4, 15.3 and 0.35. Calculate mean and C.V. [4]

SOLUTION

Given,

Median $(M_d) = 17.4$

Mode $(M_0) = 15.3$

Coefficient of skewness $(S_k) = 0.35$

Mean $(\bar{x}) = ?$

C.V = ?

We have,

$M_0 = 3M_d - 2\bar{x}$

or, $15.3 = 3 \times 17.4 - 2\bar{x}$

or, $2\bar{x} = 52.2 - 15.3$

or, $2\bar{x} = 36.9$

$\therefore \bar{x} = \frac{36.9}{2} = 18.45$

Again, we have

$S_k = \frac{\bar{x} - M_0}{\sigma}$

or, $0.35 = \frac{18.45 - 15.3}{\sigma}$

or, $0.35\sigma = 3.15$

or, $\sigma = \frac{3.15}{0.35} = 9$

C.V $= \frac{\sigma}{\bar{x}} \times 100 = \frac{9}{18.45} \times 100 = 48.78\%$

38. 2064 Q.No. 12 a

For a group of 10 items, $\Sigma x = 452$, $\Sigma x^2 = 24,270$ and mode = 43.7, find the Pearson's coefficient of skewness. **[4]**

SOLUTION

Given, $n = 10$ $\Sigma x = 452$

$\Sigma x^2 = 24270$ Mode $(M_0) = 43.7$

Mean $(\bar{x}) = \frac{\Sigma x}{n} = \frac{452}{10} = 45.2$

S.D $(\sigma) = \sqrt{\frac{\Sigma x^2}{n} - (\bar{x})^2} = \sqrt{\frac{24270}{10} - (45.2)^2} = \sqrt{2427 - 2,043.04} = \sqrt{383.96} = 19.59$

Pearsonian coefficient of skewness

$S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{45.2 - 43.7}{19.59} = 0.077$

39. 2065 Q.No 12 a

Calculate Karl Pearson's coefficient of skewness of the data: **[4]**

Marks	above 0	above 10	above 20	above 30	above 40
Frequency	150	140	100	80	80

SOLUTION

Calculation of Karl Pearson's Coefficient of Skewness

Marks	Mid value (x)	f	$d' = \frac{X - a(25)}{h(10)}$	fd'	fd'^{2}
0 – 10	5	150 – 140 = 10	–2	–20	40
10 – 20	15	140 – 100 = 40	–1	–40	40
20 – 30	25	100 – 80 = 20	0	0	0
30 – 40	35	80 – 80 = 0	1	0	0
40 – 50	45	80	2	160	320
		N = 150		$\Sigma fd' = 100$	$\Sigma fd'^2 = 400$

Mean $\bar{x} = a + \frac{fd'}{N} \times h = 25 + \frac{100}{150} \times 10 = 31.67$

S.D $(\sigma) = h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} = 10 \times \sqrt{\frac{400}{150} - \left(\frac{100}{150}\right)^2} = 10 \times \sqrt{2.67 - 0.44} = 10 \times \sqrt{2.23} = 14.93$

Here, the highest frequency is 80. So, modal class is (40 – 50)

$\therefore$ $l = 40, f_1 = 80, f_0 = 0, f_2 = 0, h = 10$

Mode $(M_0) = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h = 40 + \frac{80 - 0}{2 \times 80 - 0 - 0} \times 10 = 45$

Karl Pearson's coefficient of skewness

$S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{31.67 - 45}{14.93} = -0.89$

40. 2066 Q.No.12 b

Calculate the coefficient of skewness from the following frequency distribution: **[4]**

Investment	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Frequency	12	18	20	15	10	3	2

SOLUTION

Calculation of Coefficient of Skewness

Investment	Mid-value (x)	f	$d' = \frac{x - a(45)}{h(10)}$	fd'	fd'^{2}
10 - 20	15	12	-3	-36	108
20 - 30	25	18	-2	-36	72
30 - 40	35	20	-1	-20	20
40 - 50	45	15	0	0	0
50 - 60	55	10	1	10	10
60 - 70	65	3	2	6	12
70 - 80	75	2	3	6	18
		N = 80		$\Sigma fd' = -70$	$\Sigma fd'^2 = 240$

Mean $(\bar{x}) = a + \frac{\Sigma fd'}{N} \times h = 45 + \frac{(-70)}{80} \times 10 = 36.25$

S.D $(\sigma) = h \times \sqrt{\frac{\Sigma fd'^2}{N} - \left(\frac{\Sigma fd'}{N}\right)^2} = 10 \times \sqrt{\frac{240}{80} - \left(\frac{-70}{80}\right)^2} = 10 \times \sqrt{3 - 0.7656} = 10 \times \sqrt{2.2344} = 14.95$

Here, the highest frequency is 20. The corresponding class to this frequency is (30 – 40).

So modal class is 30 – 40.

$l = 30, f_1 = 20, f_0 = 18, f_2 = 15, h = 10$

Mode $(M_0) = l + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times h = 30 + \frac{20 - 18}{2 \times 20 - 18 - 15} \times 10 = 30 + \frac{2}{7} \times 10 = 32.86$

Coefficient of skewness, $S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{36.25 - 32.86}{14.95} = 0.227$

41. 2068 Q.No. 12 b

Consider the following distribution:

	Distribution A	Distribution B
Arithmetic mean	100	90
Median	90	80
Standard deviation	10	10

Is the distribution A same as the distribution B regarding the degree of variation and skewness? **[4]**

SOLUTION

For distribution A

$\bar{x} = 100, M_d = 90, \sigma = 10$

We have, $M_0 = 3M_d - 2\bar{x} = 3 \times 90 - 2 \times 100 = 70$

$C.V(A) = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{100} \times 100 = 10\%$

$S_k(A) = \frac{\bar{x} - M_0}{\sigma} = \frac{100 - 70}{10} = 3$

For distribution B

$\bar{x} = 90, M_d = 80, \sigma = 10$

We have, $M_0 = 3M_d - 2\bar{x} = 3 \times 80 - 2 \times 90 = 240 - 180 = 60$

$C.V(B) = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{90} \times 100 = 11.11\%$

$S_k(B) = \frac{\bar{x} - M_0}{\sigma} = \frac{90 - 60}{10} = 3$

Since C.V(B) > C.V(A), so the degree of variation in distribution B is greater that that of A but $S_k(A) = S_k(B)$ i.e. distribution A is same as the distribution B regarding to the skewness.

42. 2069 (Set A) Old Q.No. 12 b

For a group of 10 items, $\Sigma x = 452$, $\Sigma x^2 = 24270$ and mode = 43.7, find the Pearsonian coefficient of Skewness. **[4]**

- Please refer to 2064 Q.No. 12a

43. 2069 (Set B) Q.No. 8a

If $\Sigma fx = 110$, $\Sigma fx^2 = 1650$, N = 10 and $M_0 = 12.45$, find the skewness based on mean, mode and standard deviation. **[4]**

SOLUTION

Given, $\Sigma fx = 110$

$\Sigma fx^2 = 1650$

$N = 10$

$M_0 = 12.45$

Mean $(\bar{x}) = \frac{\Sigma fx}{N} = \frac{110}{10} = 11$

S.D $(\sigma) = \sqrt{\frac{\Sigma fx^2}{N} - (\bar{x})^2} = \sqrt{\frac{1,650}{10} - 11^2} = \sqrt{165 - 121} = \sqrt{44} = 6.63$

Coefficient of skewness $S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{11 - 12.45}{6.63} = -0.22$

44. 2070 Set C Q.No. 8 a

If $\sum fx = 110$, $\sum fx^2 = 1650$, N = 10 and M_o = 12.45, find the skewness based on mean, mode and standard deviation. [4]

- Please refer to 2069 Set B Q.No. 8a

45. 2070 Set D Q.No. 8 a

Consider the following distribution.

	Distribution A	Distribution B
Arithmetic mean	100	90
Median	90	80
Standard deviation	10	10

Is the distribution A same as the distribution B regarding the degree of variation and skewness? [4]

SOLUTION

For distribution A

$\bar{x} = 100$, $M_d = 90$, $\sigma = 10$

We have, $M_o = 3M_d - 2\bar{x} = 3 \times 90 - 2 \times 100 = 70$

$C.V(A) = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{100} \times 100 = 10\%$

$S_k(A) = \frac{\bar{x} - M_0}{\sigma} = \frac{100 - 70}{10} = 3$

For distribution B

$\bar{x} = 90$, $M_d = 80$, $\sigma = 10$

We have, $M_o = 3M_d - 2\bar{x} = 3 \times 80 - 2 \times 90 = 240 - 180 = 60$

$C.V(B) = \frac{\sigma}{\bar{x}} \times 100 = \frac{10}{90} \times 100 = 11.11\%$

$S_k(B) = \frac{\bar{x} - M_0}{\sigma} = \frac{90 - 60}{10} = 3$

Since C.V(B) > C.V(A), so the degree of variation in distribution B is greater that that of A but $S_k(A) = S_k(B)$ i.e. distribution A is same as the distribution B regarding to the skewness.

46. 2070 Supp. Q.No. 8 a

Find Karl Pearson's coefficient of skewness from the given data [4]

Income	10	12	14	16	20
Frequency	5	8	15	7	5

SOLUTION

Calculation of Karl Pearson's Coefficient of Skewness

Income (x)	Frequency (f)	fx	fx²
10	5	50	500
12	8	96	1,152
14	15	210	2,940
16	7	112	1,792
20	5	100	2,000
	N = 40	Σfx = 568	Σfx² = 8384

Mean $(\bar{x}) = \frac{\Sigma fx}{N} = \frac{568}{40} = 14.2$

Here, the highest frequency is 15, so mode $(M_0) = 14$

$S.D (\sigma) = \sqrt{\frac{\Sigma fx^2}{N} - \left(\frac{\Sigma fx}{N}\right)^2} = \sqrt{\frac{8384}{40} - \left(\frac{568}{40}\right)^2} = \sqrt{209.6 - 201.64} = \sqrt{7.96} = 2.82$

Karl Pearson's coefficient of skewness $S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{14.2 - 14}{2.82} = \frac{0.2}{2.82} = 0.071$

47. 2071 Set C Q.No. 8 a

Calculate the coefficient of Skewness based on mean, mode and standard deviation from the following data. [4]

Wages (in Rs.)	100	110	120	130	140
No. of persons	2	6	10	8	4

SOLUTION

Calculation of Coefficient of Skewness

Wages (x) (in Rs)	No. of persons (f)	fx	fx²
100	2	200	20,000
110	6	660	72,600
120	10	1,200	144,000
130	8	1,040	135,200
140	4	560	78,400
	N = 30	Σfx = 3660	Σfx² = 450200

Mean $(\bar{x}) = \frac{\Sigma fx}{N} = \frac{3660}{30} = 122$

Here the highest frequency is 10, mode $(M_0) = 120$

$S.D (\sigma) = \sqrt{\frac{\Sigma fx^2}{N} - (\bar{x})^2} = \sqrt{\frac{450200}{30} - 122^2} = \sqrt{15006.67 - 14884} = \sqrt{122.67} = 11.08$

Coefficient of skewness $S_k(P) = \frac{\bar{x} - M_0}{\sigma} = \frac{122 - 120}{11.08} = 0.18$

48. 2072 Supp Q.No. 8a

Calculate the coefficient of skewness based on mean, mode and the standard deviation from the following data: [4]

Wages (in Rs.)	100	110	120	130	140
No. of person	2	6	10	8	4

- Please refer to 2071 Set C Q.No. 8a

49. 2073 Set D Q.No. 8a

If $\Sigma fx = 110$, $\Sigma fx^2 = 1650$, N = 10 and $M_0 = 12.45$, find Karl Pearson's coefficient of skewness. [4]

- Please refer to 2069 Set B Q.No. 8a

C. CORRELATION

2 MARKS QUESTIONS

50. 2057 Q.No. 4 b

Calculate r_{xy} if $\Sigma x^2 = 114$, $\Sigma y^2 = 442$; $\Sigma xy = 174$. [2]

SOLUTION

Here, $\Sigma x^2 = 114$ $\Sigma y^2 = 422$ $\Sigma xy = 174$

Correlation coefficient (r) = ?

We have, $r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}} = \frac{174}{\sqrt{114}\sqrt{422}} = 0.79$

51. 2065 Q.No 4 b

If the covariance between the variable x and y is 18 and the variances of x and y are 16 and 81 respectively, find the coefficient of correlation between them. [2]

SOLUTION

Given, Covariance between X and Y = Cov(X,Y) = 18

Variance of X = var (X) = 16

Variance of Y = var (Y) = 81

Coefficient of correlation (r) = ?

We have, $r = \frac{cov(X, Y)}{\sqrt{var(X)}\sqrt{var(Y)}} = \frac{18}{\sqrt{16}\sqrt{81}} = \frac{18}{4 \times 9} = 0.5$

52. 2068 Q.No. 4b

Calculate the correlation coefficient between two variables from the following data: [2]

$\Sigma x^2 = 114$, $\Sigma y^2 = 422$ and $\Sigma xy = 174$

- Please refer to 2057 Q.No. 4b

53. 2069 (Set A) Old Q.No. 4b

If the covariance between the two variables x and y is 18, and the variances of x and y are 16 and 81 respectively, find the coefficient of correlation between them. [2]

- Please refer to 2065 Q.No. 4b

54. 2070 Set C Q.No. 4 b

If $\Sigma(X-\overline{X})^2 = 40$, $\Sigma(Y-\overline{Y})^2 = 63$ and $\Sigma(X-\overline{X})(Y-\overline{Y}) = 35$, find the correlation coefficient between the two variables. [2]

SOLUTION

Given, $\Sigma(X-\overline{X})^2 = 40$, $\Sigma(Y-\overline{Y})^2 = 63$, $\Sigma(X-\overline{X})(Y-\overline{Y}) = 35$

Correlation of coefficient, (r) = ?

We have, correlation coefficient $(r) = \frac{\Sigma(X-\overline{X})(Y-\overline{Y})}{\sqrt{\Sigma(X-\overline{X})^2}\sqrt{\Sigma(Y-\overline{Y})^2}} = \frac{35}{\sqrt{40}\sqrt{63}} = 0.697$

55. 2070 Set D Q.No. 4 b

If $n = 10$, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$, find the correlation coefficient between the two variables. [2]

- Please refer to Model Set II, Q.No. 4b

56. 2071 Set C Q.No. 4 b

If $n = 15$, $\sigma_x = 3.2$, $\sigma_y = 3.4$ and $\Sigma(X-\overline{X})(Y-\overline{Y}) = 122$, find the correlation coefficient between the two variables. [2]

SOLUTION

Here, $n = 15$ $\sigma_x = 3.2$ $\sigma_y = 3.4$

$\Sigma(X-\overline{X})(Y-\overline{Y}) = 122$

We have, correlation coefficient $(r) = \frac{\Sigma(X-\overline{x})(Y-\overline{Y})}{n\,\sigma_X\,\sigma_Y} = \frac{122}{15 \times 3.2 \times 3.4} = 0.75$

57. 2072 Supp Q.No. 4b

If $n = 10$, $\Sigma X = 18$, $\Sigma Y = 25$, $\Sigma X^2 = 90$, $\Sigma Y^2 = 120$ and $\Sigma XY = 65$, find the correlation coefficient between two variables. [2]

SOLUTION

Given, $n = 10$, $\Sigma X = 18$,

$\Sigma Y = 25$, $\Sigma X^2 = 90$,

$\Sigma Y^2 = 120$, $\Sigma XY = 65$

Correlation coefficient (r) = ?

We have,

$$r = \frac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\,\Sigma X^2 - (\Sigma X)^2}\sqrt{n\,\Sigma Y^2 - (\Sigma Y)^2}} = \frac{10 \times 65 - 18 \times 25}{\sqrt{10 \times 90 - 18^2}\sqrt{10 \times 120 - 25^2}} = \frac{200}{24 \times \sqrt{575}} = 0.35$$

58. 2073 Set D Q.No. 4b

If $\Sigma(x-\overline{x})^2 = 40$, $\Sigma(y-\overline{y})^2 = 63$ and $\Sigma(x-\overline{x})(y-\overline{y}) = 35$, find the correlation coefficient between the two variables x and y. [2]

- Please refer to 2070 Set C Q.No. 4b

59. 2057 Q.No. 4 b

Calculate r_{xy} if $\Sigma x^2 = 114$, $\Sigma y^2 = 442$; $\Sigma xy = 174$. [2]

SOLUTION

Here, $\Sigma x^2 = 114$ $\Sigma y^2 = 422$ $\Sigma xy = 174$

Correlation coefficient (r) = ?

We have, $r = \frac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}} = \frac{174}{\sqrt{114}\sqrt{422}} = 0.79$

60. 2065 Q.No 4 b

If the covariance between the variable *x* and y is 18 and the variances of *x* and *y* are 16 and 81 respectively, find the coefficient of correlation between them. [2]

SOLUTION

Given, Covariance between X and Y = Cov(X,Y) = 18

Variance of X = var (X) = 16

Variance of Y = var (Y) = 81

Coefficient of correlation (r) = ?

We have, $r = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)}\sqrt{\text{var}(Y)}} = \frac{18}{\sqrt{16}\sqrt{81}} = \frac{18}{4 \times 9} = 0.5$

61. 2068 Q.No. 4b

Calculate the correlation coefficient between two variables from the following data: $\Sigma x^2 = 114$, $\Sigma y^2 = 422$ and $\Sigma xy = 174$ [2]

- Please refer to 2057 Q.No. 4b

62. 2069 (Set A) Old Q.No. 4b

If the covariance between the two variables *x* and y is 18, and the variances of *x* and y are 16 and 81 respectively, find the coefficient of correlation between them. [2]

- Please refer to 2065 Q.No. 4b

63. 2070 Set C Q.No. 4 b

If $\Sigma(X-\overline{X})^2 = 40$, $\Sigma(Y-\overline{Y})^2 = 63$ and $\Sigma(X-\overline{X})(Y-\overline{Y}) = 35$, find the correlation coefficient between the two variables. [2]

SOLUTION

Given, $\Sigma(X-\overline{X})^2 = 40$, $\Sigma(Y-\overline{Y})^2 = 63$, $\Sigma(X-\overline{X})(Y-\overline{Y}) = 35$

Correlation of coefficient, (r) = ?

We have, correlation coefficient $(r) = \frac{\Sigma(X-\overline{X})(Y-\overline{Y})}{\sqrt{\Sigma(X-\overline{X})^2}\sqrt{\Sigma(Y-\overline{Y})^2}} = \frac{35}{\sqrt{40}\sqrt{63}} = 0.697$

64. 2070 Set D Q.No. 4 b

If $n = 10$, $\Sigma X = 60$, $\Sigma Y = 60$, $\Sigma X^2 = 400$, $\Sigma Y^2 = 580$ and $\Sigma XY = 415$, find the correlation coefficient between the two variables. [2]

- Please refer to Model Set II, Q.No. 4b

65. 2071 Set C Q.No. 4 b

If $n = 15$, $\sigma_x = 3.2$, $\sigma_y = 3.4$ and $\Sigma(X-\overline{X})(Y-\overline{Y}) = 122$, find the correlation coefficient between the two variables. [2]

SOLUTION

Here, $n = 15$ $\sigma_x = 3.2$ $\sigma_y = 3.4$

$\Sigma(X-\overline{X})(Y-\overline{Y}) = 122$

We have, correlation coefficient $(r) = \frac{\Sigma(X-\overline{x})(Y-\overline{Y})}{n\,\sigma_X\,\sigma_Y} = \frac{122}{15 \times 3.2 \times 3.4} = 0.75$

66. 2072 Supp Q.No. 4b

If $n = 10$, $\Sigma X = 18$, $\Sigma Y = 25$, $\Sigma X^2 = 90$, $\Sigma Y^2 = 120$ and $\Sigma XY = 65$, find the correlation coefficient between two variables. [2]

SOLUTION

Given, $n = 10$, $\Sigma X = 18$,

$\Sigma Y = 25$, $\Sigma X^2 = 90$,

$\Sigma Y^2 = 120$, $\Sigma XY = 65$

Correlation coefficient (r) = ?

We have,

$$r = \frac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\,\Sigma X^2 - (\Sigma X)^2}\sqrt{n\,\Sigma Y^2 - (\Sigma Y)^2}} = \frac{10 \times 65 - 18 \times 25}{\sqrt{10 \times 90 - 18^2}\sqrt{10 \times 120 - 25^2}} = \frac{200}{24 \times \sqrt{575}} = 0.35$$

67. 2073 Set D Q.No. 4b

If $\Sigma(x-\overline{x})^2 = 40$, $\Sigma(y-\overline{y})^2 = 63$ and $\Sigma(x-\overline{x})(y-\overline{y}) = 35$, find the correlation coefficient between the two variables x and y. [2]

- Please refer to 2070 Set C Q.No. 4b

4 MARKS QUESTIONS

68. 2059 Q.No. 12 b

Calculate Karl Pearson's coefficient of correlation from the following data: [4]

X	12	9	8	10	11	13	7
y	14	8	6	9	11	12	3

SOLUTION

Calculation of Karl Pearson's Coefficient of Correlation

X	Y	X^2	Y^2	XY
12	14	144	196	168
9	8	81	64	72
8	6	64	36	48
10	9	100	81	90
11	11	121	121	121
13	12	169	144	156
7	3	49	9	21
$\Sigma X = 70$	$\Sigma Y = 63$	$\Sigma X^2 = 728$	$\Sigma Y^2 = 651$	$\Sigma XY = 676$

Here, number of items (n) = 7

We have, correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} = \dfrac{7 \times 676 - 70 \times 63}{\sqrt{7 \times 728 - 70^2}\sqrt{7 \times 651 - 63^2}}$

$= \dfrac{322}{\sqrt{196}\sqrt{588}} = 0.95$

69. 2063 Q.No. 12 b

From the following table, calculate the coefficient of correlation by Karl Pearson's method.

X	6	2	10	4	8
Y	9	11	–	8	7

Arithmetic means of X and Y series are 6 and 8 respectively. [4]

SOLUTION

Let the missing item in Y-series be 'a'. Then,

$\bar{Y} = \dfrac{\Sigma Y}{n}$

or, $8 = \dfrac{9 + 11 + a + 8 + 7}{5}$

or, $40 = 35 + a$

$\therefore a = 5$

Now, Calculation of Karl Pearson's Coefficient of Correlation

X	Y	X^2	Y^2	XY
6	9	36	81	54
2	11	4	121	22
10	5	100	25	50
4	8	16	64	32
8	7	64	49	56
$\Sigma X = 30$	$\Sigma Y = 40$	$\Sigma X^2 = 220$	$\Sigma Y^2 = 340$	$\Sigma XY = 214$

Here, number of items (n) = 5

We have correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} = \dfrac{5 \times 214 - 30 \times 40}{\sqrt{5 \times 220 - 30^2}\sqrt{5 \times 340 - 40^2}}$

$= \dfrac{-130}{\sqrt{200}\sqrt{100}} = -0.92$

70. 2066 C Q.No. 12 a

Calculate the Karl Pearson's coefficient of correlation between the age and blood pressure of 8 patients: [4]

Age	23	48	43	68	70	28	35	26
Blood Pressure	115	127	123	140	145	118	121	120

SOLUTION

Calculation of Karl Pearson's Coefficient of Correlation

Age (X)	Blood Pressure (Y)	u = X-(48)	v = Y- B(127)	u^2	v^2	uv
23	115	-25	-12	625	144	300
48	127	0	0	0	0	0
43	123	-5	-4	25	16	20
68	140	20	13	400	169	260
70	145	22	18	484	324	396
28	118	-20	-9	400	81	180
35	121	-13	-6	169	36	78
26	120	-22	-7	484	49	154
		$\Sigma u = -43$	$\Sigma v = -7$	$\Sigma u^2 = 2587$	$\Sigma v^2 = 819$	$\Sigma uv = 1388$

Here, number of items (n) = 8

We have,

Correlation coefficient (r) $= \dfrac{n\Sigma uv - \Sigma u\,\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$

$= \dfrac{8 \times 1388 - (-43)(-7)}{\sqrt{8 \times 2587 - (-43)^2}\sqrt{8 \times 819 - (-7)^2}}$

$= \dfrac{10803}{\sqrt{18847}\sqrt{6503}} = 0.98$

71. 2070 (Old) Q.No. 12 b

Compute correlation and interpret about the ages of husband and wife given below: [4]

Age of husband	23	22	20	24	23	26	27	28	30	20
Age fo wife	20	18	23	20	21	21	22	24	25	26

SOLUTION

Computation of Correlation Coefficient

Age of husband (X)	Age of wife (Y)	X^2	Y^2	XY
23	20	529	400	460
22	18	484	324	396
20	23	400	529	460
24	20	576	400	480
23	21	529	441	483
26	21	676	441	546
27	22	729	484	594
28	24	784	576	672
30	25	900	625	750
20	26	400	676	520
$\Sigma X = 243$	$\Sigma Y = 220$	$\Sigma X^2 = 6007$	$\Sigma Y^2 = 4896$	$\Sigma XY = 5361$

Here, number of items (n) = 10

We have,

Correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$

$= \dfrac{10 \times 5361 - 243 \times 220}{\sqrt{10 \times 6007 - 243^2}\sqrt{10 \times 4896 - 220^2}}$

$= \dfrac{150}{\sqrt{1021}\sqrt{560}} = 0.198$

72. 2071 Set D Q.No. 8 a

Calculate Karl Pearson's correlation coefficient between the two variables height (in cms) and weight (in kg) from the data given below: [4]

Height	160	162	165	161	163
Weight	63	62	64	60	61

SOLUTION

Calculation of Correlation Coefficient

Height (X)	Weight (Y)	u = X-(163)	v = Y- B(62)	u^2	v^2	uv
160	63	-3	1	9	1	-3
162	62	-1	0	1	0	0
165	64	2	2	4	4	4
161	60	-2	-2	4	4	4
163	61	0	-1	0	1	0
		$\Sigma u = -4$	$\Sigma v = 0$	$\Sigma u^2 = 18$	$\Sigma v^2 = 10$	$\Sigma uv = 5$

Here, number of items, n = 5

We have,

Correlation coefficient (r) $= \dfrac{n\Sigma uv - \Sigma u \,\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$

$= \dfrac{5 \times 5 - (-4) \times 0}{\sqrt{5 \times 18 - (-4)^2}\sqrt{5 \times 10 - 0^2}} = \dfrac{25}{\sqrt{74}\sqrt{50}} = 0.41$

73. 2071 Old Q.No. 12 a

Calculate the coefficient of correlation between x and y series from the following data

	Series x	Series y
No. of observations	15	15
s.d	3.01	3.03

$\Sigma(x - \bar{x})(y - \bar{y}) = 122$ [4]

SOLUTION

Here, n = 15

$\sigma_X = 3.01$

$\sigma_Y = 3.03$

$\Sigma(x - \bar{x})(y - \bar{y}) = 122$

Coefficient of correlation (r) = ?

We have, correlation coefficient (r) $= \dfrac{\Sigma(x - \bar{x})(y - \bar{y})}{n\,\sigma_X\,\sigma_Y} = \dfrac{122}{15 \times 3.01 \times 3.03} = 0.89$

74. 2071 Supp. Q.No. 8a

Using the product moment formula, calculate correlation coefficient for the following series of ages of husbands (X) and wives (Y). [4]

X	41	44	45	48	40	42	44
Y	22	24	25	27	21	22	23

SOLUTION

Calculation of Correlation Coefficient

X	Y	X^2	Y^2	XY
41	22	1618	484	902
44	24	1936	576	1056
45	25	2025	625	1125
48	27	2304	729	1296
40	21	1600	441	840
42	22	1764	484	924
44	23	1936	529	1012
$\Sigma X = 304$	$\Sigma Y = 164$	$\Sigma X^2 = 13246$	$\Sigma y^2 = 3868$	$\Sigma XY = 7155$

Here, number of items (n) = 7

We have, correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X\,\Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}}$

$= \dfrac{7 \times 7155 - 304 \times 164}{\sqrt{7 \times 13246 - 304^2}\sqrt{7 \times 3868 - 164^2}}$

$= \dfrac{229}{\sqrt{306}\sqrt{180}} = 0.976$

75. 2072 Set D Q.No. 8a

Define correlation. Find Karl Pearson's coefficient of correlation of the marks of the following distribution. [4]

X	20	30	40	50	60
Y	50	46	30	24	8

SOLUTION

A statistical measure that describes the degree of relationship between two variables is called correlation. Two variables are said to be correlated when the value of one variable changes with the change in the value of the other variable.

Calculation of Karl Pearson's Coefficient of Correlation

X	Y	u = X – A(40)	v = Y – B(30)	u^2	v^2	uv
20	50	-20	20	400	400	-400
30	46	-10	16	100	256	-160
40	30	0	0	0	0	0
50	24	10	-6	100	36	-60
60	8	20	-22	400	484	-440
		$\Sigma u = 0$	$\Sigma v = 8$	$\Sigma u^2 = 1000$	$\Sigma v^2 = 1176$	$\Sigma uv = -1060$

Here, number of items (n) = 5

We have,

Correlation coefficient (r) $= \dfrac{n\Sigma uv - \Sigma u\,\Sigma v}{\sqrt{n\Sigma u^2 - (\Sigma u)^2}\sqrt{n\Sigma v^2 - (\Sigma v)^2}}$

$= \dfrac{5 \times (-1060) - 0 \times 18}{\sqrt{5 \times 1000 - 0^2}\sqrt{5 \times 1176 - (8)^2}}$

$= \dfrac{-5300}{\sqrt{5000}\sqrt{5816}} = -0.98$

76. 2073 Set C Q.No. 8a

Calculate Karl Pearson's coefficient of correlation from the following data using product moment formula. [4]

X	12	9	8	10	11
Y	12	8	6	9	10

SOLUTION

Calculation of Karl Pearson's coefficient of correlation

X	Y	$x = X - \bar{X}(10)$	$y = Y - \bar{Y}(9)$	x^2	y^2	xy
12	12	2	3	4	9	6
9	8	-1	-1	1	1	1
8	6	-2	-3	4	9	6
10	9	0	0	0	0	0
11	10	1	1	1	1	1
$\Sigma X = 50$	$\Sigma Y = 45$			$\Sigma x^2 = 10$	$\Sigma y^2 = 20$	$\Sigma xy = 14$

Here, n = 5

$\bar{X} = \dfrac{\Sigma X}{n} = \dfrac{50}{5} = 10,\ \bar{Y} = \dfrac{\Sigma Y}{n} = \dfrac{45}{5} = 9$

By using product moment formula, we have,

Correlation coefficient (r) $= \dfrac{\Sigma xy}{\sqrt{\Sigma x^2}\sqrt{\Sigma y^2}} = \dfrac{14}{\sqrt{10}\sqrt{20}} = 0.99$

77. 2073 Supp Q.No. 8a

Find the Karl Pearson's Coefficient of correlation form the following distribution. [4]

X	10	11	18	15	25	20	14	22
Y	9	14	15	16	22	20	18	24

SOLUTION

Calculation of Karl Pearson's Coefficient of Correlation

X	Y	x^2	Y^2	XY
10	9	100	81	90
11	14	121	196	154
18	15	324	225	270
15	16	225	256	240
25	22	625	484	550
20	20	400	400	400
14	18	196	324	252
22	24	484	576	528
$\Sigma x = 135$	$\Sigma y = 138$	$\Sigma x^2 = 2475$	$\Sigma Y^2 = 2542$	$\Sigma XY = 2484$

Here, n = 8

We have, correlation coefficient (r) $= \dfrac{n\Sigma XY - \Sigma X \Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} = \dfrac{8 \times 2484 - 135 \times 138}{\sqrt{8 \times 2475 - (135)^2}\sqrt{8 \times 2542 - (138)^2}}$

$= \dfrac{1242}{\sqrt{1575}\sqrt{1292}} = 0.87$

78. 2074 Set A Q.No. 8a

From the following table calculate the correlation coefficient by Karl Pearson's method. AM of X = 15. [4]

X	10	12	20	?	16	14
Y	9	12	15	18	14	16

SOLUTION

Let a be the missing item in X series.

Then, $\Sigma x = 10 + 12 + 20 + a + 16 + 14 = 72 + a$

n = 6

A.M of X ($\bar{X}$) = 15

We have, $(\bar{X}) = \dfrac{\Sigma X}{n}$

or, $15 = \dfrac{72 + a}{6}$

or, $90 = 72 + a$

$\therefore$ $a = 90 - 72 = 18$

Computation of Correlation Coefficient

X	Y	X^2	Y^2	XY
10	9	100	81	90
12	12	144	144	144
20	15	400	225	300
18	18	324	324	324
16	14	256	196	224
14	16	196	256	224
$\Sigma X = 90$	$\Sigma Y = 84$	$\Sigma X^2 = 1420$	$\Sigma Y^2 = 1226$	$\Sigma XY = 1306$

Here, n = 6

We have, Correlation coefficient

(r) $= \dfrac{n\Sigma XY^2 - \Sigma X \Sigma Y}{\sqrt{n\Sigma X^2 - (\Sigma X)^2}\sqrt{n\Sigma Y^2 - (\Sigma Y)^2}} = \dfrac{6 \times 1306 - 90 \times 84}{\sqrt{6 \times 1420 - (90)^2}\sqrt{6 \times 1226 - (84)^2}} = \dfrac{276}{\sqrt{420}\sqrt{300}} = 0.78$

79. 2075 Set A Q.No. 8a

Find the correlation coefficient between the two variables x and y from the following data. [4]

x	5	7	1	3	4
y	2	3	4	5	6

SOLUTION

Calculation of Correlation Coefficient

x	y	x^2	y^2	xy
5	2	25	4	10
7	3	49	9	21
1	4	1	16	4
3	5	9	25	15
4	6	16	36	24
$\Sigma x = 20$	$\Sigma y = 20$	$\Sigma x^2 = 100$	$\Sigma y^2 = 90$	$\Sigma xy = 74$

Here, n = 5

We have,

Correlation coefficient (r) $= \dfrac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}$

$= \dfrac{5 \times 74 - 20 \times 20}{\sqrt{5 \times 100 - (20)^2}\sqrt{5 \times 90 - (20)^2}}$

$= \dfrac{-30}{10\sqrt{50}} = -0.42$

D. REGRESSION

2 MARKS QUESTIONS

80. 2069 (Set A) Q.No. 4b

Find the regression equation of y on x when:

$\Sigma x = 15, \Sigma y = 25, \Sigma x^2 = 55, \Sigma y^2 = 140, \Sigma xy = 78, n = 5.$ [2]

SOLUTION

Given, $\Sigma x = 15$ $\Sigma y = 25$

$\Sigma x^2 = 55$ $\Sigma y^2 = 140$

$\Sigma xy = 78$ $n = 5.$

Now,

$\bar{x} = \dfrac{\Sigma x}{n} = \dfrac{15}{5} = 3$

$\bar{y} = \dfrac{\Sigma y}{n} = \dfrac{25}{5} = 5$

$b_{yx} = \dfrac{n\Sigma xy - \Sigma x \Sigma y}{n\Sigma x^2 - (\Sigma x)^2} = \dfrac{5 \times 78 - 15 \times 25}{5 \times 55 - 15^2} = \dfrac{15}{50} = 0.3$

The regression equation of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x})$

or, $y - 5 = 0.3(x - 3)$

or, $y - 5 = 0.3x - 0.9$

or, $y = 0.3x + 5 - 0.9$

$\therefore$ $y = 0.3x + 4.1$

81. 2069 (Set B) Q.No. 4b

The regression coefficient of y on x is 0.32. If the arithmetic means of x and y series are 42 and 36 respectively, find the regression equation of y on x. [2]

SOLUTION

Regression coefficient of x on y (b_{xy}) = 0.84

Regression coefficient of y on x (b_{yx}) = 0.32

A.M of x-series ($\bar{x}$) = 42

A.M of y-series ($\bar{y}$) = 26

The regression equation of y on x is

$y - \bar{y} = b_{yx}(x - \bar{x})$

or, $y - 26 = 0.32(x - 42)$

or, $y - 26 = 0.32x - 13.44$

or, $y = 0.32x + 26 - 13.44$

$\therefore$ $y = 0.32x + 12.56$

4 MARKS QUESTIONS

82. 2070 Supp. Q.No. 8 a OR

From the following data, compute the line of regression for estimating age on weight and estimate the most probable age of a weight of 37 kg. [4]

Age (x)	5	15	30	45	50	60
Weight (y)	10	35	50	65	55	45

SOLUTION

Computation of Line of Regression of X on Y

Age (X)	Weight (Y)	XY	Y^2
5	10	50	100
15	35	525	1225
30	50	1500	2500
45	65	2925	4225
50	55	2750	3025
60	45	2700	2025
$\Sigma X = 205$	$\Sigma Y = 260$	$\Sigma XY = 10,450$	$\Sigma Y^2 = 13100$

Here, number of items (n) = 6

$\bar{X} = \frac{\Sigma X}{n} = \frac{205}{6} = 34.17$

$\bar{Y} = \frac{\Sigma Y}{n} = \frac{260}{6} = 43.33$

And, $b_{xy} = \frac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma Y^2 - (\Sigma Y)^2} = \frac{6 \times 10450 - 205 \times 260}{6 \times 13100 - (260)^2} = \frac{9400}{11000} = 0.85$

The regression equation of X on Y is

$X - \bar{X} = b_{xy} (Y - \bar{Y})$

or, $X - 34.17 = 0.85 (Y - 43.33)$

or, $X - 34.17 = 0.85Y - 36.83$

or, $X = 0.85Y + 34.17 - 36.83$

or, $X = 0.85Y - 2.66$

When Y = 37, $X = 0.85 \times 37 - 2.66 = 28.79$

∴ Most probable age = 28.79

83. 2071 Supp. Q.No. 8a OR

From the following data, compute the line of regression for estimating age on weight and estimate the most probable age on a weight of 37 Kg. [4]

Age (X)	5	15	30	45	50	60
Weight (Y)	10	35	50	65	55	45

➥ Please refer to 2070 Supp Q.No. 8a OR

84. 2072 Set C Q.No. 8a

Define regression and lines of regression. Find the correlation coefficients between the two variables when $b_{xy} = 1.8$ and $b_{yx} = 0.35$ [4]

SOLUTION

Regression: It is a statistical device which is used to predict the value of one variable when the value of other is known. The variable whose value is known is called independent variable and the variable whose value is to be determined is called dependent variable. The analysis which is used to describe the average relationship between the two variables is known as a regression analysis. There must be cause and effect relationship between two variables in regression.

Lines of regression: Whenever there shows a relationship between two variables, the dots of the scatter diagram will concentrate around a certain curve. If the curve is a straight line, then it is known as the line of regression.

The regression equation of y on x is $y = a + bx$, when b is the regression coefficient of y on x. And the regression of x on y is $x = b + ay$, where a is the regression coefficient of x on y

Given, $b_{xy} = 1.8$

$b_{yx} = 0.35$

Correlation coefficient (r) = ?

We have, $r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{1.8 \times 0.35} = \sqrt{0.63} = 0.79$

85. 2072 Set E Q.No. 8a

The regression coefficients of x on y and y on x are 0.84 and 0.32 respectively. If the arithmetic means of x and y series are 42 and 26 respectively, find two equations of lines of regression. [4]

SOLUTION

Regression coefficient of x on y $(b_{xy}) = 0.84$

Regression coefficient of y on x $(b_{yx}) = 0.32$

A.M of x-series $(\bar{x}) = 42$

A.M of y-series $(\bar{y}) = 26$

The regression equation of y on x is

$y - \bar{y} = b_{yx} (x - \bar{x})$

or, $y - 26 = 0.32 (x - 42)$

or, $y - 26 = 0.32x - 13.44$

or, $y = 0.32x + 26 - 13.44$

∴ $y = 0.32x + 12.56$

Again, the regression equation of x on is

$x - \bar{x} = b_{xy} (y - \bar{y})$

or, $x - 42 = 0.84 (y - 26)$

or, $x - 42 = 0.84y - 21.84$

or, $x = 0.84y + 42 - 21.84$

or, $x = 0.84y + 20.16$

∴ $x = 0.84y + 20.16$

∴ Required equations of regression lines are $y = 0.32x + 12.56$ and $x = 0.84y + 20.16$

86. 2074 Supp Q.No. 8a

From the following pair of regression equations, find the correlation coefficient between the two variables x and y.

$4x - 5y + 33 = 0$ and $20x - 9y - 107 = 0$. [4]

SOLUTION

Given lines are

$4x - 5y + 33 = 0$ ... (i)

$20x - 9y - 107 = 0$ ... (ii)

Let eqⁿ (i) be the regression line of y on x and eqⁿ (ii) be the regression line of x on y.

Then, from (i)

$4x + 33 = 5y$

or, $y = \frac{4x + 33}{5} = \frac{4}{5}x + \frac{33}{5}$

∴ $b_{yx} = \frac{4}{5}$

Again, from (ii)

$20x - 9y - 107 = 0$

or, $20x = 9y + 107$

or, $x = \frac{9y + 107}{20} = \frac{9}{20}y + \frac{107}{20}$

∴ $b_{yx} = \frac{9}{20}$

Here, $b_{yx} \cdot b_{xy} = \frac{4}{5} \times \frac{9}{20} = \frac{9}{25} < 1$

So, our assumption is correct.

We have,

Correlation (r) $= \sqrt{b_{yx} \cdot b_{xy}}$

$= \sqrt{\frac{4}{5} \times \frac{9}{20}}$

$= \sqrt{\frac{9}{25}} = \frac{3}{5} = 0.6$

87. 2075 Set C Q.No. 8a

The regression coefficients, $b_{XY} = 1.5$, $b_{YX} = 0.65$ and arithmetic means $(\bar{X}) = 36$, $(\bar{Y}) = 52$. Find the regression equations X on Y and Y on X. Also, find the estimated value of Y when X = 60. [4]

SOLUTION

Here,

$b_{XY} = 1.5$

$b_{YX} = 0.65$

$\bar{X} = 36$

$\bar{Y} = 52$

The regression equation of Y on X is

$Y - \bar{Y} = b_{YX}(X - \bar{X})$

$Y - 52 = 0.65\,(X - 36)$

or, $Y = 0.65x - 23.4 + 52$

$Y = 0.65x + 28.6 \quad \ldots$ (i)

Again, regression equation of X on Y is

$X - \bar{X} = b_{XY}(Y - \bar{Y})$

or, $X - 36 = 1.5\,(Y - 52)$

or, $X = 1.5Y - 78 + 36$

or, $X = 1.5\,Y - 42$

The estimated value of Y when X = 60 should be calculated from the line Y on X.

When X = 60, from (i)

$Y = 0.65 \times 60 + 28.6$

$= 67.6$

6 MARKS QUESTIONS

88. 2075 Set B Q.No. 11

The equations of two regression lines are

$3X + 4Y = 65$, $3X + Y = 32$. Find,

i. the mean of x and the mean of Y.

ii. the regression coefficients.

iii. the correlation coefficients between X and Y.

iv. the ratio of standard deviations of X and Y.

SOLUTION

i. Let $\bar{X}$ and $\bar{Y}$ be the means of X and Y respectively. Then,

$3\bar{X} + 4\bar{Y} = 65 \quad \ldots$ (i)

$3\bar{X} + \bar{Y} = 32 \quad \ldots$ (ii)

Subtracting equation (ii) from equation (i), we get

$$\begin{array}{rcrcr} 3\bar{X} & + & 4\bar{Y} & = & 65 \\ 3\bar{X} & + & \bar{Y} & = & 32 \\ - & & - & & - \\ \hline & & 3\bar{Y} & = & 33 \\ & & \bar{Y} & = & 11 \end{array}$$

Substituting the value of $\bar{Y}$ in equation (ii), we get,

$3\bar{X} + 11 = 32$

or, $3\bar{X} = 21$

or, $\bar{X} = \frac{21}{3} = 7$

$\therefore$ $\bar{X} = 7, \bar{Y} = 11$

ii. Let the regression line of Y on X be $3X + 4Y = 65$, then,

$4Y = -3X + 65$

or, $Y = \frac{-3}{4}X + \frac{65}{4}$

$\therefore$ $b_{YX} = \frac{-3}{4}$

Again, the regression line of X on Y be

$3X + Y = 32$

or, $3X = -Y + 32$

$X = -\frac{1}{3}Y + \frac{32}{3}$

$\therefore$ $b_{YX} = -\frac{1}{3}$

Here,

$b_{YX} \cdot b_{XY} = -\frac{3}{4} \cdot \left(-\frac{1}{3}\right)$

$= \frac{1}{4} < 1$

So, our assumption is correct.

$\therefore$ Regression coefficient of Y on X $(b_{YX}) = -\frac{3}{4}$

Regression coefficient of X on Y $(b_{XY}) = -\frac{1}{3}$

iii. We have,

Correlation coefficient $(r) = \pm\sqrt{b_{YX} . b_{XY}}$

Since both the regression coefficient have negative sign, so we choose –ve sign in the formula.

$\therefore$ $r = -\sqrt{(-3/4) \cdot (-1/3)} = -\frac{1}{2}$

iv. We have,

$b_{YX} = r . \frac{\sigma_Y}{\sigma_X}$

or, $-\frac{3}{2} = -\frac{1}{2} . \frac{\sigma_Y}{\sigma_X}$

or, $\frac{3}{2} = \frac{\sigma_Y}{\sigma_X}$

or, $\frac{\sigma_Y}{\sigma_X} = \frac{2}{3}$

$\therefore$ $\sigma_X : \sigma_Y = 2:3$

UNIT 11
PROBABILITY

A. PROBABILITY (SIMPLE CASES)

2 MARKS QUESTIONS

1. 2057 Q.No. 4 c

A card is drawn at random from a well shuffled deck of 52 cards. Find the probability of being it (i) a red card (ii) a heart. [2]

SOLUTION

Total number of possible cases (n) = 52

(i) There are 26 red cards in a deck of 52 cards.

∴ Number of favourable cases (m) = 26

$P(\text{red card}) = \frac{m}{n} = \frac{26}{52} = \frac{1}{2}$

(ii) There are 13 heart in a deck of 52 cards.

So, number of favourable cases (m) = 13

$P(\text{a heart}) = \frac{m}{n} = \frac{13}{52} = \frac{1}{4}$.

2. 2058 Q.No. 4 c

If A and B are two independent events with $P(A) = \frac{2}{3}$ and $P(B) = \frac{3}{5}$, find $P(A \cup B)$. [2]

SOLUTION

Since A and B are independent events, so

$P(A \cap B) = P(A) \cdot P(B) = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}$

We have,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{2}{3} + \frac{3}{5} - \frac{2}{5} = \frac{10 + 9 - 6}{15} = \frac{13}{15}$

3. 2059 Q.No. 3 c

Two letters are selected at random from the word "EXAMINATION". Find the probability that both of them are same letters. [2]

SOLUTION

There are 11 letters in the word 'EXAMINATION'.

Total number of possible cases = $^{11}C_2$

In the word examination, there are 2 A's, 2 I's and 2 N's

Number of favourable cases for selecting two letters such that both of them are A = $^2C_2$

Similarly, number of favourable cases for both I = $^2C_2$

And number of favourable cases for both N = $^2C_2$.

Required probability = $\frac{^2C_2 + {^2C_2} + {^2C_2}}{^{11}C_2}$

$= \frac{1 + 1 + 1}{\frac{11!}{9!\,2!}} = \frac{3}{\frac{11 \times 10}{2}} = \frac{3}{55}$

4. 2060 Q.No. 4 c

Two dice are thrown. Determine the probability of getting a sum ≤ 5. [2]

SOLUTION

When two dice are thrown, total number of possible cases = 6 × 6 = 36

The set having a sum ≤ 5 is

{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)}

∴ Number of favourable cases (m) = 10

$P(\text{a sum} \le 5) = \frac{m}{n} = \frac{10}{36} = \frac{5}{18}$

5. 2062 Q.No. 4 c

What is the probability that an English alphabet selected at random is (i) a vowel (ii) a consonant? [2]

SOLUTION

There are 26 letters in English alphabet. There are 5 vowels and 21 consonants.

(i) n = total number of possible cases = 26

m = number of favourable cases = 5

P(vowel) = ?

We have,

$P(\text{vowel}) = \frac{m}{n} = \frac{5}{26}$

(ii) n = total number of possible cases = 26

m = number of favourable cases = 21

P(consonant) = ?

We have, $P(\text{consonant}) = \frac{m}{n} = \frac{21}{26}$

6. 2063 Q.No. 4 c

The chance that A can solve a certain problem is $\frac{1}{4}$ and the chance that B can solve it is $\frac{2}{3}$. Find the chance that the problem will be solved if they both try. [2]

➥ Please refer to Model Set II, Q.No. 8b

7. 2064 Q.No. 4 c

A bag contains 9 red, 7 white and 4 black balls. A ball is drawn at random. Find the probability of drawing (i) a white ball (ii) not a black ball. [2]

SOLUTION

Total number of balls = 9 + 7 + 4 = 20

(i) Probability of getting a white ball

$= P(\text{white}) = \frac{\text{Number of white balls}}{\text{Total number of balls}} = \frac{7}{20}$

(ii) Probability of getting a black ball

$= P(\text{black}) = \frac{\text{Number of black balls}}{\text{Total number of balls}} = \frac{4}{20} = \frac{1}{5}$

∴ Probability of getting not a black ball

$= 1 - P(\text{black}) = 1 - \frac{1}{5} = \frac{4}{5}$

8. 2065 Q.No 4 c

Given P (A) = 0.4, P (A∪B) = 0.56, P (B) = 0.3. Are A & B independent? [2]

SOLUTION

Here, P(A) = 0.4

P(B) = 0.3

P(A∪B) = 0.56

We have, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or, $0.56 = 0.4 + 0.3 - P(A \cap B)$

∴ $P(A \cap B) = 0.7 - 0.56 = 0.14$

Again, $P(A) \cdot P(B) = 0.4 \times 0.3 = 0.12$

Here, $P(A \cap B) \ne P(A) \cdot P(B)$

Hence, the events A and B are not independent events.

9. 2066 Q.No. 4 c

The chance that A can solve a problem is $\frac{1}{4}$, the chance that B can solve it is $\frac{2}{3}$. Find the probability that the problems will be solved if both of them try. [2]

➥ Please refer to Model Set II, Q.No. 8b

10. 2067 Q.No. 4c

Define mutually exclusive events and dependent cases with example while performing an experiment. [2]

SOLUTION

Mutually exclusive events: Two or more events are said to be mutually exclusive if their simultaneous occurrence is not possible. For example, if a coin is tossed either head or tail will occur, so head and tail are two mutually exclusive events.

Dependent cases: Two events are said to be dependent if the occurrence of one event in a trial affects the probability of the occurrence of the other event in other trial. For example, if two cards are drawn in succession without replacement from a deck of 52 cards, then the result of first draw will affect the result of second draw.

11. 2068 Q.No. 4c

A card is drawn at random from a well-shuffled deck of 52 cards. What is the probability that is a red 8, a red 9 or a red 10? [2]

SOLUTION

In a deck of 52 cards, there are 2 red 8, 2 red 9, and 2 red 10.

∴ m = number of favourable cases = 2 + 2 + 2

= 6

n = total number of possible cases = 52

P(a red 8, 9 or 10) = ?

We have,

$P(\text{a red 8, 9 or 10}) = \frac{m}{n} = \frac{6}{52} = \frac{3}{26}$

12. 2069 (Set A) Q.No. 4c

From 20 tickets marked from 1 to 20, one is drawn at random. Find the probability that it is a multiple of 4 or 5. [2]

SOLUTION

Total number of possible cases (n) = 20

The set of multiple of 4 is {4, 8, 12, 16, 20}

and the set of multiple of 5 is {5, 10, 15, 20}

∴ Number of possible cases (m) = 5 + 4 − 1 = 8

$P(\text{a multiple of 4 or 5}) = \frac{m}{n} = \frac{8}{20} = \frac{2}{5}$

13. 2069 (Set A) Old Q.No. 4c

What is the probability that an English alphabet selected at random is (i) a vowel (ii) a consonant? [2]

➥ Please refer to 2062 Q.No. 4 c

14. 2069 Old (Set B) Q.No. 3c

A card is drawn at random from a well shuffled deck of 52 cards. What is the probability that it is a spade? [2]

SOLUTION

In a deck of 52 cards, there are 13 spade.

∴ Total number of possible cases (n) = 52

And number of favourable cases (m) = 13

$P(\text{a spade}) = \frac{m}{n} = \frac{13}{52} = \frac{1}{4}$

15. 2070 (Old) Q.No. 4 c

If three coins are tossed simultaneously, find the probability of turning all head. [2]

SOLUTION

When three coins are tossed simultaneously, then the sample space is

{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

∴ Total number of favorable cases (n) = 8

Number of favourable cases (m) = 1

$P(\text{all heads}) = \frac{m}{n} = \frac{1}{8}$

16. 2070 Set C Q.No. 4 c

A class consists of 60 boys and 40 girls. If two students are chosen at random, what is the probability that one is boy and one girl? [2]

SOLUTION

Total number of students = 60 + 40 = 100

Total number of possible cases (n) = number of selection of 2 students from 100 = C(100,2)

Number of favourable cases (m) = number of selection of 1 boy from 60 and number of selection of 1 girl from 40 = C(60, 1) × C(40, 1)

∴ P(one boy and one girl) = $\frac{m}{n}$

$= \frac{C(60,1) \times C(40,1)}{C(100,2)} = \frac{60 \times 40}{\frac{100!}{98!\,2!}}$

$= \frac{60 \times 40 \times 2}{100 \times 99} = \frac{16}{33}$

17. 2070 Set D Q.No. 4 c

A card is drawn from a well-shuffled deck of 52 cards. What is the probability that it is a King or a Diamond? [2]

SOLUTION

In a deck of 52 cards, there are 4 kings and 13 diamonds.

∴ Total number of favourable cases (m) = 4 + 13 − 1 = 16

Total number of possible cases (n) = 52

∴ P(a king or a diamond) = $\frac{m}{n} = \frac{16}{52} = \frac{4}{13}$

18. 2070 Supp. Q.No. 4 c

A bag contains 24 balls numbered from 1 to 24. One ball is drawn at random. What is the probability that it is a multiple of 4 and 6? [2]

SOLUTION

Total number of possible cases (n) = 24

The set of multiple of 4 and 6 is {12, 24}

Total number of favourable cases (m) = 2

∴ P(a multiple of 4 and 6) = $\frac{m}{n} = \frac{2}{24} = \frac{1}{12}$

19. 2071 Old Q.No. 4 c

A coin is tossed successively three times. Determine the probability of getting 2 heads and one tail. [2]

SOLUTION

When a coin is tossed three times, the outcomes are

1st toss	H	H	H	H	T	T	T	T
2nd toss	H	H	T	T	H	H	T	T
3rd toss	H	T	H	T	H	T	H	T

∴ Number of favourable cases (m) = 3

Total number of possible cases (n) = 8

Probability of getting 2 heads and 1 tail, P(E) = ?

We have, P(E) = $\frac{m}{n} = \frac{3}{8}$.

20. 2071 Set C Q.No. 4 c

The chance that A can solve the problem is 3/5 and the chance that B can solve the problem is 2/3. Find the probability that the problem is solved. [2]

SOLUTION

The problem will be solved if A or B solve it.

Probability of solving the problem by A = P(A) = $\frac{3}{5}$

Probability of solving the problem by B = P(B) = $\frac{2}{3}$

Probability of solving the problem = P(A or B) = ?

We have,

$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A) \cdot P(B)$

$= \frac{3}{5} + \frac{2}{3} - \frac{3}{5} \cdot \frac{2}{3} = \frac{3}{5} + \frac{2}{3} - \frac{2}{5}$

$= \frac{9 + 10 - 6}{15} = \frac{13}{15}$

21. 2071 Set D Q.No. 4 c

Two coins are tossed simultaneously. Find the sample space. Find the probability that both are heads. [2]

SOLUTION

When two coins are tossed simultaneously, then the sample space is

S = {HH, HT, TH, TT}

For probability that both are heads, number of favourable cases (m) = 1

Total number of possible cases (n) = 4

∴ P(both heads) = $\frac{m}{n} = \frac{1}{4}$

22. 2071 Supp. Q.No. 4b

Two dice are thrown together. Find the probability of getting both odd digits. [2]

SOLUTION

When two dice are thrown together, total number of possible cases (n) = 6 × 6 = 36

The set of getting both odd digits is

{(1, 1), (1, 3), (1, 5), (3, 1) (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)}

∴ Total number of favourable cases (m) = 9

Probability of getting both odd digits, P(E) = ?

We have,

P(E) = $\frac{m}{n} = \frac{9}{36} = \frac{1}{4}$.

23. 2072 Set C Q.No. 4c

In rolling a pair of dice, determine the probability of obtaining a sum of 10. [2]

SOLUTION

If a pair of dice are rolled once, then total number of possible cases (n) = 6×6=36

The favourable cases of getting a total of 10 in a single throw of two dice are (4, 6), (5, 5) and (6, 4)

So, number of favourable cases (m) = 3

P (sum of 10) = ?

We have, P(sum of 10) = $\frac{m}{n} = \frac{3}{36} = \frac{1}{12}$

24. 2072 Set D Q.No. 4c

In a draw of a card from well shuffled deck of 52 cards what is the probability that it is a king or a queen? [2]

SOLUTION

In a deck of 52 cards, there are 4 kings and 4 queens.

P(a king or a queen) = ?

We have,

P(a king or a queen) = P(a king) + P(a queen)

$= \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$

25. 2072 Set E Q.No. 4c

Two dice are rolled once. What is the probability of getting a total of 8 or 7? [2]

SOLUTION

When two dice are rolled once, total number of possible cases (n) = 6 × 6 = 36

The set having a total of 8 is

{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

and the set having a total of 7 is

{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

∴ Total number of favourable cases (m) = 5 + 6 = 11

P(a total of 8 or 7) = $\frac{m}{n} = \frac{11}{36}$

26. 2072 Supp Q.No. 4c

An urn contains 4 white and 8 red balls. If two balls are drawn at random, find the probability of getting one of each colour. [2]

SOLUTION

There are 4 white and 8 red balls.

Total number of balls = 4 + 8 = 12

P(one of each colour) = ?

No. of possible cases (m) = number of selection of 1 white ball out of 4 and 1 red ball out of 8 = $^4C_1 \times {}^8C_1$ = 4 × 8 = 32

Total no. of possible cases (n)

= No. of selection of 2 balls out of 12

$= {}^{12}C_2 = \frac{12!}{10!\,2!} = \frac{12 \times 11}{2 \times 1} = 66$

∴ P (one of each colour) = $\frac{m}{n} = \frac{32}{66} = \frac{16}{33}$

27. 2073 Set C Q.No. 4c

The chance that A can solve the problem is $\frac{2}{3}$ and the chance that B can solve the problem is $\frac{1}{3}$. Find the probability that the problem is solved by A and B. [2]

SOLUTION

Here, P(A) = $\frac{2}{3}$, P(B) = $\frac{1}{3}$

Since the events are independent,

P(A and B) = P(A) · P(B) = $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

28. 2073 Supp Q.No. 4c

Three coins are tossed simultaneously. Find the sample space. Find the probability that all are heads. [2]

➥ Please refer to 2070 (Old) Q.No. 4c

29. 2074 Set B Q.No. 4c

A class consists of 30 boys and 20 girls. If two students are chosen at random what is the probability that one is boy and another is girl? [2]

SOLUTION

Total no. of Students = 30 + 20 = 50

Total no. of possible cases = no. of selections of 2 students out of 50 students

$= C(50,2) = \frac{50!}{48!\,2!} = \frac{50 \times 49}{2} = 1225$

Number of favourable cases (m) = no. of selection of 1 boy from 30 boys and 1 girl from 20 girls

= C (30, 1) × C (20, 1) = 30 × 20 = 600

P(a boy and a girl) = $\frac{m}{n} = \frac{600}{1225} = \frac{24}{49}$

30. 2074 Supp Q.No. 4c

If A and B are two independent events with P(A) = $\frac{2}{3}$ and P(B) = $\frac{3}{5}$, find P(A∪B). [2]

➥ Please refer to 2058 Q.N. 4c

31. 2075 Set B Q.No. 4c

An urn contains 4 white, 8 black, 6 red and 2 green marbles. If three balls are drawn at random, find the probability of getting 2 red and 1 green marbles. [2]

SOLUTION

Number of white marbles = 4

Number of black marbles = 8

Number of red marbles = 6

Number of green marbles = 2

Total no. of marbles = 4 + 8 + 6 + 2 = 20

Number of favourable cases (m) = No. of selection of 2 red marbles out of 6 marbles and 1 green marble out of 2 green marbles

$= {}^6C_2 \times {}^2C_1$

$= \frac{6!}{4!\,2!} \times \frac{2!}{1!\,1!} = 30$

Total no. of possible cases (n) = no. of selection of 3 marbles out of 20

$= {}^{20}C_3 = \frac{20!}{17!\,3!} = 1140$

P(2 red and 1 green marbles) = $\frac{m}{n} = \frac{30}{1140} = \frac{1}{38}$

32. 2075 Set C Q.No. 4c

If A and B are two independent events with P(A) = $\frac{3}{4}$ and P(B) = $\frac{1}{5}$, find P(A ∪ B). [2]

SOLUTION

Given,

P(A) = $\frac{3}{4}$

P(B) = $\frac{1}{5}$

P(A∪B) = ?

We have,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{3}{4} + \frac{1}{5} - P(A).\ P(B)$

[∵ A & B are independent events]

$= \frac{3}{4} + \frac{1}{5} - \frac{3}{4} \times \frac{1}{5}$

$= \frac{15 + 4 - 3}{20} = \frac{16}{20} = \frac{4}{5}$

4 MARKS QUESTIONS

33. 2057 Q.No. 8 b

State and prove the "Theorem of Total probability". [4]

SOLUTION

Statement: If A and B are two events with their respective probabilities P(A) and P(B) then the probability of occurrence of at least one of these two events, denoted by $P(A \cup B)$, is given by

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

where $P(A \cap B)$ is the probability of the simultaneous occurrence of the events A and B.

Proof: Let n = total number of possible cases

u = number of favourable cases to the event A

v = number of favourable cases to the even

w = number of favourable cases to the event A and B

Then, $P(A) = \frac{u}{n}$, $P(B) = \frac{v}{n}$ and $P(A \cap B) = \frac{w}{n}$

Since the outcomes w are common to both A and B, so the cases favourable to the event $A \cup B$ are u + v – w

Now, $P(A \cup B) = \frac{\text{Number of favourable cases}}{\text{Total number of possible cases}}$

$= \frac{u + v - w}{n} = \frac{u}{n} + \frac{v}{n} - \frac{w}{n} = P(A) + P(B) - P(A \cap B)$

34. 2058 Q.No. 8 a

State and prove the "Theorem of total probability" [4]

➥ Please refer to 2057 Q.No. 8 b

35. 2059 Q.No. 8 a

If A, B, C are three mutually exclusive events with $\frac{1}{3}P(A) = \frac{2}{3}P(B) = \frac{1}{6}P(C)$, find P(A); P(B); and P(C). [4]

SOLUTION

Here, $\frac{1}{3}P(A) = \frac{2}{3}P(B) = \frac{1}{6}P(C)$

$\therefore P(A) = \frac{1}{2}P(C)$ and $P(B) = \frac{1}{4}P(C)$

Since A, B and C are three mutually exclusive events, so

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

or, $1 = \frac{1}{2}P(C) + \frac{1}{4}P(C) + P(C)$

or, $\frac{7}{4}P(C) = 1$

$\Rightarrow P(C) = \frac{4}{7}$

$\therefore P(A) = \frac{1}{2}P(C) = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$

$P(B) = \frac{1}{4}P(C) = \frac{1}{4} \times \frac{4}{7} = \frac{1}{7}$

36. 2060 Q.No. 8 a

State and prove the "Theorem of total probability." [4]

➥ Please refer to 2057 Q.No. 8 b

37. 2061 Q.No. 8 a

State and prove the "Theorem of Compound Probability". [4]

SOLUTION

Statement: If two events A and B are independent, then the probability of their simultaneous occurrence is equal to the product of their individual probabilities

i.e. $P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$

Proof: Let n_1 and n_2 be the total number of possible cases for the events A and B respectively. Also let their respective favourable cases be m_1 and m_2. Then,

$P(A) = \frac{m_1}{n_1}$ and $P(B) = \frac{m_2}{n_2}$.

Since the total number of possible cases for the events A and B are n_1 and n_2 respectively and since we can combine each of n_1 possible cases of A with each of n_2 possible cases of B, so the total number of possible cases for their simultaneous occurrence is $n_1 \cdot n_2$. In the same way, the total number of favourable cases for the simultaneous occurrence of A and B is $m_1 \cdot m_2$.

Now,

$P(A \cap B)$ = Probability of simultaneous occurrence of A and B

$= \frac{\text{Number of favourable cases}}{\text{Total number of possible cases}}$

$= \frac{m_1 \cdot m_2}{n_1 \cdot n_2} = \left(\frac{m_1}{n_1}\right) \cdot \left(\frac{m_2}{n_2}\right)$

$= P(A) \cdot P(B)$

$\therefore P(A \cap B) = P(A) \cdot P(B)$

38. 2061 Q.No. 8 a OR

A class consists of 60 boys and 40 girls. If two students are chosen at random, what will be the probability that (a) both are boys (b) both are girls (c) one boy and one girl? [4]

SOLUTION

Total number of students = 60 + 40 = 100

Total number of possible cases (n) = Number of selection of 2 students out of 100

$= {}^{100}C_2 = \frac{100!}{98!\ 2!} = \frac{100 \times 99}{2} = 4950$

(a) If both students are boys, then 2 boys out of 60 can be selected in ${}^{60}C_2$ ways

$= \frac{60!}{58!\ 2!} = \frac{60 \times 59}{2} = 1770$

∴ Number of favourable cases (m) = 1770

$P(\text{both boys}) = \frac{m}{n} = \frac{1770}{4950} = \frac{59}{165}$

(b) If both students are girls, then 2 girls out of 40 can be selected in ${}^{40}C_2$ ways

$= \frac{40!}{38!\ 2!} = \frac{40 \times 39}{2} = 780$

∴ Number of favourable cases (m) = 780

$P(\text{both girls}) = \frac{m}{n} = \frac{780}{4950} = \frac{26}{165}$

(c) Number of favourable cases (m) = number of selection of 1 boy from 60 and number of selection of 1 girl from 40 = C(60, 1) × C(40, 1)

$\therefore P(\text{one boy and one girl}) = \frac{m}{n}$

$= \frac{C(60,1) \times C(40,1)}{4950} = \frac{60 \times 40}{4950} = \frac{16}{33}$

39. 2062 Q.No. 8 a

State and prove "The Theorem of Total Probability". [4]

➥ Please refer to 2057 Q.No. 8 b

40. 2062 Q.No. 8 a OR

A lot contains 10 items of which 3 are defective. Three items are chosen from the lot at random one after another without replacement. Find the probability that:

(i) All three are defective.

(ii) None of them are defective. [4]

SOLUTION

(i) Since the items are not replaced, so

$P(\text{first defective item}) = \frac{m}{n} = \frac{3}{10}$

$P(\text{second defective item}) = \frac{m}{n} = \frac{2}{9}$

$P(\text{third defective item}) = \frac{m}{n} = \frac{1}{8}$

Since the events are independent,

$P(\text{all three defective items}) = \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{1}{120}$

(ii) Probability that none of them are defective

= 1 – P(all three defective items)

$= 1 - \frac{1}{120} = \frac{119}{120}$

41. 2063 Q.No. 8 a

If P(A) and P(B) be the probabilities of the independent events A and B respectively, prove that: $P(A \cap B) = P(A) \cdot P(B)$ where $P(A \cap B)$ has the usual meaning. [4]

➥ Please refer to 2061 Q.No. 8 a

42. 2064 Q.No. 8 a

If P(A) and P(B) are the probabilities of the happening of the events A and B respectively, prove that: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where $P(A \cup B)$ and $P(A \cap B)$ have the usual meanings. What will be the form of the above formula if A and B are independent events? [4]

➥ Please refer to 2057 Q.No. 8b

43. 2065 Q.No 8 a

State and prove the theorem of compound probability. [4]

➥ Please refer to 2061 Q.No. 8a

44. 2066 C Q.No. 8 a

State and prove the theorem of total probability. [4]

➥ Please refer to 2057 Q.No. 8 b

45. 2066 C Q.No. 8 a OR

If A, B, C are three mutually exclusive events with $\frac{1}{3}P(A) = \frac{2}{3}P(B) = \frac{1}{6}P(C)$. Find P (A), P (B) and P (C). [4]

➥ Please refer to 2059 Q.No. 8 a

46. 2066 Q.No. 8 a OR

Five men in a group of 20 are graduates. If three men are chosen out of 20 at random, what is the probability of at least one being graduates? [4]

SOLUTION

n = Total number of possible cases

= Number of selection of 3 out of 20

$= {}^{20}C_3 = \frac{20!}{17!\ 3!} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1{,}140.$

Since 5 men out of 20 are graduates, so number of non-graduates = 20 – 5 = 15

Now, 3 men out of 15 can be selected in ${}^{15}C_3$ ways.

$= \frac{15!}{12!\ 3!} = \frac{15 \times 14 \times 13}{3 \times 2 \times 1} = 455$

$P(\text{non-graduate}) = \frac{455}{1{,}140} = \frac{91}{228}$

∴ P(at least one graduate)

$= 1 - P(\text{non-graduate}) = 1 - \frac{91}{228} = \frac{317}{228}$

47. 2067 Q.No. 8a

A class consists of 40 boys and 60 girls. If two students are chosen at random, what will be the probability that (a) both are boys (b) both are girls (c) one boy and one girl? [4]

SOLUTION

Total number of students = 40 + 60 = 100

Total number of possible cases (n) = Number of selection of 2 students out of 100

$= {}^{100}C_2 = \frac{100!}{98!\ 2!} = \frac{100 \times 99}{2} = 4950$

(a) If both students are boys, then 2 boys out of 40 can be selected in ${}^{40}C_2$ ways

$= \frac{40!}{38!\,2!} = \frac{40 \times 39}{2} = 780$

∴ Number of favourable cases (m) = 780

P(both boys) $= \frac{m}{n} = \frac{780}{4950} = \frac{26}{165}$

(b) If both students are girls, then 2 girls out of 60 can be selected in $^{60}C_2$ ways

$= \frac{60!}{58!\,2!} = \frac{60 \times 59}{2} = 1770$

∴ Number of favourable cases (m) = 1770

P(both girls) $= \frac{m}{n} = \frac{1770}{4950} = \frac{59}{165}$

(c) Number of favourable cases (m) = number of selection of 1 boy from 40 and number of selection of 1 girl from 60 = C(40, 1) × C(60, 1)

∴ P(one boy and one girl) $= \frac{m}{n}$

$= \frac{C(40,1) \times C(60,1)}{4950} = \frac{40 \times 60}{4950} = \frac{16}{33}$

48. 2068 Q.No. 8b

The chance that A can solve a certain problem is $\frac{1}{4}$. The chance that B can solve it is $\frac{2}{3}$, find the chance that the problem will be solved if they both try. **[4]**

➥ Please refer to Model Set II, Q.No. 8b

49. 2069 (Set A) Old Q.No. 8a

Suppose 4 cards are drawn at random from a well-shuffled deck of 52 cards.

i. What is the probability that all 4 are spade?

ii. What is the probability that all 4 are black? **[4]**

SOLUTION

In a deck of 52 cards, there are 13 spades.

Now, n = Total number of possible cases

= Number of selection of 4 cards out of 52

$= {}^{52}C_4$

m = number of favourable cases

= number of selection of 4 cards out of 13

$= {}^{13}C_4$

P(4 are spade) = ?

We have,

P(4 are spade) $= \frac{m}{n} = \frac{^{13}C_4}{^{52}C_4} = \frac{\frac{13!}{9!\,4!}}{\frac{52!}{48!\,4!}}$

$= \frac{13 \times 12 \times 11 \times 10}{52 \times 51 \times 50 \times 49} = \frac{11}{4165}$

(ii) In a deck of 52 cards, there are 26 black cards

Now, n = number of possible cases

= number of selection of 4 cards out of 52

$= {}^{52}C_4$

m = number of favourable cases

= number of selection of 4 black cards out of 26 black cards

$= {}^{26}C_4$

P(4 are blacks) = ?

We have,

P(4 are black) $= \frac{m}{n} = \frac{^{26}C_4}{^{52}C_4} = \frac{\frac{26!}{22!\,4!}}{\frac{52!}{48!\,4!}}$

$= \frac{26 \times 25 \times 24 \times 23}{52 \times 51 \times 50 \times 49} = \frac{46}{833}$

50. 2069 (Set B) Q.No. 8b

A bag contains 5 red and 6 white balls. Two balls are drawn at random. Find the probability that (i) both are red (ii) both are of the same colour. **[4]**

SOLUTION

Total number of balls in the bag = 5 + 6 = 11

n = Total number of possible cases

= Number of selections of 2 balls out of 11

$= C(11, 2) = \frac{11!}{9!\,2!} = \frac{11 \times 10}{2} = 55$

(i) **For both red balls,**

Number of favourable cases (m) = number of ways of drawing 2 balls out of 5 = C(5, 2)

$= \frac{5!}{3!\,2!} = \frac{5 \times 4}{2} = 10$

P(both red balls) $= \frac{m}{n} = \frac{10}{55} = \frac{2}{11}$

(ii) For the probability of both balls are of the same colour, the number of favourable cases (m) = number of ways of drawing 2 red balls out of 5 red balls + number of ways of drawing 2 white balls out of 6 white balls.

$= C(5,2) + C(6,2) = \frac{5!}{3!\,2!} + \frac{6!}{4!\,2!}$

$= \frac{5 \times 4}{2} + \frac{6 \times 5}{2} = 25$

P(both balls are of same colour) $= \frac{m}{n} = \frac{25}{55} = \frac{5}{11}$

51. 2069 Old (Set B) Q.No. 8a

State and prove the theorem of "Compound probability". **[4]**

➥ Please refer to 2061 Q.No. 8a

52. 2070 (Old) Q.No. 8 a

State and prove theorem of compound probability. **[4]**

➥ Please refer to 2061 Q.No. 8a

53. 2071 Old Q.No. 12 b

State and prove the theorem of compound probability. **[4]**

➥ Please refer to 2061 Q.No. 8a

54. 2073 Set D Q.No. 8b

A class consists of 60 boys and 40 girls. If two students are chosen at random, what is the probability that (i) both are boys (ii) one boy and one girl. **[4]**

➥ Please refer to to 2061 Q.No. 8 a OR

55. 2075 Set A Q.No. 8b

A lot contains 10 items of which 3 are defective. Three items are chosen from the lot at random one after another without replacement. Find the probability that (i) all three are defective (ii) only first one is defective. **[4]**

SOLUTION

i. Since the items are not replaced,

P(first defective item) $= \frac{m}{n} = \frac{3}{10}$

P(second defective item) $= \frac{m}{n} = \frac{2}{9}$

P(third defective item) $= \frac{m}{n} = \frac{1}{8}$

Since the events are independent,

∴ P(all three defective items) $= \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{1}{120}$

ii. When only first one is defective, the second and third are non-defective.

P(first item defective) $= \frac{m}{n} = \frac{3}{10}$

P(second item not defective) $= \frac{m}{n} = \frac{7}{9}$

P(third item not defective) $= \frac{m}{n} = \frac{6}{8}$

∴ P(only first defective items) $= \frac{3}{10} \times \frac{7}{9} \times \frac{6}{8} = \frac{7}{40}$

B. BINOMIAL DISTRIBUTION

2 MARKS QUESTIONS

56. 2061 Q.No. 4 c

Find the binomial distribution having mean = 12 and variance = 8. **[2]**

SOLUTION

Here, mean = 12

np = 12 ... (i)

Again, variance = 8

npq = 8 ... (ii)

Dividing equation (ii) by equation (i), we get

$\frac{npq}{np} = \frac{8}{12}$

$q = \frac{2}{3}$

We have, $p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

From (i)

$n \cdot \frac{1}{3} = 12$

n = 36

Required binomial distribution is

$(q + p)^n = \left(\frac{2}{3} + \frac{1}{3}\right)^{36}$

57. 2066 C Q.No. 4 c

If three dice are thrown, what is the probability of getting exactly 3 sixes? **[2]**

SOLUTION

p = probability of getting a six in one trial of a die $= \frac{1}{6}$

∴ $q = 1 - \frac{1}{6} = \frac{5}{6}$

n = 3

Probability of getting exactly 3 sixes = P(3) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

∴ $P(3) = {}^3C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{3-3} = 1 \times \frac{1}{216} \times 1 = \frac{1}{216}$

58. 2069 (Set B) Q.No. 4c

A dice is thrown 3 times. Getting a 2 or 3 is numbered as a success. Find the probabilities of getting two successes. **[2]**

SOLUTION

Here,

p = probability of getting 2 or 3 in one trial of a die

$= \frac{2}{6} = \frac{1}{3}$

∴ $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

n = 3

Probability of getting 2 successes = P(2) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

∴ $P(2) = {}^3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{3-2} = \frac{3!}{1!\,2!} \times \frac{1}{9} \times \frac{2}{3}$

$= 3 \times \frac{1}{9} \times \frac{2}{3} = \frac{2}{9}$

59. 2073 Set D Q.No. 4c

A dice is rolled 4 times. Getting an even number is considered as a success. Find the probability of getting two successes. **[2]**

SOLUTION

Probability of getting an even number in one throw of a dice (p) $= \frac{3}{6} = \frac{1}{2}$

∴ $q = 1 - p = \frac{1}{2}$

Number of trials (n) = 4

Probability of getting 2 successes P(2) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

∴ $P(2) = {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{4!}{2!\,2!} \times \frac{1}{16} = \frac{3}{8}$

60. 2074 Set A Q.No. 4c

In 8 throws of a dice, turning of 1 or 6 is considered to be a success. Find the mean and standard deviation. **[2]**

SOLUTION

Here, p = Probability of turning of 1 or 6 in a throw of a dice $= \frac{2}{6} = \frac{1}{3}$

$\therefore\ q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

n = no. of trials = 8

Mean $= np = 8 \times \frac{1}{3} = \frac{8}{3}$

S.D $= \sqrt{npq} = \sqrt{8 \times \frac{1}{3} \times \frac{2}{3}} = \frac{4}{3}$

61. 2075 Set A Q.No. 4c

The mean of a binomial distribution is 80 and standard deviation 8, find the value of p, the probability of a success. [2]

SOLUTION

Mean = 80

or, $np = 80$ ... (i)

Again, standard deviation = 8

$\sqrt{npq} = 8$

or, $npq = 64$ ... (ii)

Dividing (ii) by (i), we get,

$\frac{npq}{np} = \frac{64}{80}$

or, $q = \frac{4}{5}$.

We have,

$p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$

4 MARKS QUESTIONS

62. 2057 Q.No. 8 b OR

A dice is thrown 3 times. Getting a '5' or '6' is numbered a success. Find the probability of getting (a) 3 successes and (b) exactly 2 successes. [4]

SOLUTION

Here,

p = probability of getting '5' or '6' in 1 trial of a die $= \frac{2}{6} = \frac{1}{3}$

Then, $q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

Number of trials (n) = 3

(a) $P(3) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(3) = {}^3C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 = 1 \times \frac{1}{27} \times 1 = \frac{1}{27}$

(b) $P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(2) = {}^3C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^1 = \frac{3!}{1!\,2!} \times \frac{1}{9} \times \frac{2}{3} = \frac{2}{9}$

63. 2058 Q.No. 8 a OR

The probability of hitting a target is 1/5. If six hittings are made, find the probability that: (i) none will strike the target (ii) exactly 2 will strike the target. [4]

SOLUTION

Here, $p = \frac{1}{5}$

$\therefore\ q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

$n = 6$

(i) Probability that non will strike the target = $P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(0) = {}^6C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{6-0} = 1 \times 1 \times \left(\frac{4}{5}\right)^6 = \frac{4096}{15625}$

(ii) Probability that exactly 2 will strike the target = $P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(2) = {}^6C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{6-2} = \frac{6 \times 5}{2} \times \frac{1}{25} \times \frac{256}{625} = \frac{768}{3125}$

64. 2059 Q.No. 8 a OR

If 4 dice are thrown, what is probability of getting (i) exactly 3 sixes (ii) exactly 2 sixes and (iii) no sixes. [4]

SOLUTION

Here, p = probability of getting a six in a throw of a die $= \frac{1}{6}$

then $q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

$n = 4$

(i) Probability of getting exactly 3 sixes = $P(3) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$P(3) = {}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{4-3} = \frac{4!}{1!\,3!} \times \frac{1}{216} \times \frac{5}{6} = \frac{5}{324}$

(ii) Probability of getting 2 sixes = $P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(2) = {}^4C_2 \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^{4-2}$

$= \frac{4!}{2!\,2!} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2 = \frac{4 \times 3}{2} \times \frac{1}{36} \times \frac{25}{36} = \frac{25}{216}$

(iii) Probability of getting no sixes = $P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(0) = {}^4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{4-0} = 1 \times 1 \times \frac{625}{1296} = \frac{625}{1296}$

65. 2060 Q.No. 8 a OR

A sample of 100 fuses is known to have an average 5 defective fuses. Three fuses of sample are tested. What is the probability that (i) none of them is defective (ii) exactly one of them is defective? [4]

SOLUTION

Here, p = Probability of a defective fuse $= \frac{5}{100} = \frac{1}{20}$

$\therefore\ q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

$n = 3$

(i) $P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(0) = {}^3C_0 \left(\frac{1}{20}\right)^0 \left(\frac{19}{20}\right)^{3-0} = 1 \times 1 \times \left(\frac{19}{20}\right)^3 = \frac{6859}{8000}$

(ii) $P(1) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(1) = {}^3C_1 \left(\frac{1}{20}\right)^1 \left(\frac{19}{20}\right)^{3-1}$

$= 3 \times \left(\frac{1}{20}\right) \times \left(\frac{19}{20}\right)^2 = \frac{1083}{8000}$

66. 2063 Q.No. 8 a OR

The incidence of occupation disease in an industry is such that the workmen have a 20% chance of suffering from it. What is the probability that out of six workmen four or more will contact the disease? [4]

SOLUTION

Here, $p = 20\% = \frac{20}{100} = \frac{1}{5}$

$\therefore\ q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

$n = 6$

Probability that 4 or more will contact the disease $p(r \geq 4) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(r \geq 4) = P(4) + P(5) + P(6)$

$= {}^6C_4 \left(\frac{1}{5}\right)^4 \left(\frac{4}{5}\right)^{6-4} + {}^6C_5 \left(\frac{1}{5}\right)^5 \left(\frac{4}{5}\right)^{6-5} + {}^6C_6 \left(\frac{1}{5}\right)^6 \left(\frac{4}{5}\right)^{6-6}$

$= \frac{6!}{2!\,4!} \times \left(\frac{1}{5}\right)^4 \times \left(\frac{4}{5}\right)^2 + \frac{6!}{1!\,5!} \times \left(\frac{1}{5}\right)^5 \times \left(\frac{4}{5}\right) + 1 \times \left(\frac{1}{5}\right)^6$

$= \frac{15 \times 16}{5^6} + \frac{6 \times 4}{5^6} + \frac{1}{5^6} = \frac{240 + 24 + 1}{5^6} = \frac{265}{15625} = \frac{53}{3125}$

67. 2064 Q.No. 8 a OR

Suppose that in a certain city 60% of all recorded births are males. If we select 5 births from the population, what will be the probability that: (i) none of them is male (ii) exactly three of them are male [4]

SOLUTION

Given, $p = 60\% = \frac{60}{100} = \frac{3}{5}$

$\therefore\ q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$

$n = 5$

(i) $P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(0) = {}^5C_0 \left(\frac{3}{5}\right)^0 \left(\frac{2}{5}\right)^{5-0} = 1 \times 1 \times \frac{32}{3{,}125} = \frac{32}{3{,}125}$

(ii) $P(3) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(3) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^{5-3} = \frac{5 \times 4}{2} \times \frac{27}{125} \times \frac{4}{25} = \frac{216}{625}$

68. 2065 Q.No 8 a OR

If three dices are thrown what is the probability of getting (i) exactly 3 sixes (ii) exactly 2 sixes [4]

SOLUTION

(i) p = probability of getting a six in one trial of a die $= \frac{1}{6}$

$\therefore\ q = 1 - \frac{1}{6} = \frac{5}{6}$

$n = 3$

Probability of getting exactly 3 sixes = $P(3) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(3) = {}^3C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{3-3} = 1 \times \frac{1}{216} \times 1 = \frac{1}{216}$

(ii) $P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(2) = {}^3C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{3-2} = 3 \times \frac{1}{36} \times \frac{5}{6} = \frac{5}{72}$

69. 2066 Q.No. 8 a

The probability of hitting a target is $\frac{1}{6}$, if eight hitting are made find the probability that (i) none will strike the target, (ii) exactly two will strike the target. [4]

SOLUTION

Here, p = probability of hitting a target $= \frac{1}{6}$

$\therefore\ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

$n = 8$

(i) Probability of none will strike the target = $P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(0) = {}^8C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{8-0} = 1 \times 1 \times \left(\frac{5}{6}\right)^8$

$= \frac{390625}{1679616}$

(ii) Probability that exactly two will strike the target = $P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore\ P(2) = {}^8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^{8-2} = \frac{8 \times 7}{2} \times \frac{1}{36} \times \left(\frac{5}{6}\right)^6$

$= \frac{109375}{419904}$

70. 2067 Q.No. 8a Or

Define Binomial distribution, and its mean and variance, hence find the probability of getting three heads in five tosses of two coins. [4]

SOLUTION

Binomial distribution: Let a trial be repeated so as to make a set of n trials. Let p be the probability of success and q be the probability of failure in one trial such that p + q = 1. Then the probability of r success out of n independent trials is given by:

$P(r) = {}^nC_r\, p^r q^{n-r}$, $0 \le r \le n$

This distribution is known as binomial distribution.

The mean of the binomial distribution is given by np and variance by npq.

Next Part

p = probability of getting two head with two coins in one trial = $\frac{1}{4}$

$\therefore q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

n = 5

Probability of getting two heads thrice P(3) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(3) = {}^5C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^{5-3} = \frac{5!}{2!\,3!} \times \frac{1}{64} \times \frac{9}{16}$

$= 10 \times \frac{1}{64} \times \frac{9}{16} = \frac{45}{512}$

71. 2068 Q.No. 8b Or

A dice is thrown 3 times. Getting a '5' or '6' is numbered a success. Find the probability of getting (i) 3 successes (ii) exactly 2 successes. [4]

➥ Please refer to 2057 Q.No. 8 b OR

72. 2069 (Set A) Q.No. 8b

If 20% of the electric bulbs manufactured by a company are defective, find the probability that out of 4 bulbs chosen at random (i) 1 (ii) at most 2 bulbs will be defective. [4]

SOLUTION

Here, $p = 20\% = \frac{20}{100} = \frac{1}{5}$

$\therefore q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

n = 4

(i) P(1) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(1) = {}^4C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^3 = \frac{4!}{3!\,1!} \times \frac{1}{5} \times \frac{64}{125}$

$= 4 \times \frac{1}{5} \times \frac{64}{125} = \frac{256}{625}$

(ii) P(at most 2 defective bulbs) = P(r ≤ 2) = P(0) + P(1) + P(2)

$= {}^4C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{4-0} + {}^4C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{4-1} + {}^4C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{4-2}$

$= 1 \times 1 \times \frac{256}{625} + 4 \times \frac{64}{625} + 6 \times \frac{16}{625}$

$= \frac{256 + 256 + 96}{625} = \frac{608}{625}$

73. 2069 (Set A) Old Q.No. 8a or

If 20% of the electric bulbs manufactured by a company are defective, find the probability that out of 4 bulbs chosen at random (i) 1 (ii) 2 bulbs will be defective. [4]

➥

i. Please refer to 2069 (Set A) Q.No. 8b

ii. P(2 defective bulbs) = P(r = 2) = P(2)

$= {}^4C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{4-2} = 6 \times \frac{16}{625} = \frac{96}{625}$

74. 2069 Old (Set B) Q.No. 8a Or

If 4 dice are thrown, what is the probability of getting (i) exactly 3 sixes and (ii) no sixes. [4]

SOLUTION

Here, p = probability of getting a six in one throw of a die = $\frac{1}{6}$

then $q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

n = 4

(i) Probability of getting exactly 3 sixes = P(3) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$P(3) = {}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{4-3} = \frac{4!}{1!\,3!} \times \frac{1}{216} \times \frac{5}{6} = \frac{5}{324}$

(ii) Probability of getting no sixes = P(0) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(0) = {}^4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^{4-0} = 1 \times 1 \times \frac{625}{1296} = \frac{625}{1296}$

75. 2070 Set C Q.No. 8 b

A certain manufacturing process produces electrical fuses of which 15% are defective. Find the probability that in a sample of 10 fuses selected at random there will be (i) no defective (ii) not more than one defective. [4]

SOLUTION

Here, $p = 15\% = \frac{15}{100} = \frac{3}{20}$

$\therefore q = 1 - p = 1 - \frac{3}{20} = \frac{17}{20}$

n = 10

(i) P(0) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(0) = {}^{10}C_0 \left(\frac{3}{20}\right)^0 \left(\frac{17}{20}\right)^{10-0}$

$= 1 \times 1 \times \left(\frac{17}{20}\right)^{10} = 0.1969$

(ii) P(not more than one defective) = P (r ≤ 1)

= P(0) + P(1)

$= {}^{10}C_0 \left(\frac{3}{20}\right)^0 \left(\frac{17}{20}\right)^{10-0} + {}^{10}C_1 \left(\frac{3}{20}\right)^1 \left(\frac{17}{20}\right)^{10-1}$

$= 1 \times 1 \times \left(\frac{17}{20}\right)^{10} + 10 \times \frac{3}{20} \times \left(\frac{17}{20}\right)^9 = 0.5443$

76. 2070 Set D Q.No. 8 b

A coin is tossed 5 times. Find the probability of getting(i) two heads (ii) at least two heads. [4]

SOLUTION

p = probability of getting head in a trial of a coin = $\frac{1}{2}$

$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

n = number of trials = 5

(i) P(2) = ?

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(2) = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2} = \frac{5!}{3!\,2!} \times \frac{1}{32}$

$= \frac{5 \times 4}{2} \times \frac{1}{32} = \frac{5}{16}$

(ii) P(r ≥ 2) = P(2) + P(3) + P(4) + P(5)

$= {}^5C_2 \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 + {}^5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + {}^5C_5 \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^0$

$= \frac{1}{32} ({}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)$

$= \frac{1}{32}(10 + 10 + 5 + 1) = \frac{26}{32} = \frac{13}{16}$

77. 2070 (Old) Q.No. 8 a or

If 20% of the electric bulbs manufactured by a company are defective, find the probability that out of 4 bulbs chosen at random (i) one (ii) zero (iii) at most 2 bulbs will be defective. [4]

SOLUTION

Here, $p = 20\% = \frac{20}{100} = \frac{1}{5}$

$\therefore q = 1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

n = 4

(i) P(1) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(1) = {}^4C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^3 = \frac{4!}{3!\,1!} \times \frac{1}{5} \times \frac{64}{125}$

$= 4 \times \frac{1}{5} \times \frac{64}{125} = \frac{256}{625}$

(ii) P(0) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(0) = {}^4C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{4-0} = 1 \times 1 \times \frac{256}{625} = \frac{256}{625}$

(iii) P(at most 2 defective bulbs) = P(r ≤ 2)

= P(0) + P(1) + P(2)

$= {}^4C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{4-0} + {}^4C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^{4-1} + {}^4C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^{4-2}$

$= 1 \times 1 \times \frac{256}{625} + 4 \times \frac{64}{625} + 6 \times \frac{16}{625}$

$= \frac{256 + 256 + 96}{625} = \frac{608}{625}$

78. 2070 Supp. Q.No. 8 b

The probability of a man's hitting a target is $\frac{1}{4}$. If he fires 5 times, what is the probability of his hitting the target.

i. exactly thrice

ii. at least thrice [4]

SOLUTION

Here,

p = Probability of hitting a target = $\frac{1}{4}$

Then

$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

n = 5

(i) P(3) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$= {}^5C_3 \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^{5-3} = \frac{5!}{2!\,3!} \times \frac{1}{64} \times \frac{9}{16}$

$= \frac{5 \times 4}{2} \times \frac{1}{64} \times \frac{9}{16} = \frac{45}{512}$

(ii) P(r ≥ 3) = P(3) + P(4) + P(5)

$= {}^5C_3 \left(\frac{1}{4}\right)^3 \cdot \left(\frac{3}{4}\right)^2 + {}^5C_4 \left(\frac{1}{4}\right)^4 \cdot \left(\frac{3}{4}\right)^1 + {}^5C_5 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^0$

$= \frac{5!}{2!\,3!} \times \frac{9}{1024} + \frac{5!}{1!\,4!} \times \frac{3}{1024} + 1 \times \frac{1}{1024} \times 1$

$= 10 \times \frac{9}{1024} + 5 \times \frac{3}{1024} + \frac{1}{1024} = \frac{1}{1024}(90 + 15 + 1)$

$= \frac{106}{1024} = \frac{53}{512}$

79. 2071 Old Q.No. 12 b OR

If 20% of the electric bulbs manufactured by a company are defective, find the probability that out of 4 bulbs chosen at random (i) 1 (ii) 0 (iii) at most 2 bulbs will be defective. [4]

➥ Please refer to 2070 (Old) Q.No. 8a OR

80. 2071 Set C Q.No. 8 b

A dice is rolled 3 times. Getting 5 or 6 is numbered as success. Find the probability of getting (i) 2 successes (ii) 3 successes. **[4]**

➥ Please refer to 2057 Q.No. 8 b OR

81. 2071 Set D Q.No. 8 b

Suppose that in a certain city 60% of all recorded births are males, suppose we select 5 birth records from the population. What is the probability that: i) three of them are males ii) more than 4 are males. **[4]**

SOLUTION

Here, $p = 60\% = \frac{60}{100} = \frac{3}{5}$

$\therefore q = 1 - p = 1 - \frac{3}{5} = \frac{2}{5}$

$n = 5$

(i) $P(3) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(3) = {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 = \frac{5!}{2!\,3!} \times \frac{27}{125} \times \frac{4}{125}$

$= \frac{5 \times 4}{2} \times \frac{27}{125} \times \frac{4}{25} = \frac{216}{625}$

(ii) $P(r > 4) = P(5) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(5) = {}^5C_5 \cdot \left(\frac{3}{5}\right)^5 \cdot \left(\frac{2}{5}\right)^0 = 1 \times \frac{343}{3125} \times 1 = \frac{343}{3125}$

82. 2071 Supp. Q.No. 8b

The probability of a man's hitting a target is $\frac{1}{4}$. If he fires 5 times, what is the probability of his hitting the target
i) exactly thrice ii) none
iii) at least thrice. **[4]**

SOLUTION

(i) **exactly thrice and (iii) at least thrice**
Please refer to 2070 Supp. Q.No. 8 b

(ii) p = Probability of hitting a target $= \frac{1}{4}$

Then

$q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

$n = 5$

$P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(0) = {}^5C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{5-0} = 1 \times 1 \times \frac{243}{1024} = \frac{243}{1024}$

83. 2072 Set C Q.No. 8b

Define Binomial distribution. Find the probability of getting (i) two heads (ii) at least two heads in 5 tosses of a coin. **[4]**

SOLUTION

Binomial distribution: Let a trial be repeated so as to make a set of n trials. Let p be the probability of success and q be the probability of failure in one trial such that p + q = 1. Then the probability of r success out of n independent trials is given by:

$P(r) = {}^nC_r\, p^r q^{n-r}, \quad 0 \le r \le n$

This distribution is known as binomial distribution.

Second Part: Please refer to 2070 Set D Q.No. 8 b

84. 2072 Set D Q.No. 8b

The probability of hitting a target is found to be 0.25. If eight hits are made, find the probability that (i) none will hit the target (ii) exactly two will hit the target. **[4]**

SOLUTION

Given,

p = Probability of hitting a target

$= 0.25 = \frac{25}{100} = \frac{1}{4}$

Then, $q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

$n = 8$

(i) Probability of non-will hit the target

$= P(0) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(0) = {}^8C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^8 = 1 \times \left(\frac{3}{4}\right)^8 = \frac{6{,}561}{65{,}536}$

(ii) Probability that exactly two will hit the target

$= P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(2) = {}^8C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{8-2}$

$= \frac{8!}{6!\,2!} \times \left(\frac{1}{4}\right)^2 \times \left(\frac{3}{4}\right)^6 = \frac{7 \times 8}{2} \times \frac{1}{16} \times \frac{729}{4{,}096} = \frac{5{,}103}{16{,}384}$

85. 2072 Set E Q.No. 8b

Four coins are tossed simultaneously. What is the probability of getting
(i) 2 heads
(ii) 4 heads. **[4]**

SOLUTION

Probability of getting a head $(p) = \frac{1}{2}$

Then, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}, n = 4$

(i) $P(2) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(2) = {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{4!}{2!\,2!} \times \frac{1}{4} \times \frac{1}{4}$

$= \frac{4 \times 3}{2} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{8}$

(ii) $P(4) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(4) = {}^4C_4 \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^{4-4} = 1 \cdot \frac{1}{16} \cdot 1 = \frac{1}{16}$

86. 2072 Supp Q.No. 8b

A certain manufacturing process produce electrical fuses of which 15% are defective. Find the probability that in a sample of 10 fuses selected at random there will be (i) no defective (ii) not more than one defective. **[4]**

➥ Please refer to 2070 Set C Q.No. 8 b

87. 2073 Set C Q.No. 8b

If 4 dice are rolled simultaneously, what is the probability of getting (i) exactly 3 sixes (ii) exactly 2 sixes? **[4]**

SOLUTION

Here, p = probability of getting a six in a throw of a die $= \frac{1}{6}$

then $q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$

$n = 4$

(i) Probability of getting exactly 3 sixes = P(3) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$P(3) = {}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^{4-3} = \frac{4!}{1!\,3!} \times \frac{1}{216} \times \frac{5}{6} = \frac{5}{324}$

(ii) Probability of getting 2 sixes = P(2) = ?

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(2) = {}^4C_2 \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^{4-2} = \frac{4!}{2!\,2!} \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right)^2$

$= \frac{4 \times 3}{2} \times \frac{1}{36} \times \frac{25}{36} = \frac{25}{216}$

88. 2073 Supp Q.No. 8b

The probability of hitting a target is 0.2. If six hits are made, find the probability that (i) exactly one will hit the target (ii) exactly two will hit the target. **[4]**

SOLUTION

Here,

p = probability of hitting a target = 0.2

Then, $q = 1 - p = 1 - 0.2 = 0.8$

n = numbers of hitting = 6

P(r) = prob. of getting r successes in n trials

$= {}^nC_r p^r q^{n-r}$

i. $P(1) = {}^6C_1 p^1 q^{6-1} = \frac{6!}{(6-1)!\,1!} \times 0.2 \times (0.8)^5 = 0.3932$

ii. $P(2) = {}^6C_2 p^2 q^{6-2} = \frac{6!}{4!\,2!} \times (0.2)^2 \times (0.8)^4$

$= \frac{6 \times 5}{2} \times (0.2)^2 \times (0.8)^4 = 0.2458$

89. 2074 Set A Q.No. 8b

A company produces electronic chips by a process that manufactures normally average 20% defective products. A sample of four chips is selected at random and the parts are tested for certain characteristics, what is the probability that (i) no chip is defective (ii) one chip is defective (iii) more than one chips are defective. **[4]**

SOLUTION

(i) $P(0) = ?$

We have, $P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(0) = {}^4C_0 \left(\frac{1}{5}\right)^0 \left(\frac{4}{5}\right)^{4-0} = 1 \times 1 \times \frac{256}{625} = \frac{256}{625}$

(ii) $P(1) = ?$

We have,

$P(r) = {}^nC_r\, p^r q^{n-r}$

$\therefore P(1) = {}^4C_1 \left(\frac{1}{5}\right)^1 \left(\frac{4}{5}\right)^3 = \frac{4!}{3!\,1!} \times \frac{1}{5} \times \frac{64}{125}$

$= 4 \times \frac{1}{5} \times \frac{64}{125} = \frac{256}{625}$

(iii) Prob. of getting more than 1 chips defective

$P(r > 1) = 1 - P(0) - P(1) = 1 - \frac{256}{625} - \frac{256}{625}$

$= \frac{625 - 256 - 256}{625} = \frac{113}{625}$

90. 2074 Set B Q.No. 8b

The average percentage of failure in a certain examination is 70. What is the probability that out of 5 candidates at least 3 will pass? **[4]**

SOLUTION

Percentage of pass = 100% − 70% = 30%

Let,

p = probability of pass of a candidate $= \frac{30}{100} = \frac{3}{10}$

$\therefore q = 1 - p = 1 - \frac{3}{10} = \frac{7}{10}$

No. of candidates (n) = 5

Probability of at least 3 candidates will pass $= P(r \ge 3)$

$= P(3) + P(4) + P(5)$

$= {}^5C_3 p^3 q^{5-3} + {}^5C_4 p^4 q^{5-4} + {}^5C_5 p^5 q^{5-5}$

$= \frac{5!}{2!\,3!} \left(\frac{3}{10}\right)^3 \left(\frac{7}{10}\right)^2 + \frac{5!}{1!\,4!} \left(\frac{3}{10}\right)^4 \left(\frac{7}{10}\right) + 1 \times \left(\frac{3}{10}\right)^5$

$= 10 \times \frac{27}{1000} \times \frac{49}{100} + 5 \times \frac{81}{10000} \times \frac{7}{10} + \frac{243}{100000}$

$= 0.1323 + 0.02835 + 0.00243 = 0.16308$

91. 2074 Supp Q.No. 8b

Out of 32 students in a class, 8 students are girls. If 3 students are selected, find the probability that **[4]**
i. one student is a boy

ii. 2 students are boys and one girl.

SOLUTION

Total no. of students = 32

No. of girls = 8

$\therefore$ No. of boys = 32 - 8 = 24

Now,

p = prob. of getting a boy $= \frac{24}{32} = \frac{3}{4}$

$\therefore q = 1 - p = 1 - \frac{3}{4} = \frac{1}{4}$

Here, n = 3

i. P(1) = ?

We have,

$P(r) = {}^nC_r p^r q^{n-r}$

$\therefore P(1) = {}^3C_1 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^{3-1}$

$= \frac{3!}{(3-1)!\,1!} \cdot \frac{3}{4} \cdot \frac{1}{16}$

$= \frac{3 \times 2!}{2!} \times \frac{3}{4} \times \frac{1}{16}$

$= \frac{9}{64}$

ii. P(2) = ? [If 2 boys comes then automatically 1 girl also comes]

We have,

$P(r) = {}^nC_r p^r q^{n-r}$

$\therefore P(2) = {}^3C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^{3-2}$

$= \frac{3!}{1!\,2!} \times \frac{9}{16} \times \frac{1}{4}$

$= 3 \times \frac{9}{16} \times \frac{1}{4} = \frac{27}{64}$

92. 2075 Set B Q.No. 8b

A company produces electronics chips by a process that manufactures normally average 20% defective products. Sample of four chips is selected at random and the parts are tested for certain characteristics, what is the probability that (i) no chip is defective (ii) one chip is defective (iii) more than one chip are defective. [4]

➥ Please refer to 2074 Set A Q.No. 8b

93. 2075 Set C Q.No. 8b

The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(x \geq 1)$. [4]

SOLUTION

Given, mean = 4

$\therefore np = 4 \quad \ldots (i)$

And, variance $= \frac{4}{3}$

$\therefore npq = \frac{4}{3} \quad \ldots (ii)$

Dividing (ii) by (i), we get

$\frac{npq}{np} = \frac{4/3}{4}$

$\therefore q = \frac{1}{3}$

We have, $p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$

Substituting the value of p in (i), we get,

$n \times \frac{2}{3} = 4$

n = 6

Now,

$P(x \geq 1)$

$= P(1) + P(2) + P(3) + P(4) + P(5) + P(6)$

$= 1 - P(0)$

$= 1 - {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0} \quad [\because P(r) = {}^nC_r p^r q^{n-r}]$

$= 1 - \frac{6!}{6!\,0!} \times 1 \times \frac{1}{36}$

$= 1 - \frac{1}{729}$

$= \frac{729 - 1}{729} = \frac{728}{729}$

UNIT 12
STATICS

2 MARKS QUESTIONS

1. 2057 Q.No. 5 a

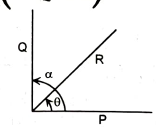
Write the expression for the magnitude and the direction of the resultant of two forces acting at a given angle. [2]

SOLUTION

Let P and Q be given forces and R be their resultant. If angle between P and Q be α then resultant $(R) = \sqrt{P^2 + Q^2 + 2PQ \cos\alpha}$

Also, if θ be the angle made by R with P then,

$\theta = \tan^{-1}\left(\frac{Q \sin\alpha}{P + Q\cos\alpha}\right)$

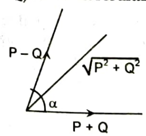


2. 2058 Q.No. 5 a

At what angle do forces equal to (P + Q) N and (P - Q) N act so that the resultant may be $\sqrt{P^2 + Q^2}$? [2]

SOLUTION

Let α be the angle between the forces (P + Q)N and (P – Q)N. Given resultant is $\sqrt{P^2+Q^2}$ N



We have,

$R^2 = P^2 + Q^2 + 2PQ \cos\alpha$

So, $\left(\sqrt{P^2+Q^2}\right)^2 = (P + Q)^2 + (P - Q)^2 + 2(P + Q)(P - Q)\cos\alpha$

or, $P^2 + Q^2 = P^2 + 2PQ + Q^2 + P^2 - 2PQ + Q^2 + 2(P^2 - Q^2)\cos\alpha$

or, $2(P^2 - Q^2)\cos\alpha = -(P^2 + Q^2)$

$\therefore \cos\alpha = -\frac{1}{2}\left(\frac{P^2 + Q^2}{P^2 - Q^2}\right)$

$\therefore \alpha = \cos^{-1}\left[-\frac{1}{2}\left(\frac{P^2 + Q^2}{P^2 - Q^2}\right)\right]$

3. 2059 Q.No. 5 a

The resultant of two forces P and Q is R. If Q is doubled the new resultant is perpendicular to P. Prove that Q = R. [2]

SOLUTION

Let α be the angle between two forces P and Q. Since R is the resultant, so

$R^2 = P^2 + Q^2 + 2PQ \cos\alpha \quad \ldots (i)$

Again, given that when Q is doubled, the new resultant is perpendicular to P, so we have

$\tan 90° = \frac{2Q \sin\alpha}{P + 2Q\cos\alpha}$

or, $\frac{1}{0} = \frac{2Q\sin\alpha}{P + 2Q\cos\alpha}$

or, $P + 2Q\cos\alpha = 0$

or, $\cos\alpha = \frac{-P}{2Q}$

Putting the value of cos α in equation (i), we get

$R^2 = P^2 + Q^2 + 2PQ\left(\frac{-P}{2Q}\right)$

or, $R^2 = P^2 + Q^2 - P^2$

or, $R^2 = Q^2$

$\therefore Q = R$

4. 2060 Q.No. 5 a

At what angle do forces equal to (P + Q) and (P – Q) act so that the resultant may be $\sqrt{P^2 + Q^2}$? [2]

➥ Please refer to 2058 Q.No. 13 a OR

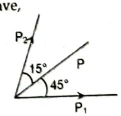
5. 2061 Q.No. 5 a

If a force P be resolved into two forces making angles of 45° and 15° with its directions. Show that the latter force is $\frac{\sqrt{6}}{3}$P. [2]

SOLUTION

Let P_1 and P_2 be the resolved forces of the given force. We have to find the later force P_2.

We have,



$P_2 = \frac{R \sin\alpha}{\sin(\alpha + \beta)}$

Here, R = P, α = 45°, β = 15°

$\therefore P_2 = \frac{P \sin 45°}{\sin 60°} = \frac{P \cdot \frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \frac{2P}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}P}{6}$

$= \frac{\sqrt{6}}{3}P.$

6. 2062 Q.No. 6 b

Two forces whose magnitudes are P and $P\sqrt{2}$ act on a particle in directions inclined at an angle of 135° to each other; find the magnitude and direction of the resultant. [2]

SOLUTION

Here, $P = PN$ $Q = P\sqrt{2}$ N
$\alpha = 135°$ $R = ?$
$\theta = ?$

We have,

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$= P^2 + (P\sqrt{2})^2 + 2\cdot P\cdot P\sqrt{2}\cdot\cos 135°$

$= P^2 + 2P^2 + 2P^2\cdot\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)$

$R^2 = P^2$

$\therefore R = P$ N

Also, we have

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\tan\theta = \frac{P\sqrt{2}\sin 135°}{P + P\sqrt{2}\cos 135°} = \frac{P\sqrt{2}\cdot\frac{1}{\sqrt{2}}}{P + P\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)}$

$= \frac{P}{P - P} = \infty = \tan 90°$

$\therefore \theta = 90°$

Hence, the resultant is PN acting at right angle with the first force.

7. 2063 Q.No. 5 b

At what angle of forces equal to (P + Q) Newton and (P – Q) newton act so that the resultant may be $\sqrt{P^2 + Q^2}$ newton? [2]

➥ Please refer to 2058 Q.No. 5 a

8. 2064 Q.No. 5 b

Forces equal to 7P, 5P and 8P acting on a particle are in equilibrium. Find the angle between the latter pair of forces. [2]

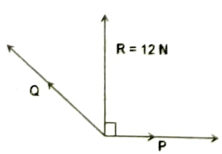
➥ Please refer to Model Set II, Q.No. 12a

9. 2065 Q.No 4 b

The sum of two forces is 18 and the resultant whose direction is perpendicular to the smaller of the two forces is 12, find the magnitude of the forces. [2]

SOLUTION

Given, $P + Q = 18$ ...(i)



Suppose P < Q.

Here, $R = 12$ N
$\theta = 90°$

We have,

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

(where α is the angle between P and Q)

or, $\tan 90° = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\frac{1}{0} = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $P + Q\cos\alpha = 0$

$\therefore \cos\alpha = \frac{-P}{Q}$

Again, we have

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

or, $12^2 = P^2 + Q^2 + 2PQ\left(\frac{-P}{Q}\right)$

or, $144 = P^2 + Q^2 - 2P^2$

or, $Q^2 - P^2 = 144$

or, $(Q + P)(Q - P) = 144$

or, $18(Q - P) = 144$ [Using (i)]

or, $Q - P = 8$ ...(ii)

Solving (i) and (ii), we get

$P = 5$N, $Q = 13$N

10. 2066 C Q.No. 5 b

The resultant of two forces P and Q is R. If Q is doubled the new resultant is perpendicular to P. Prove that Q = R. [2]

➥ Please refer to 2059 Q.No. 5 a

11. 2066 Q.No. 5 a

The resultant of two forces P and Q is R. If Q is doubled, the new resultant is perpendicular to P. Prove that Q = R. [2]

➥ Please refer to 2059 Q.No. 5 a

12. 2067 Q.No. 5a

Find the resultant of two forces equal to 3N and 6N respectively such that their diagonal is perpendicular to the first force. [2]

SOLUTION

Given, $P = 3$ N $Q = 6$ N
$\theta = 90°$ $R = ?$

We have,

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\tan 90° = \frac{6\cdot\sin\alpha}{3 + 6\cdot\cos\alpha}$

or, $\frac{1}{0} = \frac{6\sin\alpha}{3 + 6\cos\alpha}$

or, $3 + 6\cos\alpha = 0$

or, $\cos\alpha = \frac{-1}{2}$

Again, we have

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$= 3^2 + 6^2 + 2\times 3\times 6\times\left(\frac{-1}{2}\right)$

$= 9 + 36 - 18 = 27$

$\therefore R = 3\sqrt{3}$ N

13. 2068 Q.No. 5a

Two forces acting at an angle of 45° have a resultant equal to $\sqrt{10}$ N; if one of the forces be $\sqrt{2}$ N, find the other force. [2]

SOLUTION

Here, $P = \sqrt{2}$ N $R = \sqrt{10}$ N
$\alpha = 45°$ $Q = ?$

We have,

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

or, $(\sqrt{10})^2 = (\sqrt{2})^2 + Q^2 + 2\cdot\sqrt{2}\cdot Q\cdot\cos 45°$

or, $10 = 2 + Q^2 + 2\sqrt{2}\cdot Q\cdot\frac{1}{\sqrt{2}}$

or, $8 = Q^2 + 2Q$

or, $Q^2 + 2Q - 8 = 0$

or, $(Q + 4)(Q - 2) = 0$

or, $Q = -4$ (rejected)

$\therefore Q = 2$ N

14. 2069 (Set A) Q.No. 12a

Two forces whose magnitudes are P and $P\sqrt{2}$ N act on a particle in direction inclined at an angle 135° to each other, find the magnitude and the direction of the resultant. [2]

➥ Please refer to 2062 Q.No. 6 b

15. 2069 (Set A) Old Q.No. 5a

Find the resultant of two forces equal to 3N and 6N acting at an angle of 120°. [2]

SOLUTION

Given, $P = 3$ N $Q = 6$ N
$\alpha = 120°$ $R = ?$
$\theta = ?$

We have,

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

$= 3^2 + 6^2 + 2\times 3\times 6\times\cos 120°$

$= 9 + 36 + 36\left(-\frac{1}{2}\right) = 45 - 18 = 27$

$R = \sqrt{27} = 3\sqrt{3}$ N

Again,

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} = \frac{6\times\sin 120°}{3 + 6\times\cos 120°}$

$= \frac{6\times\frac{\sqrt{3}}{2}}{3 + 6\left(-\frac{1}{2}\right)} = \infty = \tan 90°$

$\therefore \theta = 90°$

The resultant is $3\sqrt{3}$ N acting at right angle with the first force.

16. 2069 (Set B) Q.No. 12a

If a force P be resolved into two forces making angles 45° and 15° with its direction; show that the latter force is $\frac{\sqrt{6}}{3}$P. [2]

➥ Please refer to 2061 Q. No. 5a

17. 2069 Old (Set B) Q.No. 5a

Find the least resultant of two forces of magnitudes 12N and 8N respectively. [2]

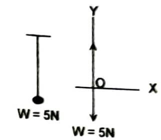
SOLUTION

The least resultant of two forces of magnitudes 12N and 8N is 12N – 8N = 4N.

18. 2069 Old (Set B) Q.No. 5c

A particle weighing 5N is suspended freely from the ceiling by a Weightless inextensible cord. Find the tension in the cord. [2]

SOLUTION



Two forces act upon the particle. The weight is W = 5 N which acts downwards and the tension T opposite to it acts upwards along vertical line (y-axis). There is no-x component.

The y-component are T and W. So, from equilibrium condition,

$\Sigma X_i = 0 \Rightarrow$

$0 = 0$

and $\Sigma Y_i = 0 \Rightarrow T - W = T - 5 = 0 \Rightarrow T = 5$ N

19. 2070 Set C Q.No. 12 a

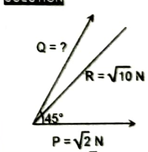
At what angle do the force equal to P + Q and P – Q act so that the resultant may be $\sqrt{P^2+Q^2}$? [2]

➥ Please refer to 2058 Q.No. 5 a

20. 2070 Set D Q.No. 12 a

Two forces acting at an angle of 45° have a resultant equal to $\sqrt{10}$ N; if one of the forces be $\sqrt{2}$ N, find the other force. [2]

SOLUTION



Given, $P = \sqrt{2}$ N, $R = \sqrt{10}$ N
$\alpha = 45°$ $Q = ?$

We have, $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

or, $(\sqrt{10})^2 = (\sqrt{2})^2 + Q^2 + 2\sqrt{2}\cdot Q\cdot\cos 45°$

or, $10 = 2 + Q^2 + 2\sqrt{2}Q\cdot\frac{1}{\sqrt{2}}$

or, $8 = Q^2 + 2Q$

or, $Q^2 + 2Q - 8 = 0$

or, $(Q + 4)(Q - 2) = 0$

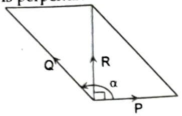
$\therefore Q = 2$N ($\because$ Q cannot be negative)

21. 2070 (Old) Q.No. 5 a

If the resultant R of two forces P and Q acting at a point is right angle to P, then prove that $R^2 = Q^2 - P^2$. [2]

SOLUTION

Let α be the angle between the forces P and Q.
Since R is perpendicular to P, θ = 90°.



We have,

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\tan 90° = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\frac{1}{0} = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $P + Q\cos\alpha = 0$

∴ $\cos\alpha = \frac{-P}{Q}$ $\Rightarrow \alpha = \cos^{-1}(-P/Q)$.

Again, we have

$R^2 = P^2 + Q^2 + 2PQ + \cos\alpha$

or, $R^2 = P^2 + Q^2 + 2PQ\left(\frac{-P}{Q}\right)$

or, $R^2 = P^2 + Q^2 - 2P^2$

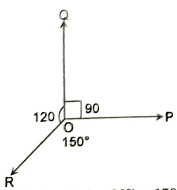
∴ $Q^2 - P^2 = R^2$

22. 2071 Set C Q.No. 12 a

Three forces acting on a particle are in equilibrium: The angle between the first and second is 90° and that between the second and third is 120°, find the ratios of the forces. [2]

SOLUTION

Let P, Q and R be the three forces acting at the point O.
Angle between third and first force



= 360° – (120° + 90°) = 150°

Since the forces are in equilibrium, so by Lami's theorem

$\frac{P}{\sin 120°} = \frac{Q}{\sin 150°} = \frac{R}{\sin 90°}$

or, $\frac{P}{\sqrt{3}/2} = \frac{Q}{1/2} = \frac{R}{1}$

or, $\frac{P}{\sqrt{3}} = \frac{Q}{1} = \frac{R}{2}$

∴ $P : Q : R = \sqrt{3} : 1 : 2$

23. 2071 Set D Q.No. 12 a

Forces equal to 7p, 5p and 8p acting on a particle are in equilibrium. Find the angle between latter pair of forces. [2]

➥ Please refer to Model Set II, Q.No. 12a

24. 2071 Old Q.No. 5 a

If a force P be resolved into two forces making angles 45° and 15° with its direction, show that the latter force is $\frac{\sqrt{6}}{3}$ P. [2]

➥ Please refer to 2061 Q.No. 5 a

25. 2072 Set C Q.No. 12 a

Find the resultant and the angle subtended by it with P when the forces P and Q act at right angle. [2]

SOLUTION

Let R be the resultant of the force P and Q and α be the angle between them. Here, α = 90°.

We have,

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

or, $R^2 = P^2 + Q^2 + 2PQ\cos 90°$

or, $R^2 = P^2 + Q^2$

∴ $R = \sqrt{P^2 + Q^2}$

If θ be the angle made by R with P, then

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} = \frac{Q\sin 90°}{P + Q\cos 90°} = \frac{Q}{P}$

∴ $\theta = \tan^{-1}\left(\frac{Q}{P}\right)$

Hence, resultant (R) $= \sqrt{P^2 + Q^2}$, angle (θ) $= \tan^{-1}\left(\frac{Q}{P}\right)$

26. 2072 Set D Q.No. 12a

Show that the resultant of two equal forces bisects the angle between them. [2]

SOLUTION

Let R be the resultant of two equal force P and P acting at an angle α.
Let θ be the angle made by R with P, then

$\tan\theta = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{\sin\alpha}{1 + \cos\alpha}$

$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$

∴ $\theta = \frac{\alpha}{2}$

This shows that the resultant of two equal forces bisects the angle between them.

27. 2072 Set E Q.No. 12a

Two forces P and 2P acting at a point have the resultant $\sqrt{3}$ P. Find the angle between the two given forces. [2]

SOLUTION

Let α be the angle between two given forces.
We have,

$R^2 = P^2 + Q^2 + 2PQ + \cos\alpha$

or, $(\sqrt{3}P)^2 = P^2 + (2P)^2 + 2 . P . 2P \cdot \cos\alpha$

or, $3P^2 = P^2 + 4P^2 + 4P^2\cos\alpha$

or, $4P^2\cos\alpha = -2P^2$

or, $\cos\alpha = \frac{-1}{2} = \cos 120°$

∴ $\alpha = 120°$

28. 2072 Supp Q.No. 12a

Two forces P and 2P acting at a point have the resultant $\sqrt{3}$ P, find the angle between the two given forces. [2]

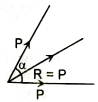
➥ Please refer to 2072 Set E Q.No. 12a

29. 2073 Set C Q.No. 12a

If the resultant of two equal forces is equal to the given force, find angle between the forces. [2]

SOLUTION

Let two equal forces be P N and P N each. By given resultant force(R) = P N. Let α be the angle between forces.



Then, we have

$P^2 = P^2 + P^2 + 2P^2\cos\alpha$

⇒ $P^2 = 2P^2(1 + \cos\alpha)$

⇒ $1 = 2 \cdot 2\cos^2\frac{\alpha}{2}$

⇒ $\cos\frac{\alpha}{2} = \frac{1}{2} = \cos 60°$

⇒ $\frac{\alpha}{2} = 60°$

∴ $\alpha = 120°$

30. 2073 Set D Q.No. 12a

Two forces P and 2P acting at a point have a resultant $\sqrt{3}$ P. Find the angle between the two forces. [2]

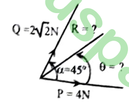
➥ Please refer to 2072 Set E Q.No. 12a

31. 2074 Set A Q.No. 12a

Two forces 4N and $2\sqrt{2}$ N act at an angle of 45°. Find their resultant. [2]

SOLUTION

Here, P = 4N, Q = $2\sqrt{2}$ N, α = 45°, R = ?, Q = ?



We have,

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$

$= \sqrt{(4)^2 + (2\sqrt{2})^2 + 2.4.2\sqrt{2}\,.\cos 45°}$

$= \sqrt{16 + 8 + 2.4.2\sqrt{2}.\frac{1}{\sqrt{2}}}$

$= \sqrt{16 + 8 + 16} = \sqrt{40} = 2\sqrt{10}$ N.

And, if the angle made by R with P be θ then

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha} = \frac{2\sqrt{2}\sin 45°}{4 + 2\sqrt{2}\cos 45°}$

$= \frac{2\sqrt{2}.\frac{1}{\sqrt{2}}}{4 + 2\sqrt{2}.\frac{1}{\sqrt{2}}} = \frac{2}{6} = \frac{1}{3}$

∴ $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

32. 2074 Set B Q.No. 12a

If the resultant R of two forces P and Q acting at a point is at right angle to P, prove that: $R^2 = Q^2 - P^2$. [2]

➥ Please refer to 2070 (Old) Q.No. 5a

33. 2074 Supp Q.No. 12a

A heavy chain has weights of 10 kg and 16 kg attached to its ends and hangs in equilibrium over a smooth pulley. If the greatest tension of the chain is 20 kg. wt., find the weight of the chain. [2]

SOLUTION

Let W_1 be the weight of the part of the chain containing the weight of 10 kg at its end.
Then,

$T = W_1 + 10$

or $20 = W_1 + 10$

∴ $W_1 = 10$ kg

Again, let W_2 be the weight of the part of the chain containing the weight of 16 kg at its end. Then,

$T = W_2 + 16$

or $20 = W_2 + 16$

∴ $W_2 = 4$

Hence,

Total weight of the chain $= W_1 + W_2$
$= 10 + 4 = 14$ kg

34. 2075 Set A Q.No. 12a

A force equal to 10N is inclined at an angle of 30° to the horizontal. Find its resolved parts in horizontal and vertical directions. [2]

SOLUTION

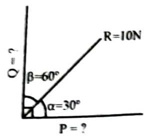
Here,
R = 10N

$\alpha = 30°$

$\therefore\ \beta = 90 - 30 = 60°$

P = ?

Q = ?



We have,

$P = \frac{R \sin\beta}{\sin(\alpha+\beta)} = \frac{10 \sin 60}{\sin 90}$

$= 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$ N

$Q = \frac{R \sin\alpha}{\sin(\alpha+\beta)} = \frac{10 \sin 30}{\sin 90} = 10 \times \frac{1}{2} = 5$N

35. 2075 Set B Q.No. 12a

Find the angle between the forces P + Q and P – Q such that their resultant may be $\sqrt{P^2 + 3Q^2}$. [2]

SOLUTION

Given $\sqrt{P^2 + 3Q^2}$ is the resultant of two forces P + Q and P – Q. Let α be the angle between the forces. Then, using the formula, $R^2 = P^2 + Q^2 + 2PQ\cos\alpha$, we have

$P^2 + 3Q^2 = (P + Q)^2 + (P - Q)^2 + 2(P + Q)(P - Q)\cos\alpha$

or, $P^2 + 3Q^2 = P^2 + 2PQ + Q^2 + P^2 - 2PQ + Q^2 + 2(P^2 - Q^2)\cos\alpha$

or, $Q^2 - P^2 = 2(P^2 - Q^2)\cos\alpha$

or, $\frac{-(P^2 - Q^2)}{2(P^2 - Q^2)} = \cos\alpha$

or, $\cos\alpha = -\frac{1}{2} = \cos 120°$

$\therefore\ \alpha = 120°$

36. 2075 Set C Q.No. 12a

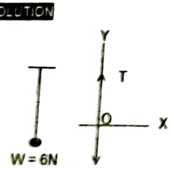
Forces equal to 7p, 5p and 8p acting on a particle are in equilibrium. Find the angle between the latter pair of forces. [2]

➥ Please refer to Model Set II Q.No. 12a

37. 2075 Set C Q.No. 12b

A particle weighing 6N is suspended freely from the ceiling by a weightless in extensible cord. Find the tension in the cord. [2]

SOLUTION



Two forces act upon the particle. The weight is W = 6 N which acts downwards and the tension T opposite to it acts upwards along vertical line (y-axis). There is no-x component.

The y-component are T and W. So, from equilibrium condition,

$\Sigma X_i = 0 \Rightarrow$

$0 = 0$

and $\Sigma Y_i = 0 \Rightarrow T - W = T - 6 = 0 \Rightarrow T = 6$ N

4 MARKS QUESTIONS

38. 2057 Q.No. 13 a

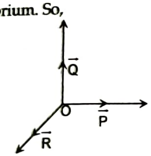
State and prove "Lami's Theorem". [4]

SOLUTION

Lami's theorem: If three forces acting at a point, be in equilibrium, each force is proportional to the sine of the angle between the other two.

Proof:

Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ be three forces acting at O be in equilibrium. So,



$\vec{P} + \vec{Q} + \vec{R} = 0$

or, $\vec{P} \times \vec{P} + \vec{P} \times \vec{Q} + \vec{P} \times \vec{R} = 0$

or, $\vec{P} \times \vec{Q} = -\vec{P} \times \vec{R}$ $(\because \vec{P} \times \vec{P} = 0)$

or, $\vec{P} \times \vec{Q} = \vec{R} \times \vec{P}$

or, $|\vec{P} \times \vec{Q}| = |\vec{R} \times \vec{P}|$

or, PQ sin (P, Q) = RP sin (R, P)

where, P, Q and R represent the magnitudes of $\vec{P}$, $\vec{Q}$ and $\vec{R}$; and sin (P, Q) is the sine of the angle between $\vec{P}$ and $\vec{Q}$.

$\therefore\ \frac{Q}{\sin(R, P)} = \frac{R}{\sin(P, Q)}$ ...(i)

Again,

$\vec{P} + \vec{Q} + \vec{R} = 0$

or, $\vec{Q} \times \vec{P} + \vec{Q} \times \vec{Q} + \vec{Q} \times \vec{R} = 0$

or, $\vec{Q} \times \vec{R} = -\vec{Q} \times \vec{P}$ $(\because \vec{Q} \times \vec{Q} = 0)$

or, $\vec{Q} \times \vec{R} = \vec{P} \times \vec{Q}$

or, $|\vec{Q} \times \vec{R}| = |\vec{P} \times \vec{Q}|$

or, QR sin (Q, R) = PQ sin (P, Q)

or, $\frac{P}{\sin(Q, R)} = \frac{R}{\sin(P, Q)}$ ... (ii)

From (i) and (ii)

$\frac{P}{\sin(Q,R)} = \frac{Q}{\sin(R,P)} = \frac{R}{\sin(P,Q)}$

39. 2057 Q.No. 13 a OR

The resultant of two forces P and Q acting at an angle α is $(2m + 1)\sqrt{P^2 + Q^2}$ when they act at an angle $(90° - \alpha)$ the resultant is $(2m - 1)\sqrt{P^2 + Q^2}$. Prove that $\tan\alpha = \frac{m-1}{m+1}$. [4]

SOLUTION

Since $(2m + 1)\sqrt{P^2 + Q^2}$ is the resultant of P and Q acting at an angle α, so

$\{(2m + 1)\sqrt{P^2 + Q^2}\}^2 = P^2 + Q^2 + 2PQ\cos\alpha$

or, $(2m + 1)^2(P^2 + Q^2) - (P^2 + Q^2) = 2PQ\cos\alpha$

or, $\cos\alpha = \frac{(P^2 + Q^2)\{(2m + 1)^2 - 1\}}{2PQ}$ ... (i)

Again, $(2m - 1)\sqrt{P^2 + Q^2}$ is the resultant of P and Q acting at an angle $(90° - \alpha)$, so

$\{(2m - 1)\sqrt{P^2 + Q^2}\}^2 = P^2 + Q^2 + 2PQ\cos(90 - \alpha)$

or, $(2m - 1)^2(P^2 + Q^2) - (P^2 + Q^2) = 2PQ\sin\alpha$

or, $\sin\alpha = \frac{(P^2 + Q^2)\{(2m - 1)^2 - 1\}}{2PQ}$ ... (ii)

Dividing equation (ii) by equation (i), we have

$\tan\alpha = \frac{(2m - 1)^2 - 1}{(2m + 1)^2 - 1} = \frac{4m^2 - 4m + 1 - 1}{4m^2 + 4m + 1 - 1}$

$= \frac{4m(m - 1)}{4m(m + 1)}$

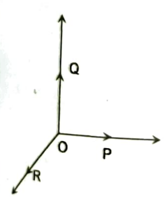
$\therefore\ \tan\alpha = \frac{m - 1}{m + 1}$

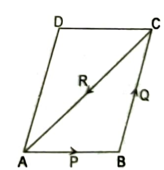
40. 2058 Q.No. 13 a

State and prove "Triangle of forces" [4]

SOLUTION

Triangle of forces: If three forces acting at a point be represented in magnitude and direction by the sides of a triangle taken in order then the forces are in equilibrium.





Proof: Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ be the three forces acting at O, be represented in magnitude and direction by AB, BC and CA of ΔABC. So, $\vec{AB} = \vec{P}$, $\vec{BC} = \vec{Q}$ and $\vec{CA} = \vec{R}$.

Let us complete the parallelogram ABCD. Since BC and AD are equal and parallel, so $\vec{AD} = \vec{BC} = \vec{Q}$.

By parallelogram law of forces, we have

$\vec{P} + \vec{Q} = \vec{AB} + \vec{AD} = \vec{AC}$

Now,

$\vec{P} + \vec{Q} + \vec{R} = \vec{AB} + \vec{BC} + \vec{CA}$

$= \vec{AB} + \vec{AD} + \vec{CA} = \vec{AC} + \vec{CA} = 0$

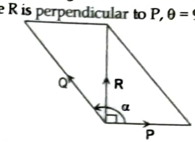
Hence, the forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are in equilibrium.

41. 2058 Q.No. 13 a OR

Two forces P and Q act at a point. Their resultant R is at right angles to P. Show that $Q^2 - P^2 = R^2$ and the angle between the forces is $\cos^{-1}\left(-\frac{P}{Q}\right)$. [4]

SOLUTION

Let α be the angle between the forces P and Q. Since R is perpendicular to P, θ = 90°.



We have,

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\tan 90° = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $\frac{1}{0} = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

or, $P + Q\cos\alpha = 0$

$\therefore\ \cos\alpha = \frac{-P}{Q} \Rightarrow \alpha = \cos^{-1}\left(-\frac{P}{Q}\right)$.

Again, we have

$R^2 = P^2 + Q^2 + 2PQ + \cos\alpha$

or, $R^2 = P^2 + Q^2 + 2PQ\left(\frac{-P}{Q}\right)$

or, $R^2 = P^2 + Q^2 - 2P^2$

$\therefore\ Q^2 - P^2 = R^2$

42. 2059 Q.No. 5 c

State 'Triangle of forces'.

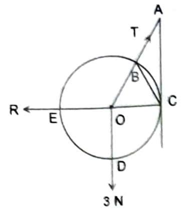
➥ Please refer to 2058 Q.No. 13 a

43. 2059 Q.No. 13 b

A uniform sphere of weight 3N rests in contact with a smooth vertical wall. It is supported by a string whose length equals the radius of the sphere joining a point on the surface of the sphere to a point of the wall. Calculate the tension in the string and the reaction of the wall. [4]

SOLUTION

Let C be the point of contact of the sphere and the vertical wall. Let O be the centre of the sphere and OD be the line of action of the weight. Also let R be the normal reaction and T be the tension of the string BA.



By question, length of string = radius of the sphere

∴ AB = OC = OB = OD

Hence ΔOBC is an equilateral triangle.

Since, the forces T, R and 3N acting at O are in equilibrium, so by Lami's theorem, we have,

$$\frac{T}{\sin \angle ROD} = \frac{R}{\sin \angle AOD} = \frac{3}{\sin \angle AOE}$$

or, $\frac{T}{\sin 90^\circ} = \frac{R}{\sin 150^\circ} = \frac{3}{\sin 120^\circ}$

or, $\frac{T}{1} = \frac{R}{\frac{1}{2}} = \frac{3}{\frac{\sqrt{3}}{2}}$

or, $T = 2R = 2\sqrt{3}$

∴ $T = 2\sqrt{3}$ N

and $2R = 2\sqrt{3}$

∴ $R = \sqrt{3}$ N

44. 2059 Q.No. 13 b OR

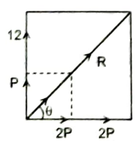
Two forces equal to 2 P and P respectively act on a particle. If the first be doubled and the second is increased by 12N, the direction of the resultant is unaltered. Find the value of P. [4]

SOLUTION

Let α be the angle between two forces 2P and P. Let θ be the angle made by the resultant R with the direction of the force 2P. Then

$$\tan\theta = \frac{P \sin\alpha}{2P + P\cos\alpha} \quad \ldots(i)$$

If the resultant R of the forces 4P and P + 12 makes the same angle θ with 4P, then



$$\tan\theta = \frac{(P+12)\sin\alpha}{4P + (P+12)\cos\alpha}$$

From (i) and (ii), we have

$$\frac{P\sin\alpha}{2P + P\cos\alpha} = \frac{(P+12)\sin\alpha}{4P + (P+12)\cos\alpha}$$

or, $\frac{1}{2+\cos\alpha} = \frac{P+12}{4P + (P+12)\cos\alpha}$

or, $4P + (P+12)\cos\alpha = 2(P+12) + (P+12)\cos\alpha$

or, $4P = 2P + 24$

or, $2P = 24$

∴ $P = 12$ N.

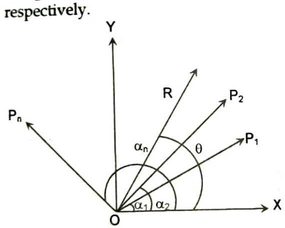
45. 2060 Q.No. 13 a

Find the resultant of n number of coplanar forces acting at a point. [4]

SOLUTION

Let the coplanar forces $P_1, P_2, \ldots, P_n$ acting at a point O be inclined at angles $\alpha_1, \alpha_2, \ldots, \alpha_n$ respectively to a line OX through the point O. Let OY be perpendicular to OX. The components of the forces $P_1, P_2, \ldots, P_n$ along OX and OY are $P_1\cos\alpha, P_2\cos\alpha_2, \ldots P_n\cos\alpha_n$ and $P_1\sin\alpha_1, P_2\sin\alpha_2, \ldots, P_n\sin\alpha_n$ respectively.

Let R be the resultant of the given system of forces and let it make an angle θ with OX. Then the resolved parts of the resultant R along OX and OY are R cos θ and R sin θ respectively.



Since its resolved parts are equal to the algebraic sum of the resolved parts of the forces along the same directions, we have

$R\cos\theta = P_1\cos\alpha_1 + P_2\cos\alpha_2 + \ldots + P_n\cos\alpha_n = X$ (say)

and $R\sin\theta = P_1\sin\alpha_1 + P_2\sin\alpha_2 + \ldots + P_n\sin\alpha_n = Y$ (say)

Squaring and adding, we get

$R^2 = X^2 + Y^2$

or, $R = \sqrt{X^2 + Y^2}$ ...(i)

Again, by division, we get

$\tan\theta = \frac{Y}{X}$

i.e. $\theta = \tan^{-1}\left(\frac{Y}{X}\right)$ ...(ii)

Hence, (i) and (ii) give the magnitude and direction of required resultant.

46. 2060 Q.No. 13 a OR

The resultant of two forces P and Q is R. If Q is doubled, the new resultant is perpendicular to P, prove that R = Q. [4]

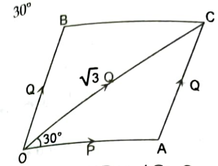
➥ Please refer to 2059 Q.No. 5 a

47. 2061 Q.No. 13 a

The resultant of two forces P & Q is equal to $\sqrt{3}$ Q and making an angle of 30° with the direction of P. Show that P is either equal to Q or is double of Q. [4]

SOLUTION

Let OA and OB represent the two forces P and Q acting at the point O. Let OC be the diagonal of the parallelogram OACB through which the resultant $\sqrt{3}$ Q act. Also, ∠AOC = 30°



Since AC = OB, so AC = Q.

Now, applying cosine law in ΔAOC, we have

$$\cos AOC = \frac{(OA)^2 + (OC)^2 - (AC)^2}{2\,OA \times OC}$$

or, $\cos 30^\circ = \frac{P^2 + 3Q^2 - Q^2}{2P \times \sqrt{3}Q}$

or, $\frac{\sqrt{3}}{2} = \frac{P^2 + 2Q^2}{2\sqrt{3}PQ}$

or, $3PQ = P^2 + 2PQ$

or, $P^2 - 3PQ + 2Q^2 = 0$

or, $(P - Q)(P - 2Q) = 0$

Either P = Q or P = 2Q

Hence, P equals to Q or is double of Q.

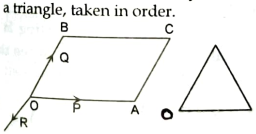
48. 2062 Q.No. 13 a

State and prove converse of triangle of forces. [4]

SOLUTION

Converse of the triangle of forces

If three forces acting at a point be in equilibrium, they can be represented in magnitude and direction by the three sides of a triangle, taken in order.



Proof:

Let $\vec{P}$, $\vec{Q}$ and $\vec{R}$ be three forces acting at O.

Let the forces be in equilibrium. Let $\vec{OA} = \vec{P}$, $\vec{OB} = \vec{Q}$ and $\vec{OD} = \vec{R}$.

Let us complete the parallelogram OACB with the magnitude of $\vec{P}$ and $\vec{Q}$ as its adjacent sides. Then, by parallelogram law of forces.

$\vec{P} + \vec{Q} = \vec{OA} + \vec{OB} = \vec{OC}$ ...(i)

Since the forces are in equilibrium, so

$\vec{P} + \vec{Q} + \vec{R} = 0$

or, $\vec{OA} + \vec{OB} + \vec{R} = 0$

or, $\vec{OC} + \vec{R} = 0$ [Using (i)]

or, $\vec{R} = -\vec{OC} = \vec{CO}$

Also, $\vec{AC} = \vec{OB} = \vec{Q}$ $\left(\because AC = OB \text{ and } AC \parallel OB\right)$

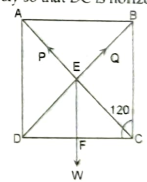
Hence, the forces $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are represented by the sides OA, AC and CO of the ΔOAC taken in order.

49. 2062 Q.No. 13 a OR

A uniform plane lamina in the form of a rhombus, one of whose angle is 120°, is supported by two forces applied at the centre in the directions of the diagonals so that one side of the rhombus is horizontal; show that if P and Q be the forces and P be the greater then $P^2 = 3Q^2$. [4]

SOLUTION

Let ABCD be a uniform plane lamina in the form of rhombus whose one angle say ∠BCD = 120°. Let E be the point of intersection of the diagonals AC and BD. The weight W acts vertically downwards as shown in the figure. Let P and Q be two forces along the diagonals EA and EB respectively so that DC is horizontal.



Since the diagonals of rhombus are at right angle to each other, so

∠BEF = 90° + 30° = 120°

∠AEB = 90° and ∠AEF = 90 + 60 = 150°

Since the lamina is in equilibrium, by Lami's theorem, we have

$$\frac{P}{\sin\angle BEF} = \frac{Q}{\sin\angle AEF}$$

or, $\frac{P}{\sin 120^\circ} = \frac{Q}{\sin 150^\circ}$

or, $\frac{P}{\frac{\sqrt{3}}{2}} = \frac{Q}{\frac{1}{2}}$

or, $P = \sqrt{3}Q$

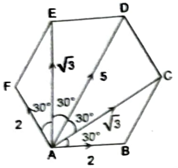
∴ $P^2 = 3Q^2$

50. 2063 Q.No. 13 a

Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2 newtons respectively act at one of the angular points of a regular hexagon towards the five other angular points. Find the magnitude and direction of the resultant. [4]

SOLUTION

Let the forces 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2 Newtons respectively act along AB, AC, AD, AE and AF of a regular hexagon ABCDEF as shown in the figure.



Since an interior angle of regular hexagon is 120°,

$\angle BAC = \angle CAD = \angle DAE = \angle EAF = 30°$.

Let R be the resultant inclined at an angle θ with AB. Now, resolving the forces along and perpendicular to AB, we have

$X = 2\cos 0° + \sqrt{3}\cos 30° + 5\cos 60° + \sqrt{3}\cos 90° + 2\cos 120°$

$= 2\cdot 1 + \sqrt{3}\cdot\frac{\sqrt{3}}{2} + 5\cdot\frac{1}{2} + \sqrt{3}\cdot 0 + 2\cdot\left(-\frac{1}{2}\right) = 5$

and $Y = 2\sin 0° + \sqrt{3}\sin 30° + 5\sin 60° + \sqrt{3}\sin 90° + 2\sin 120°$

$= 2\times 0 + \sqrt{3}\times\frac{1}{2} + 5\times\frac{\sqrt{3}}{2} + \sqrt{3}\times 1 + 2\times\frac{\sqrt{3}}{2}$

$= \frac{10\sqrt{3}}{2} = 5\sqrt{3}$

Now, resultant (R) $= \sqrt{X^2 + Y^2} = \sqrt{5^2 + (5\sqrt{3})^2}$

$= \sqrt{25 + 75} = 10$ N

Again, $\tan\theta = \frac{Y}{X} = \frac{5\sqrt{3}}{5} = \sqrt{3} = \tan 60°$

∴ θ = 60°

Hence, the direction of the resultant R is towards AD.

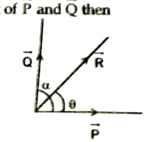
∴ R = 10 N is the resultant which act towards the opposite angular point.

51. 2063 Q.No. 13 a OR

Find the resultant of two forces P and Q acting at a point. [4]

SOLUTION

Let $\vec{P}$ and $\vec{Q}$ be two forces acting at O and α be the angle between them. If $\vec{R}$ is the resultant of $\vec{P}$ and $\vec{Q}$ then



$\vec{R} = \vec{P} + \vec{Q}$

Now,

$R^2 = \vec{R}\cdot\vec{R}$

$= (\vec{P} + \vec{Q})\cdot(\vec{P} + \vec{Q})$

$= \vec{P}^2 + \vec{Q}^2 + 2\vec{P}\cdot\vec{Q} = P^2 + Q^2 + 2PQ\cos\alpha$

where P, Q and R be the magnitudes of $\vec{P}$, $\vec{Q}$ and $\vec{R}$ respectively.

∴ $R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$ which gives the magnitude of P and Q.

Again, let θ be the angle between $\vec{P}$ and $\vec{R}$.

Here, $\vec{R} = \vec{P} + \vec{Q}$

⇒ $\vec{P}\cdot\vec{R} = \vec{P}\cdot\vec{P} + \vec{P}\cdot\vec{Q}$

⇒ $PR\cos\alpha = P^2 + PQ\cos\alpha$

⇒ $R\cos\alpha = P + Q\cos\alpha$ ...(i)

Again,

$\vec{R} = \vec{P} + \vec{Q}$

⇒ $\vec{P}\times\vec{R} = \vec{P}\times\vec{P} + \vec{P}\times\vec{Q}$

⇒ $\vec{P}\times\vec{R} = \vec{P}\times\vec{Q}$ $(\because \vec{P}\times\vec{P} = 0)$

⇒ $|\vec{P}\times\vec{R}| = |\vec{P}\times\vec{Q}|$

⇒ $PR\sin\theta = PQ\sin\alpha$

⇒ $R\sin\theta = Q\sin\alpha$ ...(ii)

Dividing equation (ii) by equation (i), we get,

$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$

∴ $\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right)$

which gives the direction of the resultant $\vec{R}$.

52. 2064 Q.No. 13 a

The resultant of two forces P & Q acting at an angle α is equal to (2m + 1) $\sqrt{P^2 + Q^2}$, when they act at an angle 90° – α, the resultant is (2m – 1)$\sqrt{P^2 + Q^2}$. Prove that $\tan\alpha = \frac{m-1}{m+1}$. [4]

➥ Please refer to 2057 Q.No. 13 a OR

53. 2064 Q.No. 13 a OR

State and prove Lami's theorem. [4]

➥ Please refer to 2057 Q.No. 13 a

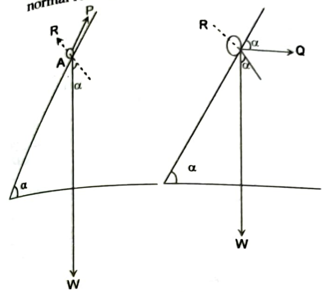
54. 2065 Q.No 13 a

Two forces P & Q acting parallel to the length and base of an inclined plane respectively, would each of them singly support a weight W on the plane, prove that: [4]

$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}$

SOLUTION

Let α be the angle of inclination of the inclined plane with the horizon. A body of weight W placed at A is supported by the force P parallel to the length as shown in the figure (i) and by force Q parallel to the base as shown in the figure (ii). Also, let R be the normal reaction.



When P is used parallel to the length, using Lami's theorem, we have

$\frac{P}{\sin(180° - \alpha)} = \frac{W}{\sin 90°}$

or, $\frac{P}{\sin\alpha} = W$

∴ $\sin\alpha = \frac{P}{W}$ ...(i)

When Q is parallel to base, using Lami's theorem, we have

$\frac{Q}{\sin(180 - \alpha)} = \frac{W}{\sin(90 + \alpha)}$

or, $\frac{Q}{\sin\alpha} = \frac{W}{\cos\alpha}$

or, $Q\cos\alpha = W\sin\alpha$

or, $Q\cos\alpha = W\cdot\frac{P}{W}$ [using (i)]

∴ $\cos\alpha = \frac{P}{Q}$ ...(ii)

Squaring and adding (i) and (ii)

$\frac{P^2}{W^2} + \frac{P^2}{Q^2} = 1$

or, $\frac{1}{W^2} + \frac{1}{Q^2} = \frac{1}{P^2}$

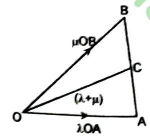
∴ $\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}$

55. 2065 Q.No 13 a OR

State and prove: λ – μ theorem. [4]

SOLUTION

Statement: The resultant of two forces, acting at a point O in the direction OA and OB and represented in magnitude and directions by λOA and μOB is represented by (λ + μ) OC, where C divides AB in the ration μ : λ.



i.e. λAC = μBC

Proof:

From ΔOAC,

$\vec{OA} + \vec{AC} = \vec{OC}$

or, $\lambda\vec{OA} + \lambda\vec{AC} = \lambda\vec{OC}$ ...(i)

Similarly, from ΔOBC,

$\vec{OB} + \vec{BC} = \vec{OC}$

∴ $\mu\vec{OB} + \mu\vec{BC} = \mu\vec{OC}$ ...(ii)

Adding (i) and (ii), we get

$\lambda\vec{OA} + \mu\vec{OB} + \lambda\vec{AC} + \mu\vec{BC} = (\lambda + \mu)\vec{OC}$

But λAC = μBC and since λAC and μBC are acting in opposite directions along the same line, so they cancel out.

So, $\lambda\vec{OA} + \mu\vec{OB} = (\lambda + \mu)\vec{OC}$

56. 2066 C Q.No. 13 a

State and prove converse of triangle of forces. [4]

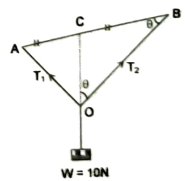
➥ Please refer to 2062 Q.No. 13a

57. 2066 C Q.No. 13 a OR

A body of weight 10 kg is suspended by two strings 7 and 24 metres in length, their other ends being fastened to the extremities of a rod of length 25 metres. If the rod be so hold that the body hangs immediately below its middle point. Find the tensions of the strings. [4]

SOLUTION

Let OA and OB be two strings such that OA = 7m and OB = 24 m.



By question, AB = 25 m

Here, $OA^2 + OB^2 = 7^2 + 24^2 = 625 = 25^2 = AB^2$

So, ∠AOB = 90°

Here, OC is vertical and meets AB at C such that AC = BC.

Since ∠AOB = 90° and AC = CB, so OC = AC = BC.

Let ∠OBC = θ then ∠BOC = θ.

Let T_1 and T_2 be the tensions along the strings OA and OB. Since T_1, T_2 and W = 10 N acting at O are in equilibrium, then by Lami's theorem

$$\frac{T_1}{\sin(180-\theta)} = \frac{T_2}{\sin(180-(90-\theta))} = \frac{W}{\sin 90^\circ}$$

or, $\frac{T_1}{\sin\theta} = \frac{T_2}{\cos\theta} = \frac{10}{1}$

$\therefore$ $T_1 = 10\sin\theta = 10 \times \frac{7}{25} = \frac{14}{5}$ kg – wt

$T_2 = 10\cos\theta = 10 \times \frac{24}{25} = \frac{48}{5}$ kg-wt

58. 2066 Q.No.13 a

Forces 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 newtons respectively act at one of the angular points of a regular hexagon towards the five other points. Find the magnitude and direction of the resultant. [4]

➡ Please refer to 2063 Q.No. 13 a

59. 2066 Q.No.13 a OR

State and prove the theorem on Triangle of forces for three forces acting at a point. [4]

➡ Please refer to 2058 Q.No. 13 a

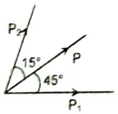
60. 2067 Q.No. 13a

If a force P be resolved into two forces making angles of 45° and 15° with its direction; show that the latter force is $\sqrt{\frac{2}{3}}$P. [4]

SOLUTION

Let P_1 and P_2 be the resolved forces of the given force P. We have to find the later force P_2.

We have,



$$P_2 = \frac{R\sin\alpha}{\sin(\alpha+\beta)}$$

Here, R = P, α = 45°, β = 15°

$$\therefore\ P_2 = \frac{P\sin 45^\circ}{\sin 60^\circ} = \frac{P\cdot\frac{1}{\sqrt{2}}}{\frac{\sqrt{3}}{2}} = \frac{P}{\sqrt{2}} \times \frac{2}{\sqrt{3}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}P = \sqrt{\frac{2}{3}}P$$

61. 2067 Q.No. 13a OR

State and prove converse of the Triangle of forces. [4]

➡ Please refer to 2062 Q.No. 13 a

62. 2068 Q.No. 13 a

Find the resultant of two forces P and Q when the angle between them is α. [4]

➡ Please refer to 2063 Q.No. 13 a OR

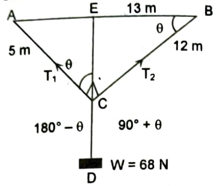
63. 2068 Q.No. 13 a Or

A body of weight 65 N is suspended by two strings of length 5m and 12m attached to two points in the same horizontal line whose distance apart is 13m; find the tension of the strings. [4]

SOLUTION

Let CA and CB be the strings so that CA = 5m and CB = 12m and AB = 13m.

Since $5^2 + 12^2 = 13^2$, so ∠ACB = 90°



Let the line of action of weight W = 68 N be produced to meet AB at E. Let ∠ACE = θ so that ∠CBE = θ. Let T_1 and T_2 be the tensions along the strings CA and CB respectively. Since the three forces T_1, T_2 and W = 68 N acting at C are in equilibrium, so by Lami's theorem,

$$\frac{T_1}{\sin BCD} = \frac{T_2}{\sin ACD} = \frac{68}{\sin ACB}$$

or, $\frac{T_1}{\sin(90+\theta)} = \frac{T_2}{\sin(180-\theta)} = \frac{68}{\sin 90^\circ}$

or, $\frac{T_1}{\cos\theta} = \frac{T_2}{\sin\theta} = 68$

$\therefore$ $T_1 = 68\cos\theta$ and $T_2 = 68\sin\theta$

But $\cos\theta = \frac{BC}{BA} = \frac{12}{13}$ and $\sin\theta = \frac{AC}{AB} = \frac{5}{13}$

Hence, $T_1 = 65 \times \frac{12}{13} = 60$ N

$T_2 = 65 \times \frac{5}{13} = 25$ N

64. 2069 (Set A) Q.No. 13a

The resultant of two forces P and Q acting at an angle α is equal to (2m + 1) $\sqrt{P^2+Q^2}$. When they act at an angle (90° – α) the resultant is (2m - 1) $\sqrt{P^2+Q^2}$. Prove that: $\tan\alpha = \frac{m-1}{m+1}$. [4]

➡ Please refer to 2057 Q.No. 13a OR

65. 2069 (Set A) Old Q.No. 13a

State and prove Lami's theorem. [4]

➡ Please refer to 2057 Q.No. 13 a

66. 2069 (Set A) Old Q.No. 13a or

Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 newtons respectively act at one of the angular points of a regular hexagon towards the five other points. Find the magnitude and the direction of the resultant. [4]

➡ Please refer to 2063 Q. No. 13a

67. 2069 (Set B) Q.No. 13a

A body of weight 65N is suspended by two strings of lengths 5m and 12m attached to two points in the same horizontal line whose distance apart is 13m, find the tension of the strings. [4]

➡ Please refer to 2068 Q. No. 13a OR

68. 2069 (Set B) Q.No. 13a Or

Find the resultant of two forces P and Q acting at a point when the angle between them is α. [4]

➡ Please refer to 2063 Q.No. 13a OR

69. 2069 Old (Set B) Q.No. 13a

State and prove "Lami's theorem". [4]

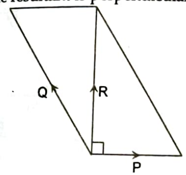
➡ Please refer to 2057 Q.No. 13 a

70. 2069 Old (Set B) Q.No. 13a Or

If the resultant of two forces acting on a particle be at right angle to one of them and its magnitude be half of the magnitude of the other, show that the ratio of the greater force to the smaller is 2: $\sqrt{3}$. [4]

SOLUTION

Let α be the angle between the two forces P and Q (P < Q). The magnitude of their resultant is $\frac{1}{2}$Q and the resultant is perpendicular to P.



So, $\tan 90^\circ = \frac{Q\sin\alpha}{P+Q\cos\alpha}$

or, $\frac{1}{0} = \frac{Q\sin\alpha}{P+Q\cos\alpha}$

or, $P + Q\cos\alpha = 0$

$\therefore$ $\cos\alpha = \frac{-P}{Q}$ ...(i)

Again, we have,

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$

or, $\left(\frac{1}{2}Q\right)^2 = P^2 + Q^2 + 2PQ\left(\frac{-P}{Q}\right)$ [Using (i)]

or, $\frac{Q^2}{4} = P^2 + Q^2 - 2P^2$

or, $\frac{Q^2}{4} = Q^2 - P^2$

or, $Q^2 = 4Q^2 - 4P^2$

or, $4P^2 = 3Q^2$

or, $\frac{Q^2}{P^2} = \frac{4}{3}$

or, $\frac{Q}{P} = \frac{2}{\sqrt{3}}$

$\therefore$ $Q : P = 2 : \sqrt{3}$

71. 2070 Set C Q.No. 13 a

Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2N respectively act at one of the angular points of a regular hexagon towards the five other points. Find the magnitude and direction of the resultant. [4]

➡ Please refer to 2063 Q.No. 13 a

72. 2070 Set D Q.No. 13 a

A body of weight 65N is suspended by two strings of lengths 5 and 12 m attached to two points in the same horizontal line whose distance apart is 13m; find the tensions of the string. [4]

➡ Please refer to 2068 Q.No. 13a OR

73. 2070 Set D Q.No. 13 a Or

State and prove Lami's theorem. [4]

➡ Please refer to 2057 Q.No. 13 a

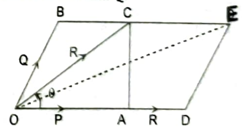
74. 2070 (Old) Q.No. 13 a Or

If the resultant R of two focus P and Q inclined to one another at any given angle, makes an angle θ with the direction of P, show that the resultant of the forces P + R and Q acting at the same given angle, will make an angle θ/2 with the direction of P + R. [4]

SOLUTION

Let the forces P and Q be represented by OA and OB of the parallelogram OACB respectively. Also let the resultant R be represented by OC.

Let us produce OA to D making AD = OC so that OD represents the force P + R. Now, complete the parallelogram ODEB. Then the diagonal OE represents the new resultant of the forces P + R and Q.



Here, AD = OC

Also, AD = CE

$\therefore$ CE = OC

∠COE = ∠CEO

Again, ∠CEO = ∠DOE

$\therefore$ ∠COE = ∠DOE

But ∠AOC = θ

$\therefore$ ∠COE = ∠DOE = $\frac{\theta}{2}$

75. 2071 Set C Q.No. 13 a

The resultant of two forces P and Q acting at an angle α is equal to (2m + 1) $\sqrt{P^2+Q^2}$. When they act at an angle (90° – α) the resultant is (2m – 1) $\sqrt{P^2+Q^2}$. Prove that: $\tan\alpha = \frac{m-1}{m+1}$ [4]

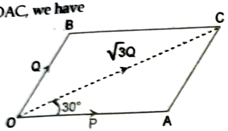
➡ Please refer to 2057 Q.No. 13 a OR

76. 2071 Set D Q.No. 13 a

The resultant of two forces P and Q is equal to $\sqrt{3}$Q and makes an angle of 30° with the direction of P; Show that P is either equal to Q or is double of Q. **[4]**

SOLUTION

Let OA and OB represent the force P and Q. Let us complete the parallelogram OACB, then the diagonal OC represents the resultant i.e. OC = $\sqrt{3}$ Q. Since OACB is parallelogram, so AC = OB = Q. Applying cosine law in ΔOAC, we have



$$\cos 30° = \frac{(OA)^2 + (OC)^2 - (AC)^2}{2 \cdot OA \cdot OC}$$

or, $\frac{\sqrt{3}}{2} = \frac{P^2 + 3Q^2 - Q^2}{2 \cdot P \cdot \sqrt{3}Q}$

or, $\frac{\sqrt{3}}{2} = \frac{P^2 + 2Q^2}{2\sqrt{3}PQ}$

or, $P^2 - 3PQ + 2Q^2 = 0$

or, $(P - Q)(P - 2Q) = 0$

⇒ P = Q or P = 2Q

∴ Hence P equals to Q or is double of Q.

77. 2071 Set D Q.No. 13 a OR

State and prove Lami's theorem. **[4]**

➥ Please refer to 2057 Q.No. 13 a

78. 2071 Old Q.No. 13 a

State and prove Lami's theorem. **[4]**

➥ Please refer to 2057 Q.No. 13 a

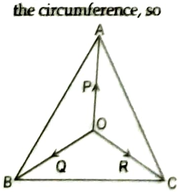
79. 2071 Old Q.No. 13 a OR

O is the circumcentre of the triangle ABC. Forces P, Q and R acting along OA, OB and OC are in equilibrium. Show that: **[4]**

$$\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

SOLUTION

Let O be the circumcentre of ΔABC. Since the angle at the centre O is the twice the angle at the circumference, so



∠BOC = 2A

∠COA = 2B

and ∠AOB = 2C

Since the forces P, Q and R acting along OA, OB and OC are in equilibrium, so by Lami's theorem

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB}$$

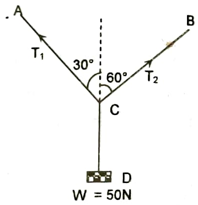
$$\therefore \frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$$

80. 2072 Set C Q.No. 13a

Two men carry a weight 50N supported by two strings; one string is inclined at 30° to the vertical and other at 60°, find the tension of each string. **[4]**

SOLUTION

Let CA and CB be two strings and CD be the line of action of weight 50 N. Let T_1 and T_2 be the tensions of the strings CA and CB respectively. Since T_1, T_2 and W acting at C are in equilibrium, so by Lami's theorem, we have



$$\frac{T_1}{\sin \angle BCD} = \frac{T_2}{\sin \angle ACD} = \frac{W}{\sin \angle ACB}$$

or, $\frac{T_1}{\sin 120°} = \frac{T_2}{\sin 150°} = \frac{50}{\sin 90°}$

$$\therefore T_1 = \frac{50 \sin 120°}{\sin 90°} = \frac{50 \times \frac{\sqrt{3}}{2}}{1} = 25\sqrt{3}\text{ N},$$

$$\text{and } T_2 = \frac{50 \sin 150°}{\sin 90°} = \frac{50 \times 1/2}{1} = 25\text{ N}$$

Hence the required tensions are $25\sqrt{3}$ N and 25 N.

81. 2072 Set D Q.No. 13a

State and prove Lami's theorem. [4]

➥ Please refer to 2057 Q.No. 13 a

82. 2072 Set D Q.No. 13a OR

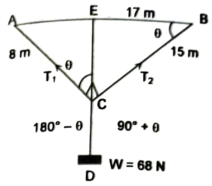
A body of weight 68N is suspended by two strings of length 8 m and 15 m respectively, and the other ends of the strings are attached to two fixed points in a horizontal line 17 m apart, find the tensions of the strings. **[4]**

SOLUTION

Let CA and CB be the strings so that CA = 8m, CB = 15 m and AB = 17 m.

Since, $8^2 + 15^2 = 17^2$, so ∠ACB = 90°.

Let the line of action of weight 68 N be produced to meet AB at E. Let ∠ACE = θ so that ∠CBE = θ. Let T_1 and T_2 be the tensions along the strings CA and CB respectively. Since T_1, T_2 and 68 N acting at C are in equilibrium, so by Lami's theorem,



$$\frac{T_1}{\sin \angle BCD} = \frac{T_2}{\sin \angle ACD} = \frac{68}{\sin ACB}$$

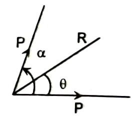
or, $\frac{T_1}{\sin (90 + \theta)} = \frac{T_2}{\sin (180 - \theta)} = \frac{68}{\sin 90}$

∴ $T_1 = 68 \cos \theta$ and $T_2 = 68 \sin \theta$.

But $\cos \theta = \frac{BC}{BA} = \frac{15}{17}$ and $\sin \theta = \frac{AC}{AB} = \frac{8}{17}$

∴ $T_1 = 68 \times \frac{15}{17} = 60$ N

$T_2 = 68 \times \frac{8}{17} = 32$ N



Hence the required tensions are 60 N and 32 N.

83. 2072 Set E Q.No. 13a

A body of weight 65 N is suspended by two strings of lengths 5 and 12 m attached to two points in the same horizontal line whose distance apart is 13m; find the tensions of the string. [4]

➥ Please refer to 2068 Q.No. 13 a Or

84. 2072 Supp Q.No. 13a

Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$, 2 N respectively act at one of the angular points of a regular hexagon towards the five other points. Find the magnitude and the direction of the resultant. **[4]**

➥ Please refer to 2063 Q.No. 13 a

85. 2072 Supp Q.No. 13a OR

State and prove Lami's theorem. **[4]**

➥ Please refer to 2057 Q.No. 13 a

86. 2073 Set C Q.No. 13a

State and prove triangle of forces. **[4]**

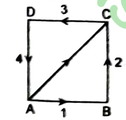
➥ Please refer to 2058 Q.No. 13 a

87. 2073 Set C Q.No. 13a OR

ABCD is a square. Forces 1, 2, 3, 4 and $2\sqrt{2}$ newton's act at a point in directions AB, BC, CD, DA and AC respectively. Find the resultant.

SOLUTION

Let the forces 1, 2, 3, 4 and $2\sqrt{2}$ newtons act along AB, BC, CD, DA and AC of a square ABCD respectively.



Now, resolving the forces along and perpendicular to AB, we have

X = 1 cos 0° + 2 cos 90° + 3 cos 180° + 4 cos 270° + $2\sqrt{2}$ cos 45°

$= 1 + 2 \times 0 + 3 \times (-1) + 4 \times 0 + 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

and Y = 1 sin 0° + 2 sin 90° + 3 sin 180° + 4 cos 270° + $2\sqrt{2}$ sin 45°

$= 1 \times 0 + 2 \times 1 + 3 \times 0 + 4 \times (-1) + 2\sqrt{2} \times \frac{1}{\sqrt{2}} = 0$

If R be the resultant, then $R = \sqrt{X^2 + Y^2}$

$= \sqrt{0^2 + 0^2} = 0$

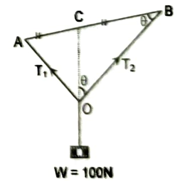
Hence, the forces are in equilibrium.

88. 2073 Set D Q.No. 13a

A body of weight 100 kg is suspended by two things 7m and 24 m in length; their other ends being fastened to the extremities of a rod of length 25m. If the rod is so held that the body hangs immediately below its middle point, find the tensions of the strings. **[4]**

SOLUTION

Let OA and OB be two strings such that OA = 7m and OB = 24 m.



By question, AB = 25 m

Here, $OA^2 + OB^2 = 7^2 + 24^2 = 625 = 25^2 = AB^2$

So, ∠AOB = 90°

Here, OC is vertical and meets AB at C such that AC = BC.

Since,

∠AOB = 90° and AC = CB, so OC = AC = BC.

Let ∠OBC = θ then ∠BOC = θ.

Let T_1 and T_2 be the tensions along the strings OA and OB. Since T_1, T_2 and W = 100 N acting at O are in equilibrium, then by Lami's theorem

$$\frac{T_1}{\sin(180-\theta)} = \frac{T_2}{\sin(180-(90-\theta))} = \frac{W}{\sin 90^\circ}$$

or, $\frac{T_1}{\sin\theta} = \frac{T_2}{\cos\theta} = \frac{100}{1}$

∴ $T_1 = 100\sin\theta = 100\times\frac{7}{25} = 28\text{kg - wt}$

$T_2 = 100\cos\theta = 100\times\frac{24}{25} = 96\text{kg-wt}$

89. 2073 Set D Q.No. 13a OR

Find the resultant of two forces P and Q acting at a point when the angle between them is α.

- Please refer to 2063 Q.No. 13 a OR

90. 2073 Supp Q.No. 13a

If a force P be resolved into two forces making angles 45° and 15° with its directions, show that the later force is $\sqrt{\frac{2}{3}}$ P. [4]

- Please refer to 2067 Q.No. 13a

91. 2074 Set A Q.No. 13a

State and prove 'Triangle Law of forces'. [4]

- Please refer to 2058 Q.No. 13a

92. 2074 Set B Q.No. 13a

State and prove Lami's theorem. [4]

- Please refer to 2057 Q.No. 13a

93. 2074 Set B Q.No. 13a OR

A body of weight 10kg is suspended by two strings of lengths 7m and 24m, their other ends being fastened to the extremities of a rod of length 25m. If the rod be so held that the body hangs immediately below its middle point, find the tensions of the strings. [4]

- Please refer to 2066 C Q. No. 13a or

94. 2074 Supp Q.No. 13a

Forces of 2, $\sqrt{3}$, 5, $\sqrt{3}$ and 2 N respectively act at one of the angular points of a regular hexagon towards the five other points. Find the magnitude and direction of the resultant. [4]

- Please refer to 2063 Q.N. 13a

95. 2074 Supp Q.No. 13a OR

Find the resultant of two forces P and Q acting at a point when the angle between them is α.

- Please refer to 2063 Q.N. 13a OR

96. 2075 Set A Q.No. 13a

A body of weight 68N is suspended by two strings of lengths 8 m and 15m respectively and the other ends of the strings are attached to two fixed points in a horizontal line 17m apart. Find the tensions of the strings. [4]

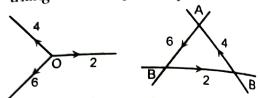
- Please refer to 2072 Set D Q.No. 13a OR

97. 2075 Set B Q.No. 13a

Forces 2, 4, 6 units act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find their resultant. [4]

SOLUTION

Let the forces 2, 4 and 6 units act at O parallel to the sides BC, CA and AB of an equilateral triangle ABC respectively.



Now, resolving the forces along and perpendicular to BC, we have

X = 2. cos 0° + 4. cos 120° + 6.cos 240°

$= 2\times 1 + 4\times\left(-\frac{1}{2}\right) + 6\times\left(-\frac{1}{2}\right)$

$= 2 - 2 - 3 = -3$

Y = 2. sin 0° + 4. sin 120° + 6.sin 240°

$= 2\times 0 + 4\times\frac{\sqrt{3}}{2} + 6\times\left(-\frac{\sqrt{3}}{2}\right)$

$= 2\sqrt{3} - 3\sqrt{3}$

$= -\sqrt{3}$

Let R be the resultant force making an angle θ with the force 2 units

Then,

$R = \sqrt{X^2 + Y^2}$

$= \sqrt{(-3)^2 + (-\sqrt{3})^2}$

$= \sqrt{12} = 2\sqrt{3}$ units

and $\tan\theta = \frac{Y}{X} = \frac{-\sqrt{3}}{-3} = \frac{1}{\sqrt{3}} = \tan 210^\circ$ (∵ X & Y are both –ve)

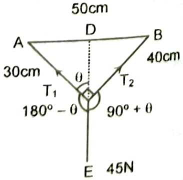
Hence the resultant is $2\sqrt{3}$ units at an angle of 210° with 2 units force.

98. 2075 Set B Q.No. 13a OR

A body of weight 45 N is suspended by two strings of length 30 cm and 40 cm respectively and the other ends of the strings are attached to the extremities of a rod of length 50 cm. The rod is so held that the body hangs immediately below its middle point. Find the tensions of the strings.

SOLUTION

Let CA and CB be the strings so that CA = 30 cm, CB = 40 cm and AB = 50cm.



Here,

$30^2 + 40^2 = 900 + 1600 = 2500 = 50^2$

∴ ∠ ACB = 90°

Let the line of action of the weight 45N be produced to meet AB at D. Let ∠ACD = θ so that ∠CBD = θ. Let T_1 and T_2 be the tensions along the strings CA and CB respectively. Since T_1, T_2 and 45N acting at C are in equilibrium, by Lami's theorem,

$$\frac{T_1}{\sin\angle ECB} = \frac{T_2}{\sin\angle ECA} = \frac{45}{\sin\angle ACB}$$

or, $\frac{T_1}{\sin(90^\circ+\theta)} = \frac{T_2}{\sin(180^\circ-\theta)} = \frac{45}{\sin 90}$

or, $\frac{T_1}{\cos\theta} = \frac{T_2}{\sin\theta} = 45.$

∴ $T_1 = 45\cos\theta$, $T_2 = 45\sin\theta$

Here, $\cos\theta = \frac{BC}{AB} = \frac{40}{50} = \frac{4}{5}$

$\sin\theta = \frac{AC}{AB} = \frac{30}{50} = \frac{3}{5}$

∴ $T_1 = 45\times\frac{4}{5} = 36$N

$T_2 = 45\times\frac{3}{5} = 27$ N

99. 2075 Set C Q.No. 13a OR

State and prove λ – μ theorem. [4]

- Please refer to 2065 Q.No. 13a OR

6 MARKS QUESTIONS

100. 2070 Supp. Q.No. 14

Two forces equal to 2P and P respectively act on a particle. If the first be doubled and the second increased by 12N, the direction of the resultant is unaltered. Find the value of P. [6]

- Please refer to 2059 Q.No. 13 b OR

101. 2070 Supp. Q.No. 14 OR

Let P and Q be two forces acting on a particle, whose directions include an angle of α(≠0 or π). Derive the magnitude and direction of their resultant. [6]

- Please refer to 2063 Q.No. 13a OR

102. 2071 Supp. Q.No. 14

Prove that if three forces acting on a particle are in equilibrium, then each is proportional to the sine of the angle between the other two. Also, if a body of weight w is suspended by strings of length 3m and 4m attached to two points in the same horizontal line whose distance apart is 5m, find the tensions along the strings. [6]

- **First Part:** Please refer to 2057 Q.No. 13 a

Second Part:

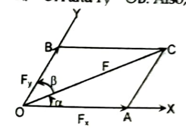
Please refer to Model Set I, Q.No. 13a

103. 2071 Supp. Q.No. 14 OR

Let force F make angles α and β with directions OX and OY respectively. Prove that if F_x and F_y are the components of the force in the directions of OX and OY respectively then $F_x = \frac{F\sin\beta}{\sin(\alpha+\beta)}$, $F_y = \frac{F\sin\alpha}{\sin(\alpha+\beta)}$. Also, if Q is doubled and the new resultant is perpendicular to P, prove that Q = R, where R is the resultant of forces P and Q. [6]

SOLUTION

Let F be a given force represented by OC. We have to resolve F in the direction of OX and OY such that ∠XOC = α and ∠COY = β. Let us draw AC parallel to OY and BC parallel to OX so that OACB is a parallelogram. Then F_x = OA and F_y = OB. Also,



AC = OB, ∠OAC = π – (α + β) and ∠ACO = β.

By sine law, we have

$$\frac{OA}{\sin\angle ACO} = \frac{AC}{\sin\alpha} = \frac{OC}{\sin(\pi-(\alpha+\beta))}$$

or, $\frac{F_x}{\sin\beta} = \frac{F_y}{\sin\alpha} = \frac{F}{\sin(\alpha+\beta)}$

∴ $F_x = \frac{F\sin\beta}{\sin(\alpha+\beta)}$ and

$F_y = \frac{F\sin\alpha}{\sin(\alpha+\beta)}$

Next part

Let α be the angle between two forces P and Q. Since R is the resultant, so

$R^2 = P^2 + Q^2 + 2PQ\cos\alpha$ ... (i)

Again, given that when Q is doubled, the new resultant is perpendicular to P, so we have

$\tan 90^\circ = \frac{2Q\sin\alpha}{P + 2Q\cos\alpha}$

or, $\frac{1}{0} = \frac{2Q\sin\alpha}{P + 2Q\cos\alpha}$

or, $P + 2Q\cos\alpha = 0$

or, $\cos\alpha = \frac{-P}{2Q}$

Putting the value of cos α in equation (i), we get

$R^2 = P^2 + Q^2 + 2PQ\left(\frac{-P}{2Q}\right)$

or, $R^2 = P^2 + Q^2 - P^2$

or, $R^2 = Q^2$

∴ Q = R

UNIT 13

STATICS (CONTINUED)

2 MARKS QUESTIONS

1. 2057 Q.No. 6 a

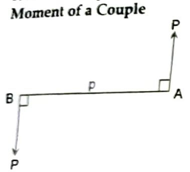
Define a couple and the moment of a couple. Express the moment of a couple mathematically. [2]

SOLUTION

Arm of the Couple

The arm of the couple is the perpendicular distance between the lines of action of the two forces forming a couple.

Moment of a Couple



The moment of a couple is the product of the magnitude of one of the forces forming a couple and the arm of the couple.

Mathematically if the forces P, P are measured in newton and the arm p in metre then the moment of couple is given by P × p.

2. 2057 Q.No. 6 b

Replace a force of magnitude 50 kg wt by two like parallel forces one at a distance of 2m and other at 8 m from the given force. [2]

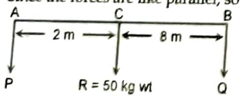
SOLUTION

Let P and Q be two like parallel forces acting at A and B respectively. Let R = 50 kg wt be given resultant force acting at C. By given,

AC = 2m, BC = 8m. Then,

AB = AC + BC = 2 + 8 = 10 m

Since the forces are like parallel, so



$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{P}{8} = \frac{Q}{2} = \frac{50}{10}$

∴ P = 8 × 5 = 40 kg wt

Q = 5 × 2 = 10 kg wt

3. 2058 Q.No. 6 a

Define a couple. What do you mean by arm of a couple? [2]

SOLUTION

Couple

Two equal unlike parallel forces not having the same line of action and acting on a rigid body are said to form a couple.

Arm of a couple

The arm of a couple is the perpendicular distance between the lines of action of the two forces forming a couple.

4. 2058 Q.No. 6 b

A straight uniform rod is 3m long, when a load of 5N is placed at one end it balances about a point 25cm from that end. Find the weight of the road. [2]

SOLUTION

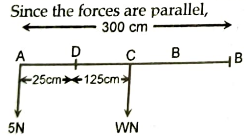
Let AB be a rod of weight W N. The weight W N acts at the middle point C of AB. If a load of 5 N is placed at the point A, then AB will be balanced at D where AD = 25 cm. Here,

AB = 3m = 300 cm. So,

AC = $\frac{300}{2}$ = 150 cm

∴ CD = AC – AD = 150 – 25 = 125 cm

Since the forces are parallel,



$\frac{5}{CD} = \frac{W}{AD}$

or, $\frac{5}{125} = \frac{W}{25}$

or, $\frac{1}{25} = \frac{W}{25}$

∴ W = 1N

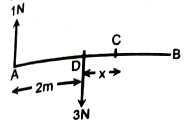
5. 2059 Q.No. 6 b

A uniform bar 4m long and weighting 3N passes over a prop and is supported in horizontal position by a force of 1 N vertically upwards at the other end. Find the distance of the prop from the centre of the bar. [2]

SOLUTION

Let AB be the uniform bar of length 4m and weight 3N acting through D. Let it pass over a prop C. It is supported in a horizontal position by a force of 1 N acting vertically upward at A. Let DC = x.

Then,



$\frac{3}{AC} = \frac{1}{DC}$

or, 3 × DC = 1 × AC

or, 3 × x = 1 × (x + 2)

or, 2x = 2

∴ x = 1

Hence, the distance of the prop from the centre of bar is 1m.

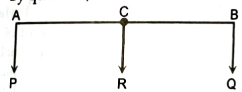
6. 2060 Q.No. 6 a

Find two like parallel forces, acting at a distance of 2.5m apart, which are equivalent to a given force of 30N, the line of action of one being at a distance of 50cm from the given force. [2]

SOLUTION

Let P and Q be the two like parallel forces acting at A and B respectively. Let the resultant R act at C.

By question,



AB = 2.5 m, AC = 50 cm = 0.5 m. R = 30N

Then, CB = AB – AC

= 2.5 –0.5 = 2m

Since the forces are like parallel, so

$\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{P}{2} = \frac{Q}{0.5} = \frac{30}{2.5}$

∴ P = $\frac{30 \times 2}{2.5}$ = 24N and Q = $\frac{30 \times 0.5}{2.5}$ = 6N

Hence, the required forces are 24N and 6N.

7. 2060 Q.No. 6 b

Define arm of a couple and the moment of a couple. [2]

➥ Please refer to 2057 Q.No. 6a

8. 2061 Q.No. 6 a

A straight uniform rod is 3m long. When a load of 5N is placed at one end it balances about a point 25cm from that end. Find the weight of the rod. [2]

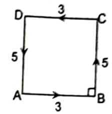
➥ Please refer to 2058 Q.No. 6b

9. 2061 Q.No. 6 b

Forces equal to 3, 5, 3 and 5 newtons respectively act along the sides of a square taken in order, find their resultant. [2]

SOLUTION

Let the forces 3, 5, 3, 5 Newtons act along the sides AB, BC, CD and DA respectively of the square ABCD. Also, let length of side = a meters.



Now, the forces 3N along AB and 3N along CD forms a couple of moment

= 3 × a = 3a Nm

Again, the forces 5N along BC and 5N along DA from a couple of moment 5 × a = 5a Nm.

The resultant of two couples is a couple.

Hence, the resultant of all forces is equivalent to a couple whose moment

= (3a + 5a) Nm = 8a Nm

10. 2062 Q.No. 5 a

Define moment of a force about a point. Give the geometrical meaning of the moment of a force about a point. [2]

➥ Please refer to Model Set II, Q.No. 15

11. 2063 Q.No. 6 b

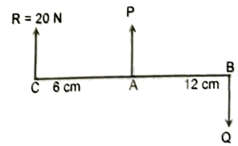
Find the two unlike parallel forces acting at a distance of 12 cm which are equivalent to a force of 20N, the greater of the two forces being at a distance of 6 cm from the given force. [2]

SOLUTION

Let P and Q (P > Q) be two unlike parallel forces whose resultant (R) = 20N. Also let P, Q and R act at the point A, B and C respectively.

Here, CA = 6 cm, AB = 12 cm and BC =12 + 6 =18 cm

Now, we have



$\frac{P}{CB} = \frac{Q}{CA} = \frac{R}{AB}$

or, $\frac{P}{18} = \frac{Q}{6} = \frac{20}{12}$

or, P = $\frac{20 \times 18}{12}$, Q = $\frac{6 \times 20}{12}$

∴ P = 30 N, Q = 10N

12. 2064 Q.No. 6 b

A straight uniform rod is 3 m long. When a load of 5N is placed at one end, it balances about a point 25 cm from that end. Find the weight of the rod. [2]

➥ Please refer to 2058 Q.No. 6b

13. 2065 Q.No 6 b

A uniform bar 4m long and weighting 3N passes over a prop and is supported in a horizontal position by a force of 1N acting vertically upwards at the other end. Find the distance of the prop from the centre of the bar. [2]

➥ Please refer to 2059 Q.No. 6b

14. 2066 C Q.No. 6 b

Define a couple. What do you mean by arm of a couple? [2]

➥ Please refer to 2058 Q.No. 6a

15. 2066 Q.No. 6 b

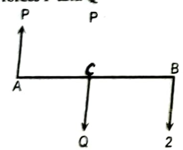
A straight weightless rod, 48 cms in length, rests in a horizontal position between two pegs placed at a distance of 6 cm apart, one peg being at one end of the rod, and a weight of 2 kg is suspended from the other end. Find the pressures on the pegs. [2]

SOLUTION

Let AB be a straight weightless rod resting between two pegs at A and C so that AB = 48 cm and AC = 6 cm. Then,

CB = AB – AC = 48 – 6 = 42 cm

Let P and Q be the pressure on the pegs at A and C. The system is in equilibrium when a weight of 2 kg be suspended at the end B. So, 2 kg wt is the resultant of two unlike parallel forces P and Q



$\therefore \frac{P}{CB} = \frac{Q}{AB} = \frac{2}{AC}$

or, $\frac{P}{42} = \frac{Q}{48} = \frac{2}{6}$

or, $P = \frac{2}{6} \times 42 = 14$ kg wt

and $Q = \frac{2}{6} \times 48 = 16$ kg wt

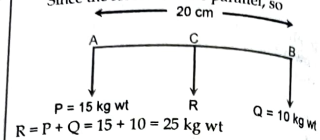
16. 2067 Q.No. 6a

Find the resultant of two parallel forces of 15 kg wt. and 10 kg wt acting at a distance 20 cm apart in the same direction. [2]

SOLUTION

Let P = 15 kg wt and Q = 10 kg wt be acting at the points A and B respectively. Let R be the resultant acting at C. Given AB = 20 cm.

Since the forces are like parallel, so



R = P + Q = 15 + 10 = 25 kg wt

Also,

$\frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{10}{AC} = \frac{25}{20}$

or, $AC = \frac{10 \times 20}{25} = 8$ cm.

Hence, the resultant is 25 kg wt acting at a distance of 8 cm from the force 15 kg wt.

17. 2068 Q.No. 6a

Find two unlike parallel forces acting at a distance of 12cm which are equivalent to a force of 20N, the greater of the two forces being at a distance of 6cm from the given force. [2]

➥ Please refer to 2063 Q.No. 6b

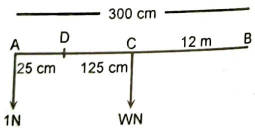
18. 2069 (Set A) Q.No. 12b

A straight uniform rod is 3 m long. When a load of 10N is placed at one end it balances about a point 25 cms from that end. Find the weight of the rod. [2]

SOLUTION

Suppose AB is a straight uniform rod whose length is 3m = 300 cm. Let the weight of the rod be W newtons. Then the weight W acts at the middle C of AB. If a load of 5N is placed at the point A, then AB will be balanced about the point D where AD = 25 cm. Hence CD = AC – AD = 150 – 25 = 125 cm.

Since the forces are like parallel, so



$\frac{10}{CD} = \frac{W}{AD}$

or, $\frac{10}{125} = \frac{W}{25}$

$\therefore$ W = 2N

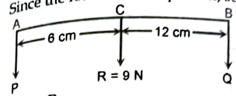
19. 2069 (Set A) Old Q.No. 6a

Two like parallel forces P and Q act at points 18m apart. If the resultant force be 9N and acts at a distance of 6m from P, find Q. [2]

SOLUTION

Suppose P and Q be two like parallel forces acting at the points A and B such that AB = 18 m. Let the resultant R act at C where AC = 6m and R = 9N.

Since the forces are like parallel, so



$\frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{Q}{6} = \frac{9}{6+12}$

or, $Q = \frac{6 \times 9}{18} = 3$

$\therefore$ Q = 3N

20. 2069 (Set B) Q.No. 12b

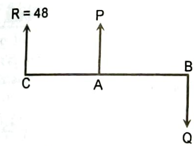
Replace a force of magnitude 48 kgwt by two unlike parallel forces, one at a distance of 2m and other at 8m from the given force. [2]

SOLUTION

Let P and Q (P > Q) be two unlike parallel forces whose resultant is 48 kg wt. Let the forces P, Q and R = 50 kg wt act at the points A, B and C respectively.

Here, CB = 8m and AC = 2m

Since the forces from unlike parallel forces, so



$\frac{P}{CB} = \frac{Q}{CA} = \frac{R}{AB}$

or, $\frac{P}{8} = \frac{Q}{2} = \frac{48}{6}$

or, $\frac{P}{8} = \frac{Q}{2} = 8$

$\therefore$ P = 8 × 8 = 64 kg wt

Q = 8 × 2 = 16 kg wt

21. 2069 Old (Set B) Q.No. 5b

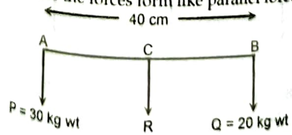
Two parallel forces of 30 kg wt and 20 kg wt are acting at a distance 40cm apart. Find their resultant and its position if forces are like. [2]

SOLUTION

Let R be the resultant of two like parallel forces P = 30 kg wt and Q = 20 kg wt. Let P, Q and R be acting at A, B and C respectively.

Here, AB = 40 cm

Since the forces form like parallel forces, so



R = P + Q = 30 + 20 = 50 kg wt

Also,

$\frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{20}{AC} = \frac{50}{40}$

or, $AC = \frac{20 \times 40}{50} = 16$ cms

Hence the resultant is 50 kg wt and it acts at a distance of 16 cm from the point at which the weight of 30 kg wt acts.

22. 2070 Set D Q.No. 12 b

Find two like parallel forces acting at a distance of 2.5 m apart, which are equivalent to a given force of 30 N. The lines of action of one being at a distance of 50 cm from the given force. [2]

➥ Please refer to 2060 Q.No. 6a

23. 2070 Supp. Q.No. 12 a

If two like parallel forces are 16N and 12N, find their resultant acting at a distance of 90 cm from the greater force and the distance between the forces. [2]

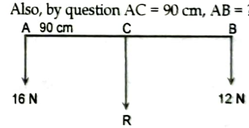
SOLUTION

Let two forces P = 16N and Q = 12N act at A and B respectively. Let R be the resultant acting at C.

Since the forces are like parallel,

R = P + Q = 16 + 12 = 28N.

Also, by question AC = 90 cm, AB = ?



We have,

$\frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{12}{90} = \frac{28}{AB}$

or, 12AB = 28 × 90

$\therefore AB = \frac{28 \times 90}{12} = 210$ cm = 2.1m

24. 2071 Set D Q.No. 2 b

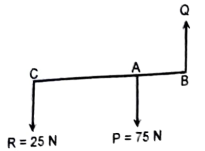
Two unlike parallel forces, the greater of which is 75N, have a resultant 25N. Find the ratio of the distances of the resultant from the component forces. [2]

SOLUTION

Let P and Q be two unlike forces (P > Q) acting at A and B respectively. Let the resultant R act at C.

By question, P = 75N, R = 25N, Q = ?

Since the forces are unlike,



R = P – Q

or, 25 = 75 – Q

∴ Q = 50 N

Also we have

$\frac{P}{BC} = \frac{Q}{AC}$

or, $\frac{75}{BC} = \frac{50}{AC}$

or, $\frac{AC}{BC} = \frac{50}{75} = \frac{2}{3}$

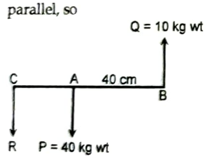
AC : BC = 2 : 3

25. 2071 Old Q.No. 5 b

Two parallel forces of 40kg wt and 10 kg wt are acting at a distance 40 cms apart. Find their resultant if forces are unlike. [2]

SOLUTION

Let two unlike parallel forces P and Q (P > Q) be acting at A and B respectively. Let resultant R act at C. Since the forces are unlike parallel, so



R = P – Q = 40 – 10 = 30 kg wt

Again, we have

$\frac{Q}{CA} = \frac{R}{AB}$

or, $\frac{10}{CA} = \frac{30}{40}$

or, 30CA = 400

∴ CA = $\frac{400}{30} = 13\frac{1}{3}$

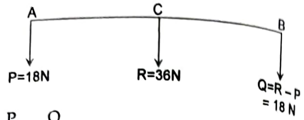
Hence, the resultant is 30 kg wt acting at a distance of $13\frac{1}{3}$ cm from the greater force.

26. 2071 Supp. Q.No. 12a

If one of two like parallel forces and their resultant are 18N and 36N respectively, find the ratio of distances of the resultant from the forces. [2]

SOLUTION

Since P and Q are like parallel, so



$\frac{P}{BC} = \frac{Q}{AC}$

or, $\frac{18}{BC} = \frac{18}{AC}$

i.e. $\frac{AC}{BC} = \frac{1}{1}$

AC : BC = 1 : 1

27. 2072 Set C Q.No. 12b

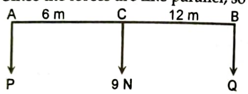
Two like parallel forces P and Q act at points 18 m apart, if the resultant force is 9N and acts at a distance 12 m from Q, find P. [2]

SOLUTION

Let two like parallel forces P and Q act at the point A and B. Let the resultant 9N acts at C on AB such that BC = 12m.

Then AC = 18 – 12 = 6 m.

Since the forces are like parallel, so



$\frac{P}{BC} = \frac{Q}{AC} = \frac{9}{AB}$

or, $\frac{P}{12} = \frac{Q}{6} = \frac{9}{18}$

From first and last ratios, we have

$\frac{P}{12} = \frac{9}{18}$

∴ P = 6N

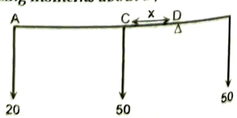
28. 2072 Set D Q.No. 12b

A uniform beam AB is 16 m long and weighs 50 kg weights of 20 kg and 50 kg are suspended from A and B respectively. At what point must the beam be supported so that it may rest horizontally? [2]

SOLUTION

AB is a uniform beam whose weight acts through the middle point C such that AC = CB = 8 metres. When the weights of 20 kg and 50 kg be suspended from A and B let the beam be supported at D so that the beam may rest horizontally, let CD = x.

Taking moments about D,



20 · AD + 50 · CD = 50 · DB

or, 20 (8 + x) + 50x = 50 (8 – x)

or, 16 + 2x + 5x = 40 – 5x

or, 12x = 24

∴ x = 2 m

The point D must be at a distance of (8 + 2) m i.e. 10 m from the end A from which a weight of 20 kg is suspended.

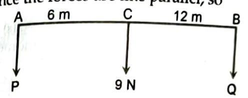
29. 2072 Supp Q.No. 12c

Two like parallel forces P and Q act at points 18m apart, if the resultant force be 9N and acts at a point 6 m from P, find Q. [2]

SOLUTION

Let two like parallel forces P and Q act at the point A and B. Let the resultant 9N acts at C on AB such that AC = 6m. Then BC = 18 – 6 = 12 m.

Since the forces are like parallel, so



$\frac{P}{BC} = \frac{Q}{AC} = \frac{9}{AB}$

or, $\frac{P}{12} = \frac{Q}{6} = \frac{9}{18}$

From last two ratios, we have

$\frac{Q}{6} = \frac{9}{18}$

∴ Q = 3N

30. 2073 Set D Q.No. 12b

Two like parallel forces P and Q act at points 18m apart, if the resultant force be 9N and acts at a distance 6m from P, find Q. [2]

➥ Please refer to 2072 Supp Q.No. 12c

31. 2074 Set A Q.No. 12b

Forces equal to 5N, 2N, 5N, 2N act along the sides AB, CB, CD and AD of a square ABCD whose side is 3m. Find the moment of the couple that will give equilibrium. [2]

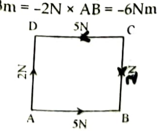
SOLUTION

The forces 5N along AB & 5N along CD from a couple in anticlockwise direction. Hence the moment of this couple = 5N × AD

= 5N × 3m = 15 Nm

Also, the force 2N along CB & 2N along AD form a couple in clockwise direction. Hence the moment of this couple

= –2N × 3m = –2N × AB = –6Nm



Now, sum of moments of the couples

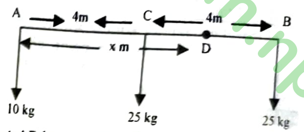
= 15Nm – 6Nm = 9Nm

∴ Hence the moment of the couple that will give equilibrium = 9 Nm.

32. 2074 Set B Q.No. 12b

A uniform beam AB is 8m long and weights 25kg. Weights of 10kg and 25 kg are suspended from A and B respectively. At what point must the beam be supported so that it may rest horizontally? [2]

SOLUTION



Let AB be a uniform beam of length 8m and weight 25 kg. Also let the weights 10 kg & 25 kg are suspended from A & B respectively. Suppose that the beam AB is supported at D so that AB remains horizontal. Here C is the midpoint of AB.

Let AD = x,

Then, CD = AD – AC = x – 4

BD = AB – AD = 8 – x

Now, taking moment about the point D,

10 × AD + 25 × CD = 25 × BD

or, 10x + 25(x – 4) = 25(8 – x)

or, 10x + 25x – 100 = 200 – 25x

or, 60x = 300

or, x = $\frac{300}{60}$ = 5

Hence the beam should be supported at a distance of 5m from A.

33. 2074 Supp Q.No. 12b

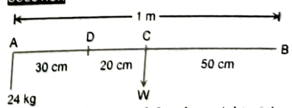
Two like parallel forces P and Q act at points 18 m apart; if the resultant force be 9N and acts at a distance 6m from P, find Q. [2]

➥ Please refer to 2072 Supp. Q.N. 12c

34. 2075 Set B Q.No. 12b

A straight uniform beam 1m long, when a load of 24 kg is placed at one end, it balances about a point 30 cm from that end. Find the weight of the beam. [2]

SOLUTION



Suppose AB is a rod. Let the weight of the rod be W kg. Then, the weight W kg act at the middle point C of AB. If a load 24 kg is placed at the point A, then AB will be balanced about the point D, where AD = 30cm. Then CD = 20 cm. Since the forces are parallel, we have,

$\frac{24}{CD} = \frac{W}{AD}$

or, $\frac{24}{20} = \frac{W}{30}$

or, $W = \frac{24 \times 30}{20}$

$\therefore$ W = 36 kg

4 MARKS QUESTIONS

35. 2057 Q.No. 14 a

Prove that the algebraic sum of the moments of any two forces, meeting at a point, about any point in their plane is equal to the moment of their resultant about the same point. [4]

SOLUTION

Let P and Q be two forces and R be their resultant.

Let O be a given point. Let us draw OC parallel to P to meet the line of action of Q at C. Let AC represent Q in magnitude and AB represent P in magnitude. Now, complete the parallelogram ABDC.

Then AD represents the resultant R of P and Q.

Join OA and OB.

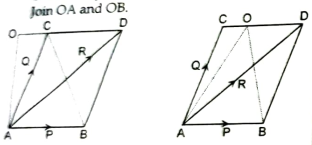


fig (i) fig (ii)

The point O may lie outside the ∠BAC in fig (i) and inside ∠BAC in fig (ii). The moment of P about O is 2 ΔOAB and is positive in both figures.

The moment of Q about O is 2ΔOAC which is positive in fig (i) and negative in fig (ii)

In fig (i)

The sum of moments of P and Q about O

= 2ΔOAB + 2ΔOAC

= 2ΔADB + 2ΔOAC (∵ ΔOAB = ΔADB)

= 2ΔOAD

= moment of R about O

In fig (ii),

The moment of P and Q about O

= 2ΔOAB − 2ΔOAC

= 2ΔADB + 2ΔOAC (∵ ΔOAB = ΔADB)

= 2ΔOAD

= moment of R about O.

36. 2058 Q.No. 14 a

Prove that the algebraic sum of moments of two like parallel forces, about any point in their plane is equal to the moment of their resultant about the same point. [4]

SOLUTION

Let P and Q be like parallel forces and R be their resultant. Let O be any point in their plane. Let us draw a line through O and perpendicular to the lines of actions of the forces P, Q and R to meet them in A, B and C respectively. Since the forces are like parallel, so

R = P + Q

and P · AC = Q · BC ...(i)

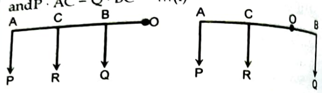


fig (i) fig (ii)

In fig (i), the algebraic sum of the moments of P and Q about O

= P ·AO + Q · BO

= P(AC + OC) + Q (OC − BC)

= (P + Q) OC + P · AC − Q ·BC

= R · OC [using (i)]

= moment of R about O.

In figure (ii),

The algebraic sum of the moments of P and Q about O

= P ·AO − Q ·OB

= P(AC + OC) − Q(BC − OC)

= P · AC − Q · BC + (P + Q) · OC

= R · OC [using (i)]

= moment of R about O.

37. 2059 Q.No. 13 a

Find the resultant of two like parallel forces acting on a rigid body. [4]

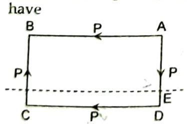
➥ Please refer to Model Set I, Q.No. 14

38. 2061 Q.No. 14 a

ABCD is a square along AB, CB, AD and DC of equal forces, P act. Show that the magnitude of their resultant is equal to double of any component and acts along DC. [4]

SOLUTION

Let equal forces P act along the sides AB, CB, AD and DC of the square ABCD. Resolving the forces along and perpendicular to CD, we have



X = −P − P = −2P

Y = −P + P = 0

Let R be the resultant. Then

$R = \sqrt{X^2 + Y^2} = \sqrt{(-2P)^2 + 0^2} = 2P$

Hence, the resultant is double of P.

Let θ be the angle made by R with CD.

Then $\tan\theta = \frac{Y}{X} = \frac{0}{-2P} = 0 = \tan 180°$

$\therefore$ θ = 180°

Again, let the resultant R cuts AD at E such that DE = x and CD = a. Now, taking moment about E, we have

−P × DE + P × EA − P × CD = 0

or, −P × x + P × (a − x) −P × a = 0

or, −Px + Pa − Px − Pa = 0

or, −2Px = 0

$\therefore$ x = 0

$\therefore$ The resultant acts along DC.

39. 2060 Q.No. 14 a

Prove that the algebraic sum of moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point. [4]

➥ Please refer to 2057 Q.No. 14a

40. 2061 Q.No. 13 a OR

Find the resultant of like parallel forces. [4]

➥ Please refer to Model Set I, Q.No. 14

41. 2062 Q.No. 14 a

The wire passing round a telegraph pole is horizontal and the two portion attached to the pole are inclined at an angle 60° to one another. The pole is supported by a wire attached to the middle point of the pole and inclined at 60° to the horizon; show that the tension of this wire is $4\sqrt{3}$ times that of the telegraph wire. [4]

SOLUTION

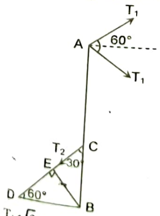
Let C be the midpoint of the telegraph wire AB. Let tension of the telegraph wire at A be T_1 and tension of the wire attached at C be T_2

Resultant tension of the two portions of the wire attached at A

$= 2 \times T_1 \cos\frac{60}{2} = 2T_1 \cos 30 = 2T_1 \times \frac{\sqrt{3}}{2} = \sqrt{3}\,T_1$

Draw BE perpendicular to DC.

Now, taking moment about B,



$T_1\sqrt{3} \times AB - T_2 \times BE = 0$

or, $T_1\sqrt{3} \times AB = T_2 \times BC \sin 30°$

or, $T_1\sqrt{3} \times AB = T_2 \times \frac{AB}{2} \times \frac{1}{2}$

or, $T_1\sqrt{3} = \frac{T_2}{4}$

or, $T_2 = 4\sqrt{3}\,T_1$

$\therefore$ The tension of the wire attached at C is $4\sqrt{3}$ times that of the tension of the telegraph wire.

42. 2063 Q.No. 14 a

Prove that the algebraic sum of the moments of any two like parallel forces about a point in their plane is equal to the moment of their resultant force about the same point. [4]

➥ Please refer to 2058 Q.No. 14a

43. 2064 Q.No. 14 a

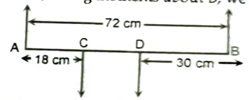
A light rod of length 72 cms has equal weight attached to it, one at 18 cms from one end and the other at 30 cms from the other end. It is supported by two vertical strings attached to its ends and if the string can not support a tension greater than the weight of 50 kg, what is the greatest magnitude of the equal weights? [4]

SOLUTION

Let AB be a light rod of length 72 cms. Let W be the equal weight suspended through C and D such that AC = 18 cms and BD = 30 cms.

Also, CB = AB − AC = 72 − 18 = 54 cms

Now, taking moments about B, we have,



50 × AB = W × CB + W × DB

or, 50 × 72 = W × 54 + W × 30

or, 50 × 72 = 84W

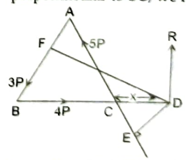
$\therefore W = \frac{50 \times 72}{84} = \frac{300}{7} = 42\frac{6}{7}$ kg

44. 2065 Q.No 14 a

Forces equal to 3P, 4P, 5P act along the sides AB, BC and CA of an equilateral triangle ABC, find the magnitude, direction and line of action of the resultant. [4]

SOLUTION

Let a be the side of an equilateral triangle ABC. Resolving the forces along and perpendicular to BC, we have



X = 4P cos 0° + 5P cos 120° + 3P cos 240°

$= 4P \times 1 + 5P \times \left(\frac{-1}{2}\right) + 3P \times \left(\frac{-1}{2}\right)$

$= 4P - 8P \times \frac{1}{2} = 0$

$Y = 4P \sin 0° + 5P \sin 120° + 3P \sin 240°$

$= 4P \times 0 + 5P \times \frac{\sqrt{3}}{2} + 3P \times \left(-\frac{\sqrt{3}}{2}\right) = P\sqrt{3}$

Let R be the resultant. Then

$R = \sqrt{X^2 + Y^2} = \sqrt{0^2 + (P\sqrt{3})^2} = P\sqrt{3}$ N

If the resultant R is inclined at an angle θ with BC, then

$\tan\theta = \frac{Y}{X} = \frac{P\sqrt{3}}{0} = \infty = \tan 90°$

∴ $\theta = 90°$

Hence, the resultant $R = P\sqrt{3}$ N is in the direction at right angles to BC.

Let the resultant cut BC produced at D where CD = x. Now, taking moments about D, we have

$5P \times DE - 3P \times DF = 0$

or, $5P \times x \sin 30 - 3P \times (a + x) \sin 60° = 0$

or, $5Px \times \frac{1}{2} = 3P(a + x)\frac{\sqrt{3}}{2}$

or, $5x = 3a + 3x$

or, $x = \frac{3a}{2}$

∴ $CD = \frac{3BC}{2}$

45. 2066 C Q.No. 14 a

Prove that the algebraic sum of the moments of any two forces, meeting at a point, about any point in their plane is equal to the moment of their resultant about the same point. [4]

- Please refer to 2057 Q.No. 14a

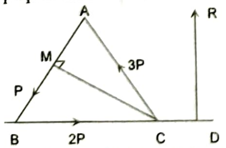
46. 2066 Q.No.14 a

Three forces P, 2P and 3P act along the sides AB, BC and CA of an equilateral triangle ABC of side a. Find the magnitude, direction and line of action of the resultant. [4]

SOLUTION

Let the forces P, 2P and 3P act along the sides AB, BC and CA of an equilateral triangle of side a,

Now, resolving the forces along and perpendicular to BC, we have



$X = 2P \cos 0° + 3P \cos 120° + P \cos 240°$

$= 2P \cdot 1 + 3P \cdot \left(\frac{-1}{2}\right) + P\left(\frac{-1}{2}\right)$

$= 2P - \frac{3P}{2} - \frac{P}{2} = 0$

$Y = 2P \sin 0° + 3P \sin 120° + P \sin 240°$

$= 2P \times 0 + 3P \times \frac{\sqrt{3}}{2} + P \times \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}\,P$

Let R be the resultant. Then,

$R = \sqrt{X^2 + Y^2} = \sqrt{0^2 + (\sqrt{3}P)^2} = \sqrt{3}\,P$

Let θ be the angle made by the resultant with BC. Then,

$\tan\theta = \frac{Y}{X} = \frac{\sqrt{3}P}{0} = \infty = \tan 90°$

∴ $\theta = 90°$

Let the resultant force meets the side BC in D where CD = x.

Taking moment about C, we have

$P \times CM - R \times CD = 0$, where CM is perpendicular to AB.

or, $P \times BC \sin 60° - \sqrt{3}P \times x = 0$

or, $\frac{a\sqrt{3}}{2} - \sqrt{3}x = 0$

or, $x = \frac{a}{2}$, where a is side of equilateral triangle.

47. 2067 Q.No. 14a

Define parallel forces. Deduce the resultant of two like parallel forces. [4]

SOLUTION

Parallel Forces

If two forces act along parallel lines, they are said to be parallel forces.

Next part

Please refer to Model Set I, Q.No. 14

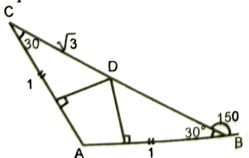
48. 2068 Q.No. 14 a

A light rod of length 72cm has equal weights attached to it, one at 18cm from one end and the other at 30cm from the other end; if it be supported by two vertical strings attached to its ends and if the strings cannot support a tension greater than the weight of 50kg, what is the greatest magnitude of the equal weight? [4]

SOLUTION

Let ΔABC be an isosceles triangle where ∠A = 120°. Then ∠ABC = ∠ACB = 30°

Now, resolving the forces along and perpendicular to AB, we have



$X = 1 \times \cos 0° + \sqrt{3} \times \cos 150° + 1 \times \cos 120°$

$= 1 \times 1 + \sqrt{3} \times \left(-\frac{\sqrt{3}}{2}\right) + 1 \times \left(\frac{-1}{2}\right)$

$= 1 - \frac{3}{2} - \frac{1}{2} = -1$

and $Y = 1 \times \sin 0° + \sqrt{3} \times \sin 150° + 1 \times \sin 120°$

$= 1 \times 0 + \sqrt{3} \times \frac{1}{2} + 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

Let R be the resultant. Then

$R = \sqrt{X^2 + Y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

Again, let R make an angle θ with AB. Then,

$\tan\theta = \frac{Y}{X} = \frac{\sqrt{3}}{-1} = -\sqrt{3} = \tan 120°$

∴ $\theta = 120°$

Hence, the resultant is parallel to AC. Let the resultant cut BC at D. Draw perpendicular from D to AB and AC.

Taking moment about D, we have

$1 \cdot MD - 1 \cdot DN = 0$

or, $MD - DN = 0$

or, $BD \sin 30° = CD \sin 30°$

or, $BD = CD$

Hence, the resultant bisects BC at D.

49. 2069 (Set A) Old Q.No. 14a

ABC is an isosceles triangle whose angle A is 120° and forces of magnitude 1, 1 and $\sqrt{3}$ N act along AB, AC and BC, show that the resultant bisects BC and is parallel to one of the other sides of the triangle. [4]

- Please refer to 2069 Set 'A', Q.No. 15

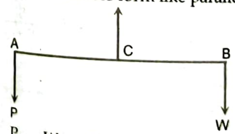
50. 2069 Old (Set B) Q.No. 13b

A man carries a bundle at the end of a stick which is placed over his shoulder. If the distance between his hand and shoulder be changed, how does the pressure on his shoulder change? [4]

SOLUTION

Let AB be a stick of length *l*. Let C be the position of the shoulder. If W be the weight of the bundle suspended at the end B and P be the pressure due to the hand at A, then the shoulder is pressed due to the resultant of P and W. Let R be the reaction on the shoulder C such that AC = x(say).

Since the forces form like parallel forces, so



$\frac{P}{CB} = \frac{W}{AC} = \frac{R}{AB}$

or, $\frac{P}{l - x} = \frac{W}{x} = \frac{R}{l}$

∴ $R = \frac{Wl}{x}$

Since *Wl* is constant, so R depends on x. R will be least when x will be greatest. But the greatest value of x is *l*. Hence R will be least when the distance between the hand and the shoulder is equal to the length of the stick.

51. 2070 (Old) Q.No. 6 a

The resultant of two like parallel forces is 12N and it acts at a distance 2 meter from the larger force equal to 8N. Find the distance of the resultant from the smaller force. [4]

SOLUTION

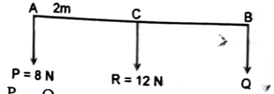
Let P and Q (P > Q) be two like parallel forces acting at A and B respectively. Let R be the resultant force acting at C. By question,

P = 8N, R = 12 N, AC = 2m, BC = ?

Since the forces are like parallel, so

Q = R − P = 12 − 8 = 4N

Now, we have



$\frac{P}{BC} = \frac{Q}{AC}$

or, $\frac{8}{BC} = \frac{4}{2}$

or, $8 = 2BC$

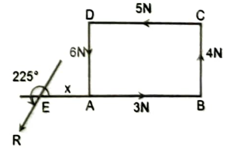
∴ $BC = 2m$

52. 2070 (Old) Q.No. 14 a

Forces equal to 3,4,5,6 N respectively act along the sides of a square ABCD taken in order, find the magnitude, direction and line of action of their resultant. [4]

SOLUTION

Let a be the length of the side of the square ABCD. Let the forces 3, 4, 5 and 6N act along AB, BC, CD and DA respectively. Resolving the forces along and perpendicular to AB, we have



$X = 3 - 5 = -2$

$Y = 4 - 6 = -2$

Let R be the resultant. Then,

$R = \sqrt{X^2 + Y^2} = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$ N

If the resultant R is inclined at an angle θ with AB, then

$\tan\theta = \frac{Y}{X} = \frac{-2}{-2} = 1 = \tan 225°$

[∵ X and Y are both negative]

∴ $\theta = 225°$

Let the resultant R cut BA produced at E where AE = x. Taking moment about E, we have

$4 \times BE + 5 \times DA - 6 \times AE = 0$

or, $4(a + x) + 5a - 6x = 0$

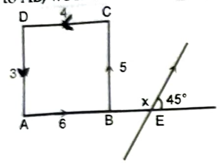
$\therefore \quad x = \frac{9}{2}a$

53. 2070 Supp. Q.No. 13 a

Forces of 6, 5, 4, 3 kg – wts respectively act along the sides of a square ABCD taken in order. Find the magnitude, direction and line of action of their resultant. [4]

SOLUTION

Let the forces 6, 5, 4, 3,kg wts respectively act along the sides AB, BC, CD and DA. Resolving the forces along and perpendicular to AB, we have



$X = 6 - 4 = 2,\ Y = 5 - 3 = 2$

Let R be the resultant, then

$R = \sqrt{X^2 + Y^2} = \sqrt{2^2 + 2^2} = 4\sqrt{2}$ kg wts

Let θ be the angle made by the resultant R with AB, then

$\tan\theta = \frac{Y}{X} = \frac{2}{2} = 1 = \tan 45°$

$\therefore \quad \theta = 45°$

Hence, the resultant is parallel to AC.

Let the resultant cut AB produced at E where BE = x. Let a be the length of the side of square ABCD.

Now, taking moment about E, we have

$-5 \times BE + 4 \times BC + 3 \times AC = 0$

or, $-5 \times x + 4 \times a + 3 \times (a + x) = 0$

or, $-5x + 4a + 3a + 3x = 0$

or, $-2x = -7a$

or, $x = \frac{7a}{2}$

$\therefore \quad BE = \frac{7a}{2}$ where a = side of a square

54. 2071 Old Q.No. 13 b

Prove that the algebraic sum of the moments of any two parallel forces about any point in their plane is equal to the moment of their resultant about the same point. [4]

➥ Please refer to 2058 Q.No. 14a

55. 2071 Supp. Q.No. 13a

Forces 1, 2, 4, 5 kg wts act along the sides of a square taken in order. Prove that their resultant is parallel to a diagonal and find where it cuts the side along which the first force acts. [4]

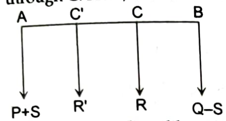
➥ Please refer to Model Set I, Q.No. 14 OR

56. 2075 Set C Q.No. 13a

Two like parallel forces of magnitudes P and Q are acting at the end points A and B of a rod AB of length ℓ. If two opposite forces each of magnitude S are added to P and Q, then prove that the line of action at the new resultant will be displaced through a distance $\frac{S\ell}{P + Q}$. [4]

SOLUTION

Let P and Q be two like parallel forces acting at the points A and B respectively. Let R be their resultant whose line of action passes through C. Here, AB = l and R = P + Q.



Since the forces form like parallel forces, so,

$\frac{P}{CB} = \frac{Q}{AC} = \frac{R}{AB}$

or, $\frac{P}{CB} = \frac{Q}{AC} = \frac{P + Q}{l}$

$\therefore \quad CB = \frac{Pl}{P + Q}$

When the forces S and –S be added to P and Q respectively, let R' be the new resultant whose line of action passes through the point C'.

Now,

$R' = (P + S) + (Q - S) = P + Q$

Then,

$\frac{P + S}{C'B} = \frac{Q - S}{AC'} = \frac{R'}{AB}$

or, $\frac{P + S}{C'B} = \frac{Q - S}{AC'} = \frac{P + Q}{l}$

$\therefore \quad C'B = \frac{(P + S).l}{P + Q}$

Now,

$CC' = C'B - CB$

$= \frac{(P + S)l}{P + Q} - \frac{P.l}{P + Q} = \frac{(P + S - P)l}{P + Q} = \frac{Sl}{P + Q}$

6 MARKS QUESTIONS

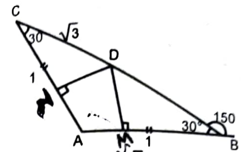
57. 2069 (Set A) Q.No. 15

ABC is an isosceles triangle whose angle A is 120° and forces of magnitudes, 1, 1 and $\sqrt{3}$ N act along AB, AC and BC; show that the resultant bisects BC and is parallel to one of the other side of the triangle. [6]

SOLUTION

Let ΔABC be an isosceles triangle where ∠A = 120°. Then ∠ABC = ∠ACB = 30°

Now, resolving the forces along and perpendicular to AB, we have



$X = 1 \times \cos 0° + \sqrt{3} \times \cos 150° + 1 \times \cos 120°$

$= 1 \times 1 + \sqrt{3} \times \left(-\frac{\sqrt{3}}{2}\right) + 1 \times \left(\frac{-1}{2}\right)$

$= 1 - \frac{3}{2} - \frac{1}{2} = -1$

and $Y = 1 \times \sin 0° + \sqrt{3} \times \sin 150° + 1 \times \sin 120°$

$= 1 \times 0 + \sqrt{3} \times \frac{1}{2} + 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \sqrt{3}$

Let R be the resultant. Then

$R = \sqrt{X^2 + Y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

Again, let R make an angle θ with AB. Then,

$\tan\theta = \frac{Y}{X} = \frac{\sqrt{3}}{-1} = -\sqrt{3} = \tan 120°$

$\therefore \quad \theta = 120°$

Hence, the resultant is parallel to AC. Let the resultant cut BC at D. Draw perpendicular from D to AB and AC.

Taking moment about D, we have

$1 \cdot MD - 1 \cdot DN = 0$

or, $MD - DN = 0$

or, $BD \sin 30° = CD \sin 30°$

or, $BD = CD$

Hence, the resultant bisects BC at D.

58. 2069 (Set A) Q.No. 15 or

Find the resultant of two like parallel forces. [6]

➥ Please refer to Model Set I, Q.No. 14

59. 2069 (Set B) Q.No. 15

Forces equal to 3p, 4p and 5p and along the sides AB, BC and CA of an equilateral triangle ABC, find the magnitude, direction and the line of action of the resultant. [6]

➥ Please refer to 2065 Q.No. 14a

60. 2070 Set C Q.No. 15

Define moment of a force about a point. Prove that the algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point. [6]

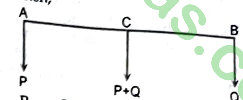
➥ Please refer to Model Set II Q.No. 15

61. 2070 Set C Q.No. 15 or

P and Q are like parallel forces. If P is moved parallel to itself through a distance x, show that the resultant of P and Q moves a distance $\frac{P.x}{P + Q}$. [6]

SOLUTION

Let P and Q be like parallel forces acting at A and B. Let their resultant P + Q be acting at C. Then,



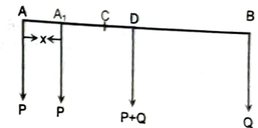
$\frac{P}{BC} = \frac{Q}{AC} = \frac{P + Q}{BC + AC}$

or, $\frac{P}{BC} = \frac{P + Q}{AB}$

or, $BC = P.\frac{AB}{P + Q}$ ...(i)

If the force P is moved parallel to itself through a distance x to A_1, then the resultant acts at D where $AA_1 = x$

Then



$\frac{P}{BD} = \frac{Q}{A_1D} = \frac{P + Q}{BD + A_1D}$

or, $\frac{P}{BD} = \frac{P + Q}{A_1B}$

or, $BD = \frac{P}{P + Q} A_1B$ ...(ii)

Now, the required distance

$CD = BC - BD$

$= \frac{P}{P+Q}AB - \frac{P}{P+Q}A_1B$ [using (i) and (ii)]

$= \frac{P}{P+Q}(AB - A_1B)$

$= \frac{P}{P+Q}AA_1$

$= \frac{Px}{P+Q}$

62. 2070 Set D Q.No. 15

Define moment of a force about a point. Prove that the algebraic sum of the moments of the moment of two like parallel forces about any point in their plane is equal to the moment of their resultant about the same point. [6]

➥ First part:

Please refer to Model Set I Q.No. 14 OR

Second part:

Please refer to 2058 Q.No. 14a

63. 2071 Set C Q.No. 15

Three forces p, 2p and 3p act along the sides AB, BC and CA of an equilateral triangle ABC. Find the magnitude, direction and line of action of the resultant. [6]

➥ Please refer 2066 Q.No. 14a

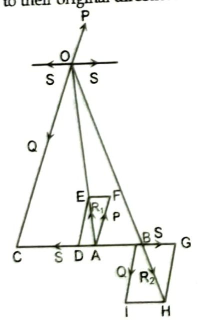
64. 2071 Set C Q.No. 15 OR

Find the resultant of two unlike parallel forces. A man carries a bundle at the end of a stick 75 cm long which is placed on his shoulder. What should be the distance between his hand and shoulder, in order that the pressure on the shoulder may be three times the weight of the bundle? [6]

SOLUTION

Let P and Q (P > Q) be two unlike parallel forces acting at the points A and B respectively on a rigid body. Let the forces be represented by the lines AF and BI. Join AB.

Let us introduce two equal and opposite forces each equal to S along BA and AB respectively. These two forces being equal and opposite have no effect upon the system. Let these forces be represented by AD and BG. Complete the parallelogram AFED and BIHG. Let their diagonals AE and HB be produced to meet at O. Through O, draw OC parallel to AF or be produced BI to meet BA produced at C. Now the forces P and S at A have a resultant R_1 represented by AE. Let its point of application be transferred to O. In the same way, the forces Q and S at B have resultant R_2 represented by BH. Let its point of application be transferred to O. Let the force R_1 be resolved into components P and S parallel to their original directions.



Similarly, R_2 is resolved into components Q and S each acting at O. Now the two equal and opposite forces S and S at O balance each other. Finally, we have the resultant P − Q acting along CO i.e. along the direction of greater force.

Since the triangles OCA and EDA are similar,

$$\frac{OC}{CA} = \frac{ED}{DA} = \frac{P}{S}$$

or, $\frac{OC}{CA} = \frac{P}{S}$

or, $P \times CA = S \times OC$ ...(i)

Again, since the triangles OCB and BIH are similar,

$$\frac{OC}{CB} = \frac{BI}{IH} = \frac{Q}{S}$$

or, $Q \times CB = S \times OC$ ...(ii)

From (i) and (ii), we have

$P \times CA = Q \times CB$

or, $\frac{CA}{CB} = \frac{Q}{P}$

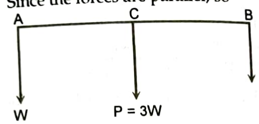
i.e. C dividends AB externally in inverse ratio of the forces.

Second Part

Let the bundle be placed at the end A of a stick AB = 25 cm which rests on the shoulder at C and the man holds at the other end B. If W be the weight of the bundle and P be the pressure on the shoulder. So,

P = 3W, BC = ?

Since the forces are parallel, so



$$\frac{W}{BC} = \frac{P}{AB}$$

or, $\frac{W}{BC} = \frac{3W}{75}$

or, $BC = \frac{75}{3} = 25$

∴ Distance between the hand and the shoulder is 25 cm.

65. 2071 Set D Q.No. 15

Define moment of a force about a point. Prove that the algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same point. [6]

➥ Please refer to Model Set II Q.No. 15

66. 2072 Set C Q.No. 14

Three forces P, 2P and 3P act along the sides AB, BC and CA of an equilateral triangle of side a, find the magnitude, direction and line of action of the resultant. [6]

➥ Please refer to 2066 Q.No. 14a

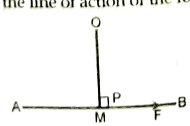
67. 2072 Set C Q.No. 14 OR

Define moment. State and prove Varignon's theorem. [6]

SOLUTION

Moment

The moment of a force about a given point is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force.



Thus if F be a force and p be the perpendicular distance of the point O from AB then the moment of F about O is F × OM = F × p

Varignon's Theorem

The algebraic sum of the moments of any two forces about any two point in their plane is equal to the moment of their resultant about the same point.

Proof: Let P and Q be two forces and let R be their resultant. Then the following two cases arise

Case I: When P and Q meet at a point

Let O be a given point. Let us draw OC parallel to P to meet the line of action of Q at C. Let AC represent Q in magnitude and AB represent P in magnitude. Now, complete the parallelogram ABDC.

Then AD represents the resultant R of P and Q.

Join OA and OB.

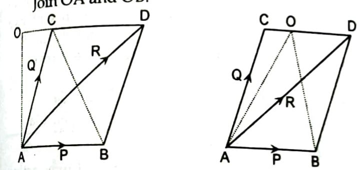


fig (i) fig (ii)

The point O may lie outside the ∠BAC in fig (i) and inside ∠BAC in fig (ii). The moment of P about O is 2 ΔOAB and is positive in both figures.

The moment of Q about O is 2ΔOAC which is positive in fig (i) and negative in fig (ii)

In fig (i)

The sum of moments of P and Q about O

= 2ΔOAB + 2ΔOAC

= 2ΔADB + 2ΔOAC (∵ ΔOAB = ΔADB)

= 2ΔOAD

= moment of R about O

In fig (ii),

The moment of P and Q about O

= 2ΔOAB − 2ΔOAC

= 2ΔADB + 2ΔOAC (∵ ΔOAB = ΔADB)

= 2ΔOAD

= moment of R about O.

Case II: When P and Q are parallel

Let P and Q be like parallel forces and R be their resultant. Let O be any point in their plane. Let us draw a line through O and perpendicular to the lines of actions of the forces P, Q and R to meet them in A, B and C respectively. Since the forces are like parallel,

R = P + Q

and P · AC = Q · BC ...(i)

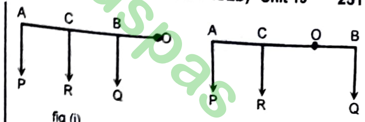


fig (i) fig (ii)

In fig (i), the algebraic sum of the moments of P and Q about O

= P · AO + Q · BO

= P(AC + OC) + Q (OC − BC)

= (P + Q) OC + P · AC − Q · BC

= R · OC [using (i)]

= moment of R about O.

In figure (ii),

The algebraic sum of the moments of P and Q about O

= P · AO − Q · OB

= P(AC + OC) − Q(BC − OC)

= P · AC − Q · BC + (P + Q) · OC

= R · OC [using (i)]

= moment of R about O.

68. 2072 Set D Q.No. 14

Define coplanar forces. Forces equal to P, 2P, 3P and 4P act along the sides of a square ABCD taken in order, find the magnitude, direction and the line of action of the resultant. [6]

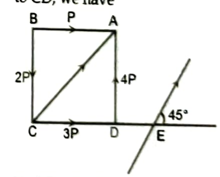
SOLUTION

Coplanar Forces:

A system of forces whose lines of action lie on the same plane are called coplanar forces.

Next part:

Let the forces P, 2P, 3P and 4P act along the side AB, BC, CD and DA respectively. Resolving the forces along an perpendicular to CD, we have



X = 3P − P = 2P

Y = 4P − 2P = 2P

Let θ be the angle made by R with AB, we have

$$\tan\theta = \frac{Y}{X} = \frac{2P}{2P} = 1 = \tan 45°$$

∴ θ = 45°

Hence, the resultant is parallel to CA. Let the resultant cut CD produced at E where DE = x. Let CD = a. Taking moment about E, we have

−4P × DE + P × DA + 2P × CE = 0

or, −4x + 1 × a + 2(a + x) = 0

or, −2x = −3a

∴ $x = \frac{3a}{2}$

i.e. $DE = \frac{3a}{2}CD$

69. 2072 Set E Q.No. 15

Find the resultant of two like parallel forces. [6]

➥ Please refer to Model Set I, Q.No. 14

70. 2072 Set E Q.No. 15 OR

ABCD is a square; along AB, CB, AD and DC equal forces P act; show that the magnitude of their resultant is equal to double of any components and acts along DC. [6]

➥ Please refer to 2061 Q.No. 14a

71. 2072 Supp Q.No. 15

Define moment of a force about a point. What does it represent geometrically? Prove that the algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about the same points. [6]

➥ Please refer to Model Set II, Q.No. 15

72. 2073 Set C Q.No. 15

Define like and unlike parallel forces. A man carries a bundle at the end of a stick which is placed over his shoulder, if the distance between his hand and shoulder be changed how does the pressure on his shoulder change? [6]

SOLUTION

First part: Two parallel forces are said to be like when they act in the same direction and they are said to be unlike when they act in opposite direction.

Next part:

Please refer to 2069 (Old) Set B Q.No. 13b

73. 2073 Set D Q.No. 15

Forces equal to P, 2P, 3P and 4P act along the sides of a square ABCD taken in order. Find the magnitude, direction and the line of action of the resultant. [6]

➥ Please refer to 2072 Set D Q.No. 14

74. 2073 Supp Q.No. 14

Define parallel forces. Deduce the resultant of two parallel forces. [6]

➥ Please refer to Model Set I Q.No. 14

75. 2073 Supp Q.No. 14 or

Geometrically interpret moment of a force. Also state and prove Varignon's theorem.

➥ Please refer 2072 Set C Q.No. 14 OR

76. 2074 Set A Q.No. 14

Define the moment of a force. Prove that the algebraic sum of the moments of two parallel forces about a point in their plane is equal to the moment of their resultant about the same point. [6]

SOLUTION

Moment

The moment of a force about a given point is the product of the magnitude of the force and the perpendicular distance of the point from the line of action of the force.

Thus if F be a force and p be the perpendicular distance of the point O from AB then the moment of F about O is F × OM = F × p

Geometric Meaning of the Moment of a Force

Last Part

Please refer to 2058 Q.No. 14a

77. 2074 Set B Q.No. 14

Define moment of a force about a point. State and prove Varignon's theorem for two interesting forces. [6]

➥ Please refer to Model Set II Q.No. 15

78. 2074 Supp Q.No. 15

Define moment of a force about a point. Prove that the algebraic sum of the moments of two like parallel forces about any point in their plane is equal to the moment of their resultant about the same point. [6]

➥ **First part:** Please refer to 2062 Q.N. 5a

Second part: Please refer to 2058 Q.N. 14a

79. 2075 Set A Q.No. 15

P and Q (P>Q) are two like parallel forces acting at A and B. Show that if they interchange positions, the point of application of the resultant is displaced a distance $\frac{P-Q}{P+Q}$ AB. [6]

➥ Please refer to Model Set II Q.No. 13a

80. 2075 Set A Q.No. 15 OR

Three forces P, 2P and 3P act along the sides AB, BC and CA of an equilateral triangle ABC of side a; find the magnitude, direction and the line of action of the resultant. [6]

➥ Please refer to 2066 Q.No. 14a

81. 2075 Set B Q.No. 14

Prove that the algebraic sum of the moments of two intersecting forces about a point in their plane is equal to the moment of their resultant about the same point. [6]

➥ Please refer to Model Set II Q.No. 15

82. 2075 Set B Q.No. 14 OR

Define the moment of a force. Forces 1, 2, 4, 5 kg wts act along the sides of a square taken in order. Prove that their resultant is parallel to a diagonal and find where it cuts the side along which the first force acts. [6]

➥ Please refer to Model Set I Q.No. 14a OR

UNIT 14
DYNAMICS

A. MOTION IN A STRAIGHT LINE

2 MARKS QUESTIONS

1. 2063 Q.No. 5 a

A train moving with a velocity of 360 km/hr has the uniform acceleration 40 m/s². Obtain the distance covered by the train in $\frac{1}{2}$ minute. [2]

SOLUTION

Initial velocity (u) = 360 km/hr

$= \frac{360 \times 1,000}{60 \times 60} = 100\ m/s$

Acceleration (a) = 40 m/s²

Time (t) = $\frac{1}{2}$ minute = 30s

Distance covered (s) = ?

We have,

$s = ut + \frac{1}{2}at^2 = 100 \times 30 + \frac{1}{2} \times 40 \times 30^2$

$= 3,000 + 18,000 = 21,000\ m = 21\ km$

2. 2069 Old (Set B) Q.No. 6a

A car moving with a velocity of 15ms⁻¹ has a uniform acceleration of 2ms⁻². If it moves for 2.5 sec, find its final velocity. [2]

SOLUTION

Given, $u = 15\ ms^{-1}$ $a = 2\ ms^{-2}$

$t = 2.5\ s$ $v = ?$

We have,

$v = u + at = 15 + 2 \times 2.5 = 20\ ms^{-1}$

Required final velocity is 20 ms⁻¹.

3. 2072 Supp Q.No. 12b

A cyclist travelling with a velocity of 72 km/hr accelerates at the rate of 4m/s² until it describes a distance of 48 m. Find the time taken. [2]

SOLUTION

Here, $u = 72\ km/hr = \frac{72 \times 1,000}{60 \times 60} = 20\ m/s$

$a = 4\ m/s^2$

$s = 48\ m$

$t = ?$

We have,

$S = ut + \frac{1}{2}at^2$

or, $48 = 20 \cdot t + \frac{1}{2} \cdot 4 \cdot t^2$

or, $48 = 20t + 2t^2$

or, $t^2 + 10t - 24 = 0$

or, $(t + 12)(t - 2) = 0$

Either t = –12 (not possible)

or t = 2

∴ t = 2 s

4. 2073 Set D Q.No. 12c

An aeroplane land on the runway with a velocity of 108 km/hr. If then its velocity slows down at the rate of 25m/s²; find the distance covered by the aeroplane before coming to rest. [2]

SOLUTION

Here, $u = 108\ km/hr = \frac{108 \times 1000}{60 \times 60}\ m/s = 30\ m/s$

$v = 0\ m/s$

$a = -25\ m/s^2$

$s = ?$

We have,

$v^2 = u^2 + 2as$

or, $0^2 = 30^2 + 2 \cdot (-25) \cdot s$

or, $50s = 900$

∴ $s = \frac{900}{50} = 18\ m$

∴ Required distance covered = 18 m.

5. 2075 Set C Q.No. 12c

A car moving with a velocity of 20 ms⁻¹ has a uniform acceleration of 2 ms⁻². If it moves for 2.5 sec, find the final velocity. [2]

SOLUTION

Given,

Initial velocity (u) = 20ms⁻¹

Acceleration (a) = 2ms⁻²

Time taken (t) = 2.5 sec

Final velocity (v) = ?

We have,

$v = u + at$

$= 20 + 2 \times 2.5$

$= 25\ m^{-1}$

4 MARKS QUESTIONS

6. 2057 Q.No. 13 b

If a, b, c be the space described by a particle during the pᵗʰ, qᵗʰ and rᵗʰ seconds of its motion respectively, prove that:

a(q – r) + b(r – p) + c(p – q) = 0. [4]

SOLUTION

Let u and f be the initial velocity and the uniform acceleration of the particle respectively. By question, we have

$s_p = u + \left(\frac{2p-1}{2}\right) f$

or, $a = u + \left(\frac{2p-1}{2}\right) f$ ...(i)

Similarly,

$b = u + \left(\frac{2q-1}{2}\right) f$ ...(ii)

$c = u + \left(\frac{2r-1}{2}\right) f$ ...(iii)

Now,

L.H.S. $= a(q-r) + b(r-p) + c(p-q)$

$= \left\{u + \frac{(2p-1)}{2}f\right\}(q-r) + \left\{u + \frac{(2q-1)}{2}f\right\}(r-p) + \left\{u + \frac{(2r-1)}{2}f\right\}(p-q)$ [using (i), (ii) and (iii)]

$= u(q-r) + \frac{1}{2}(2p-1)(q-r)f + u(r-p) + \frac{1}{2}(2q-1)(r-p)f + u(p-q) + \frac{1}{2}(2r-1)(p-q)f$

$= u(q-r+r-p+p-q) + \frac{1}{2}f\{(2p-1)(q-r) + (2q-1)(r-p) + (2r-1)(p-q)\}$

$= 0 + \frac{1}{2}f(2pq - 2pr - q + r + 2qr - 2pq - r + p + 2pr - 2qr - p + q)$

$= \frac{1}{2}f \times 0 = 0 =$ R.H.S.

7. 2059 Q.No. 14 a

Prove that for a particle moving with uniform acceleration f in a straight line $f = \dfrac{2\left(\frac{s'}{t'} - \frac{s}{t}\right)}{t+t'}$ where s is the space described in t secs and s' during the next t' secs. [4]

SOLUTION

Let u be the initial velocity of the particle. Then

$s = ut + \frac{1}{2}at^2$

$\frac{s}{t} = u + \frac{1}{2}at$ ...(i)

Again, let v be the velocity at the end of time t then v = u + at which is the initial velocity for the second part of the journey. So, for second part of journey, we have

$s' = (u + at)t' + \frac{1}{2}at'^2$

or, $\frac{s'}{t'} = u + at + \frac{1}{2}at'$ ...(ii)

Subtracting (i) from (ii)

$\frac{s'}{t'} - \frac{s}{t} = \left(u + at + \frac{1}{2}at'\right) - \left(u + \frac{1}{2}at\right)$

or, $\frac{s'}{t'} - \frac{s}{t} = \frac{1}{2}at + \frac{1}{2}at' = \frac{1}{2}a(t + t')$

$\therefore a = \dfrac{2\left(\frac{s'}{t'} - \frac{s}{t}\right)}{t+t'}$

8. 2060 Q.No. 13 b

A body moves for 3 seconds with a constant acceleration during which it describes 24.30 metres, the acceleration then ceases and during the next 3 seconds, it describes 21.60 metres. Find the initial velocity and the acceleration. [4]

SOLUTION

Let u and a be the initial velocity and uniform acceleration respectively. For first 3s, s = 24.30m

We have,

$s = ut + \frac{1}{2}at^2$

or, $24.30 = u \times 3 + \frac{1}{2}a \times 3^2$

or, $6u + 9a = 48.60$

$\therefore 2u + 3a = 16.20$ ...(i)

Let v be the velocity at the end of 3s.

$v = u + at = u + a \times 3 = u + 3a$

which is the initial velocity for next 3 s.

For next 3s, t = 3s, s = 21.6m, a = 0

We have,

$s = ut + \frac{1}{2}at^2$

or, $21.6 = (u + 3a)\times 3 + \frac{1}{2} \times 0 \times 3^2$

or, $u + 3a = 7.2$ ...(ii)

Solving (i) and (ii), we get

$u = 9ms^{-1}$ and $a = -0.6\ ms^{-2}$

$\therefore$ Initial velocity (u) = $9ms^{-1}$

Acceleration (a) = $-0.6\ ms^{-2}$

9. 2062 Q.No. 13 b

A railway train goes from one station to another moving during the first part of the journey with uniform acceleration f, when steam is shut off and the breaks are applied, it moves with retardation f^1. If 'a' be the distance between the stations, show that the time the train takes is: $\sqrt{\frac{2a(f+f^1)}{ff^1}}$ [4]

SOLUTION

Let the railway train goes from A to C. At the time of journey, the velocity at B is maximum and the velocity at A and C are zero. Let t_1 be the time taken from A to B and t_2 be the time taken from B to C. Then,

AB + BC = a

and $t_1 + t_2 = t$ (suppose)

Also let AB = s_1 and BC = s_2 and v be the velocity at B.

For AB

u = 0

We have,

$v = u + at$

or, $v = 0 + ft_1$

$\therefore t_1 = \frac{v}{f}$ ...(i)

Again,

$v^2 = u^2 + 2as$

or, $v^2 = 0^2 + 2fs_1$

$\therefore s_1 = \frac{v^2}{2f}$ ...(ii)

For BC

$v = u + at$

or, $0 = b - f't_2$

$\therefore t_2 = \frac{v}{f'}$ ...(iii)

and $v^2 = u^2 + 2as$

or, $0 = v^2 - 2f's_2$

$\therefore s_2 = \frac{v^2}{2f'}$ ...(iv)

Adding (i) and (ii),

$t_1 + t_2 = \frac{v}{f} + \frac{v}{f'}$

or, $t = v\left(\frac{1}{f} + \frac{1}{f'}\right)$ ($\because t_1 + t_2 = t$)

or, $v = \dfrac{t}{\frac{1}{f} + \frac{1}{f'}}$ ...(v)

Again, adding (ii) and (iv)

$s_1 + s_2 = \frac{v^2}{2f} + \frac{v^2}{2f'}$

or, $a = \frac{v^2}{2}\left(\frac{1}{f} + \frac{1}{f'}\right)$

$= \frac{1}{2}\left\{\dfrac{t}{\frac{1}{f} + \frac{1}{f'}}\right\}^2\left(\frac{1}{f} + \frac{1}{f'}\right)$ [using (v)]

$= \dfrac{t^2}{2\left(\frac{1}{f} + \frac{1}{f'}\right)}$

or, $t^2 = 2s\left(\frac{1}{f} + \frac{1}{f'}\right) = \frac{2s(f+f')}{ff'}$

$\therefore t = \sqrt{\frac{2s(f+f')}{ff'}}$

10. 2064 Q.No. 13 b

If a, b, c, are the spaces described by the particles during the p^{th}, q^{th}, r^{th} seconds of its motion respectively, prove that: a(q − r) + b(r − p) + c(p − q) = 0. [4]

➥ Please refer to 2057 Q.No. 13 b

11. 2064 Q.No. 13 b/ 2065 Q.No 13 b

Prove that for a particle moving with uniform acceleration a in a straight line $a = \dfrac{2\left(\frac{s^1}{t^1} - \frac{s}{t}\right)}{t+t^1}$, where s is the space described in t seconds and s^1 during the next t^1 seconds. [4]

➥ Please refer to 2059 Q.No. 14 a

12. 2066 C Q.No. 13 b

If a, b, c be the space described by a particle during the p^{th}, q^{th} and r^{th} seconds of its motion respectively, prove that:

a (q−r) + b (r −p) + c (p − q) = 0. [4]

➥ Please refer to 2057 Q.No. 13 b

13. 2068 Q.No. 13 b

If a, b, c be the spaces described by a particle during the p^{th}, q^{th} and r^{th} second of its motion respectively, prove that:

a(q − r) + b(r − p) + c(p − q) = 0 [4]

➥ Please refer to 2057 Q.No. 13 b

14. 2069 (Set A) Old Q.No. 13b

Prove that for a particle moving with uniform acceleration 'a' in a straight line is $a = \dfrac{2\left(\frac{s'}{t'} - \frac{s}{t}\right)}{t+t'}$ where s is the space described in t sec. and s' during the next t' seconds. [4]

➥ Please refer to 2059 Q.No. 14 a

15. 2070 (Old) Q.No. 13 b

A body moves along a straight line with uniform acceleration. The body covers a distance of 18 m in the first three seconds and 22 m in the next 5 seconds. Find the velocity at the end of 10 seconds and the distance covered in 10^{th} second. [4]

SOLUTION

Let u be the initial velocity and a be the uniform acceleration of the body. For the first 3 sec, s = 18 m, t = 3s.

We have,

$s = ut + \frac{1}{2}at^2$

or, $18 = u \cdot 3 + \frac{1}{2} \cdot a \cdot 3^2$

$\therefore 2u + 3a = 12$ ...(i)

Let v be the velocity at the end of 3s. Then,

$v = u + at = u + 3a$

This will be the initial velocity for next 5s. For next 5s, given s = 22m.

We have,

$s = ut + \frac{1}{2}at^2$

or, $22 = (u + 3a) \cdot 5 + \frac{1}{2} \cdot a \cdot 5^2$

or, $10u + 55a = 44$ ...(ii)

Solving (i) and (ii), we have

$u = \frac{33}{5}$ m/s and $a = \frac{-2}{5}$ m/s²

Velocity at the end of 10s

$u = \frac{33}{5}$ m/s, $a = \frac{-2}{5}$ m/s², t = 10s,

v = ?

We have,

$v = u + at = \frac{33}{5} + \left(\frac{-2}{5}\right) \times 10 = 2.6$ m/s

Distance covered in 10^{th} second

$u = \frac{33}{5}$ m/s, $a = \frac{-2}{5}$ m/s², $t = 10$s,

$S_t = ?$

We have,

$S_t = u + \frac{2t-1}{2}a$

$S_{10} = \frac{33}{5} + \left(\frac{2\times 10-1}{2}\right)\left(\frac{-2}{5}\right) = \frac{33}{5} - \frac{19}{5} = \frac{14}{5}$

$= 2.8$ m

16. 2070 Set D Q.No. 13 b

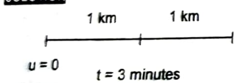
If a, b, c be the spaces described by a particle during the p^{th}, q^{th}, r^{th} seconds of its motion respectively, prove that:
$a(q - r) + b(r - p) + c(p - q) = 0$ **[4]**

➥ Please refer to 2057 Q.No. 13 b

17. 2075 Set B Q.No. 13b

A car starting from rest, moves with uniform acceleration and describes the first kilometer in 3 minutes. If it now moves with uniform velocity, how long will it take to describe another kilometer? **[4]**

SOLUTION



First part

$u = 0$

$s = 1$ km $= 1000$ m

$t = 3$ minutes

$= 3 \times 60$ seconds $= 180$ seconds

We have,

$S = ut + \frac{1}{2}at^2$

or, $1000 = 0 \times t + \frac{1}{2} \times a \times 180^2$

or, $1000 = 16200a$

∴ $a = \frac{1000}{16200} = \frac{5}{81}$

Again,

$V = u + at$

$= 0 + \frac{5}{81} \times 180$

$= \frac{100}{9}$

Second part

$u = \frac{100}{9}$ m, sec

$s = 1$ km $= 1000$ m

$t = ?$

We have,

$u = \frac{s}{t}$

or, $t = \frac{s}{u}$

∴ $t = \frac{1000}{\frac{100}{9}} = \frac{1000 \times 9}{100}$

$= 90$ seconds $= 1$ minute 30 seconds

6 MARKS QUESTIONS

18. 2070 Set C Q.No. 14

A railway train goes from one station to another moving during the first part of the journey with uniform acceleration a; when steam is shut off and the brakes are applied, it moves with retardation a^1. If s be the distance between the stations, show that the time, the train takes is $\sqrt{\frac{2s(a + a^1)}{aa^1}}$. **[6]**

➥ Please refer to 2062 Q.No. 13 b

19. 2070 Supp. Q.No. 15

A body starting with initial velocity of 15m/sec, moves with a uniform acceleration of 5m/sec².

i. What is the velocity after 10 sec?

ii. How far will it in 10 sec?

iii. What will be its velocity when it has traveled 10m?

iv. What will be the distance moved in the 10^{th} second? **[6]**

SOLUTION

Given, $u = 15$ m/s $a = 5$ m/s²

i. $t = 10$ sec

$v = ?$

We have,

$v = u + at = 15 + 5 \times 10 = 65$ m/s

ii. $t = 10$ s

$s = ?$

We have,

$s = ut + \frac{1}{2}at^2 = 15 \times 10 + \frac{1}{2} \times 5 \times 10^2 = 400$ m

iii. $s = 10$ m

$v = ?$

We have,

$v^2 = u^2 + 2as$

$v^2 = 15^2 + 2 \times 5 \times 10$

or, $v^2 = 325$

$v = \sqrt{325} = 5\sqrt{13}$ m/s.

iv. Distance travelled in 10^{th} second $(s_{10}) = ?$

$t = 10$ s

We have,

$s_t = u + \frac{2t-1}{2}a$

$s_{10} = 15 + \frac{2 \times 10 - 1}{2} \times 5 = 62.5$ m

20. 2071 Set C Q.No. 14

If a, b, c be the spaces described by a particle during the p^{th}, q^{th}, r^{th} seconds of its motion respectively, prove that:
$a(q - r) + b(r - p) + c(p - q) = 0$ **[6]**

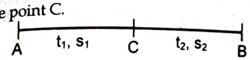
➥ Please refer to 2057 Q.No. 13 b

21. 2071 Supp. Q.No. 15

A bus starts from station A and stops at station B. The velocity increases uniformly till it reaches maximum velocity v and then decrease uniformly, show that the time taken by the bus to run from A to B is $\frac{2x}{v}$ where x is the distance between the two stations. **[6]**

SOLUTION

Let the bus reaches its maximum velocity at the point C.



Now, from A to C

Let, $t = t_1$, $s = s_1$

We have,

$v = u + at$

$v = 0 + at_1$

$t_1 = \frac{v}{a}$ ...(i)

And,

$v^2 = u^2 + 2as$

or, $v^2 = 0 + 2as_1$

∴ $s_1 = \frac{v^2}{2a}$ ...(ii)

Again, from C to B

Let $t = t_2$, $s = s_2$

We have,

$v = u + at$

$0 = v + (-a)t_2$

$t_2 = \frac{v}{a}$ ...(iii)

And,

$v^2 = u^2 + 2as$

or, $0 = v^2 + 2(-a)s_2$

∴ $s_2 = \frac{v^2}{2a}$ ...(iv)

Adding (ii) and (iv)

$s_1 + s_2 = \frac{v^2}{2a} + \frac{v^2}{2a}$

$x = \frac{2v^2}{2a} = \frac{v^2}{a}$ (∵ total distance $= x$)

∴ $a = \frac{v^2}{x}$

Again from (i) and (iii)

$t_1 + t_2 = \frac{v}{a} + \frac{v}{a}$

∴ Total time taken $= \frac{2v}{a} = \frac{2v}{v^2/x}$ $\left[∵ a = \frac{v^2}{x}\right]$

$= \frac{2x}{v}$

22. 2074 Set A Q.No. 15

A bus starts from station A and stops at station B. The Velocity increases uniformly till it reaches maximum velocity v and then decreases uniformly. Show that the time taken by the bus to run from A to B is $\frac{2x}{v}$ where x is the distance between the two stations. **[6]**

➥ Please refer to 2071 Supp Q.No. 15

23. 2075 Set A Q.No. 14

A point moving with uniform acceleration describes in the last second of its motion $\frac{9}{25}$ of the whole distance. If it started from rest, how long was it in motion and through what distance did it move, if it described 15 cms in the first second? **[6]**

SOLUTION

Let s be the total distance and t be the time taken.

By question, distance covered in last second

$= \frac{9}{25} \times s = \frac{9s}{25}$ ... (1)

If a be the uniform acceleration and u be the initial velocity then $u = 0$.

Distance covered in last second

$= u + \frac{2t-1}{2}a$ ... (2)

and $s = ut + \frac{1}{2}t^2$ ... (3)

Substituting (2) & (3) in (1), we get,

$u + \frac{(2t-1)a}{2} = \frac{9}{25}\left[ut + \frac{1}{2}at^2\right]$

or, $0 + \left(\frac{2t-1}{2}\right)a = \frac{9}{25}\left[0.t + \frac{1}{2}at^2\right]$

or, $\left(\frac{2t-1}{2}\right)a = \frac{9}{50}at^2$

or, $50t - 25 = 9t^2$

or, $9t^2 - 50t + 25 = 0$

or, $t = \frac{-(-50) \pm \sqrt{(-50)^2 - 4.9.25}}{2.9}$

$= \frac{50 \pm 40}{18}$

Taking +ve sing

$t = \frac{50 + 40}{18}$

$= 5$ sec

Again, taking -ve sing,

$t = \frac{50 - 40}{18}$

$= \frac{10}{18} = \frac{5}{9} < 1$ (not possible)

Again, the point describes 15 cm in the first second,

$\therefore\ s = ut + \frac{1}{2}at^2$

or, $15 = 0 \times t + \frac{1}{2} \times a \times 1^2$

or, $15 = \frac{a}{2}$

$\therefore\ a = 30\ cm/sec^2$

Required distance,

$s = ut + \frac{1}{2}at^2$

$= 0 \times 5 + \frac{1}{2} \times 30 \times 5^2$

$= 375$ cms

24. 2075 Set C Q.No. 14

A point moving in a straight line with uniform acceleration describes a and b metres in successive intervals at time t_1 and t_2 seconds. Prove that the acceleration is $\frac{2(bt_1 - at_2)}{t_1t_2(t_1 + t_2)}$. [6]

SOLUTION

Let u be the initial velocity and f be the uniform acceleration of a point moving in a straight line.

Then, $a = ut_1 + \frac{1}{2}f\,t_1^2$

or, $\frac{a}{t_1} = u + \frac{1}{2}f\,t_1$ ... (i)

The velocity at the end of time t_1 is $u + ft_1$

$(\because v = u + at)$

This will be the initial velocity for the next t_2 sec.

Now,

$b = (u + ft_1).t_2 + \frac{1}{2}f\,t_2^2$

or, $\frac{b}{t_2} = u + ft_1 + \frac{1}{2}f\,t_2$ ... (ii)

Subtracting (i) from (ii), we have

$\frac{b}{t_2} - \frac{a}{t_1} = ft_1 - \frac{1}{2}ft_1 + \frac{1}{2}ft_2$

or, $\frac{bt_1 - at_2}{t_1t_2} = \frac{1}{2}ft_1 + \frac{1}{2}f\,t_2$

or, $\frac{bt_1 - at_2}{t_1t_2} = \frac{f}{2}(t_1 + t_2)$

$\therefore\ f = \frac{2(bt_1 - at_2)}{t_1t_2(t_1 + t_2)}$

B. MOTION UNDER GRAVITY

2 MARKS QUESTIONS

25. 2058 Q.No. 5 b

A body is projected vertically with a velocity of 9.8 ms^{-1}, how long it takes to return to the point of projection? (g = 9.8 ms^{-2}). [2]

SOLUTION

Let the body take t secs to return to the point of projection. As the particle return to the point of projection, total displacement (h) = 0 in t secs.

Here, u = 9.8 ms^{-1}, g = 9.8 ms^{-2}

We have,

$h = ut - \frac{1}{2}gt^2$

or, $0 = 9.8t - \frac{1}{2} \times 9.8t^2$

or, $4.9t^2 - 9.8t = 0$

or, $4.9t(t - 2) = 0$

Either $4.9t = 0 \Rightarrow t = 0$

which gives initial time (not required)

or $t = 2$

$\therefore\ t = 2sec$

26. 2062 Q.No. 5 b

A stone is projected vertically upwards from the foot of the tower with a velocity just sufficient to carry it to 78.4 m. Find the velocity of the stone with which it is projected. (g = 9.8 m/s^2). [2]

SOLUTION

Initial velocity (u) = ?

v = 0 m/s

h = 78.4 m

We have,

$v^2 = u^2 - 2gh$

or, $0 = u^2 - 2 \times 9.8 \times 78.4$

or, $u^2 = 1536.64$

$\therefore\ u = \sqrt{1536.64} = 39.2\ m/s$

27. 2064 Q.No. 5 a

A ball thrown up vertically returns to the thrower after 6 seconds. Find the velocity with which it was thrown up. (g) = 10 m/s^2). [2]

SOLUTION

Since time of ascent = time of fall, so time to reach the maximum height $= \frac{6}{2} = 3s$.

Let u be the initial velocity.

v = 0

We have,

$v = u - gt$

or, $0 = u - 10 \times 3$

$\therefore\ u = 30\ m/s$

28. 2066 Q.No. 5 b

A stone is dropped from a balloon at a height 116.4 m above the ground and it reaches the ground in 6 sec. Find the velocity with which the balloon was rising. [2]

SOLUTION

Let u be the velocity of the balloon when the stone was dropped. At the instant of dropping a stone from the balloon, the velocity of the balloon is equal to the velocity of the stone. Taking upward direction positive,

$-h = ut - \frac{1}{2}at^2$

or, $-116.4 = u \times 6 - \frac{1}{2} \times 9.8 \times 6^2$

or, $-116.4 = 6u - 176.4$

or, $6u = 176.4 - 116.4$

or, $6u = 60$

or, $u = \frac{60}{6} = 10$

$\therefore\ u = 10\ m/s$

29. 2068 Q.No. 6b

A stone is projected vertically upwards from the foot of the tower with a velocity just sufficient to carry it to 78.4m. Find the velocity of the stone with which it is projected. (g = 9.8m/s^2) [2]

Please refer to 2062 Q.No. 5 b

30. 2069 (Set A) Q.No. 12c

A body is projected vertically upwards from the foot of the tower with a velocity just sufficient to carry it to 78.4m. Find the velocity of the stone with which it is projected. (g = 9.8 m/s^2) [2]

Please refer to 2062 Q.No. 5 b

31. 2069 (Set A) Old Q.No. 6b

A ball thrown up vertically upwards returns to the thrower after 6 seconds. Find its position after 4 sec. (g = 10 m/s^2) [2]

SOLUTION

Total time taken = 6s

So, time of ascent = time of decent $= \frac{6}{2} = 3s$

When the ball is thrown up,

v = 0, t= 3s, h = ?

We have,

$v = u - gt$

or, $0 = u - 10 \times 3$

$\therefore\ u = 30\ m/s$

Again, we have

$v^2 = u^2 - 2gh$

or, $0 = 30^2 - 2 \times 10 \times h$

$\therefore\ h = 45$ m

Let H be the position of the ball after 4s. So, t = 4s.

We have,

$H = ut - \frac{1}{2}gt^2 = 30 \times 4 - \frac{1}{2} \times 10 \times 4^2 = 40$ m

$\therefore$ In 4 s, the ball will be (45 – 40)m i.e. 5 m below the highest point.

32. 2069 (Set B) Q.No. 12c

A ball thrown up vertically return to the thrower after 6 secs. Find the velocity with which it was thrown up. [2]

Please refer to 2064 Q.No. 5 a

33. 2069 Old (Set B) Q.No. 6b

A ball is thrown vertically upward at a speed of 4ms^{-1}. Find the maximum height reached and the time taken to attain this height. [2]

SOLUTION

Given, u = 4ms^{-1} v = 0

h = ? t = ?

We have,

$v = u - gt$

or, $0 = 4 - 10 \times t$

$\therefore\ t = \frac{2}{5}s$

Again, we have

$v^2 = u^2 - 2gh$

or, $0 = 4^2 - 2 \times 10 \times h$

or, $20h = 16$

$\therefore\ h = \frac{16}{20} = \frac{4}{5}$ m

34. 2070 Set D Q.No. 12 c

A ball is projected vertically upwards with a velocity of 40 m/s. Find its velocity and position at the end of 3s. (g = 10 m/s^2). [2]

SOLUTION

Given, u = 40 m/s a = g = 10 m/s^2, t = 3s

v = ? h = ?

We have,

$v = u - gt = 40 - 10 \times 3 = 10\ m/s$

Again, we have

$h = ut - \frac{1}{2}gt^2$

or, $h = 40 \times 3 - \frac{1}{2} \times 10 \times 3^2$

$= 75$

$\therefore$ h= 75m

35. 2071 Set D Q.No. 12 c

A ball is thrown vertically upwards with a velocity of 30m/s. Find the time taken by the ball to reach the ground again. (g = 10m/s^2) [2]

SOLUTION

Given, u = 30 m/s

a = g = 10 m/s^2

Let the time taken to reach the maximum height be t.

At the maximum height, v= 0

We have,

$v = u - gt$

or, $0 = 30 - 10 \times t$

t = 3s

Since, time of fall = time of ascent

Here, time of fall = 3s

So total time = 2 × 3s = 6s

36. 2072 Set C Q.No. 12 c

A ball is thrown vertically upwards at a rate of 40 ms^{-1}. Find the time taken to attain the maximum height. (g = 10 ms^{-2}) [2]

SOLUTION

Here, initial velocity (u) = 40ms^{-1}

Acceleration = a = g = 10m^{-2}

Let the time taken to reach the maximum height be t

Here, v = 0

We have,

$v = u - gt$

or, $0 = 40 - 10t$

$\therefore\ t = 4s$

37. 2074 Set B Q.No. 12c

If a ball is projected vertically upward at a rate of $40ms^{-1}$, find the time taken to attain the maximum height. ($g = 10\ sm^{-2}$) [2]

➥ Please refer to 2072 Set C Q. No. 12 c.

38. 2075 Set B Q.No. 12c

A body falls from rest from the top of a tower and after 5 sec it reaches the ground. Find the striking velocity of the body and height of the tower. [$g = 9.8\ m/s^2$] [2]

SOLUTION

Here,
Initial velocity (v) = 0 m/s
Time taken (t) = 5 sec.
$g = 9.8\ m/s^2$
Striking velocity (v) = ?
Height of the tower (h) = ?
We have, for falling body,

$v = u + gt$
$= 0 + 9.8 \times 5 = 49$ m/sec.

And,

$h = ut + \frac{1}{2}gt^2$
$= 0 \times 5 + \frac{1}{2} \times 9.8 \times 5^2$
$= 122.5m$

4 MARKS QUESTIONS

39. 2058 Q.No. 13 b

A body falls from rest from the top of a tower and during the last second it falls $\frac{16^{th}}{25}$ of the whole height. Find the height of the tower ($g = 10ms^{-2}$). [4]

SOLUTION

Let h be the height and t be the time taken to descend the height h. Then,

$h = \frac{1}{2}gt^2$ [∵ u = 0]
$= \frac{1}{2} \times 10 \times t^2 = 5t^2$ ...(i)

Again,

$h_t = \left(\frac{2t-1}{2}\right)g = \left(\frac{2t-1}{2}\right)10 = 5\ (2t - 1)$ [∵ u = 0]

By given, $h_t = \frac{16}{25}h$

or, $5(2t - 1) = \frac{16}{25} \cdot 5t^2$
or, $16t^2 = 50t - 25$
or, $16t^2 - 50t + 25 = 0$
or, $16t^2 - 40t - 10t + 25 = 0$
or, $8t\ (2t - 5) - 5(2t - 5) = 0$
or, $(2t - 5)\ (8t - 5) = 0$
∴ $t = \frac{5}{2}$ or $\frac{5}{8}$

As time of falling is greater than 1s, so t cannot be less than 1s. So t cannot be equal to $\frac{5}{8}$ s.

So, $t = \frac{5}{2}$ s

Now, putting the value of t in (i), we get

$h = 5\left(\frac{5}{2}\right)^2 = 31.25$ m

40. 2061 Q.No. 13 b

A body falls from rest from the top of a tower and during the last second it falls $\frac{16}{25}$ th of the whole height. Find the height of the tower. ($g = 10m/sec^2$) [4]

➥ Please refer to 2058 Q.No. 13 b

41. 2063 Q.No. 13 b

A stone is dropped from the top of a tower 200 m high and at the same time another is projected vertically upwards from the ground with a velocity of 50 m/s. Find where and when the two will meet? ($g = 9.8\ m/s^2$). [4]

SOLUTION

Let two stones meet at a height of h from the bottom or H from the top after t seconds.
So, h + H = 200. Also, given $u = 50\ ms^{-1}$. Then,

$h = ut - \frac{1}{2}gt^2$ ...(i)

and $H = 0 + \frac{1}{2}gt^2$ ...(ii)

Adding (i) and (ii), we have
h + H = ut
or, 200 = 50t
∴ $t = \frac{200}{50} = 4s$

Now, from (i)

$h = 50 \times 4 - \frac{1}{2} \times 9.8 \times 4^2$
$= 200 - 78.4$
$= 121.6$ m

Thus, the two stones meet at the height 121.6 m from the ground after 4s.

42. 2067 Q.No. 13b

A stone is dropped from the top of a tower 200 m high and at the same time another is projected vertically upwards from the ground with a velocity of $50\ ms^{-1}$. Find where and when the two will meet ($g = 9.8\ ms^{-2}$). [4]

➥ Please refer to 2063 Q.No. 13 b

43. 2071 Old Q.No. 14 a

A stone is dropped into a well and the sound of its striking the water is heard in $4\frac{2}{9}$ seconds. If the velocity of the sound is $352.8\ ms^{-1}$. Find the depth of the well. ($g = 9.8\ ms^{-2}$) [4]

SOLUTION

Let t be the time taken by stone and T be the time taken by sound.

Then, $t + T = 4\frac{2}{9} = \frac{38}{9}$ ...(i)

If h is the depth of well, then

$h = 0 \times t + \frac{1}{2}gt^2 = \frac{1}{2} \times 9.8t^2 = 4.9t^2$ ...(ii)

Again, $h = uT$ $\left(∵ u = \frac{h}{T}\right)$
$= 352.8T$ ...(iii)

From (ii) and (iii), we have
$352.8T = 4.9t^2$
∴ $T = \frac{4.9}{352.8}t^2 = \frac{1}{72}t^2$

Now, from (i)

$t + \frac{t^2}{72} = \frac{38}{9}$
or, $\frac{72t + t^2}{72} = \frac{38}{9}$
or, $t^2 + 72t - 304 = 0$
or, $t^2 + 76t - 4t - 304 = 10$
or, $(t + 76)\ (t - 4) = 0$
∴ $t = -76, 4$

Since t = –76 is not possible, so t = 4
∴ t = 4s

Now, putting the value of t in (ii), we have
$h = 4.9 \times 4^2 = 78.4m$

44. 2073 Set C Q.No. 13b

A body falls from rest from the top of a tower and during the last second it falls $\frac{16}{25}$ th of the whole height. Find the height of the tower. ($g = 10ms^{-2}$) [4]

➥ Please refer to 2058 Q.No. 13 b

45. 2073 Supp Q.No. 13b or

A stone is dropped form the top of a tower 200 m high and at the same time another is projected vertically upwards form the ground with a velocity of $50ms^{-1}$. Find where and when the two will meet? ($g = 9.8\ ms^{-2}$) [4]

➥ Please refer 2063 Q.No. 13b

46. 2074 Set B Q.No. 13b

A stone is dropped from the top of a tower 200m high and at the same time another is projected vertically upwards from the ground with a velocity of $50ms^{-1}$. Find where and when the two will meet? ($g = 9.8\ ms^{-2}$) [4]

➥ Please refer to 2063 Q.No. 13b

6 MARKS QUESTIONS

47. 2072 Set E Q.No. 14

A body is projected vertically upward with velocity u and t seconds afterwards another body is projected similarly with the same velocity. Show that they meet a height $\frac{4u^2 - g^2t^2}{8g}$ from the point of projection after $\left(\frac{u}{g} - \frac{t}{2}\right)$ secs from the instant of projection of the second body. [6]

SOLUTION

Let two bodies meet at a height h after t_1 secs from the instant of projection of the second body.
Then, for second body,

$h = ut_1 - \frac{1}{2}gt_1^2$ ...(i)

and for first body,

$h = u(t + t_1) - \frac{1}{2}g(t + t_1)^2$ ...(ii)

Equating (i) and (ii), we have

$ut_1 - \frac{1}{2}gt_1^2 = u(t + t_1) - \frac{1}{2}g(t + t_1)^2$

or, $ut_1 - \frac{1}{2}gt_1^2 = ut + ut_1 - \frac{1}{2}gt^2 - gtt_1 - \frac{1}{2}gt_1^2$

or, $gtt_1 = ut - \frac{1}{2}gt$

or, $gt_1 = u - \frac{gt}{2}$

or, $t_1 = \left(\frac{u}{g} - \frac{t}{2}\right)$ sec.

Putting the value of t_1 in (i), we have

$h = u\left(\frac{u}{g} - \frac{t}{2}\right) - \frac{1}{2}g\left(\frac{u}{g} - \frac{t}{2}\right)^2$
$= \frac{u^2}{g} - \frac{ut}{2} - \frac{1}{2}g\left(\frac{u^2}{g^2} - \frac{ut}{g} + \frac{t^2}{4}\right)$
$= \frac{u^2}{g} - \frac{ut}{2} - \frac{u^2}{2g} + \frac{ut}{2} - \frac{gt^2}{8}$
$= \frac{u^2}{2g} - \frac{gt^2}{8} = \frac{4u^2 - g^2t^2}{8g}$

Hence, $h = \frac{4u^2 - g^2t^2}{8g}$

and $t_1 = \left(\frac{u}{g} - \frac{t}{2}\right)$ s

48. 2074 Supp Q.No. 12c

A stone is projected vertically upwards from the foot of the tower with a velocity just sufficient to carry it to 78.4 m. Find the velocity of the stone with which it is projected. ($g = 9.8\ m/s^2$) [2]

➥ Please refer to 2062 Q.N. 5b

C. MOTION DOWN AN INCLINED PLANE

2 MARKS QUESTIONS

49. 2059 Q.No. 5 b

A particle slides down an inclined plane 30 m. long and acquires a velocity of $\sqrt{300\sqrt{3}}$ ms^{-1}. Find the inclination of the plane. (g = 10ms^{-2}). [2]

SOLUTION

Given, u = 0 m/s, $v = \sqrt{300\sqrt{3}}$ m/s

l = 30 m, g = 10 ms^{-2}

θ = ?

We have,

$v^2 = u^2 + 2(g \sin\theta)\, l$

$\left(\sqrt{300\sqrt{3}}\right)^2 = 0^2 + 2 \times 10 \times \sin\theta \times 30$

or, $300\sqrt{3} = 600 \sin\theta$

or, $\sin\theta = \frac{300\sqrt{3}}{600} = \frac{\sqrt{3}}{2} = \sin 60°$

∴ θ = 60°

50. 2065 Q.No 5 a

A particle slides down an inclined plane 20m long and acquires a velocity of $10\sqrt{2}$ m/sec. Find the inclination of the plane, (g = 10m/sec^2) [2]

SOLUTION

Given, u = 0 m/s l = 20 m

$v = 10\sqrt{2}$ m/s g = 10 m/s^2

θ = ?

We have,

$v^2 = u^2 + 2(g \sin\theta)\, l$

or, $(10\sqrt{2})^2 = 0^2 + 2 \times 10 \times \sin\theta \times 20$

or, $200 = 400 \sin\theta$

or, $\sin\theta = \frac{1}{2} = \sin 30°$

∴ θ = 30°

51. 2066 C Q.No. 5 a

A body slides down an inclined plane 39.24m. long and acquires a velocity of 19.6 m/sec. Find the inclination of the plane. (take g = 10m/sec^2) [2]

SOLUTION

Given, l = 39.24 u = 0 m/s

v = 19.6 m/s θ = ?

g = 10 m/s^2

We have,

$v^2 = u^2 + 2(g \sin\theta)\, l$

or, $(19.6)^2 = 0 + 2 \times 10 \times \sin\theta \times 39.24$

or, $384.16 = 784.8 \sin\theta$

or, $\sin\theta = \frac{384.16}{784.8} = \sin 29.3°$ (approx)

∴ θ = 29.3°

52. 2070 Set C Q.No. 12 c

A ball is projected up a smooth inclined plane with velocity 25 m/s. If the inclination of the plane to the horizon be 30°, find the velocity of the ball when it travels a distance of 22.5m [g=10m/s^2]. [2]

SOLUTION

Here,

Initial velocity (u) = 25 m/s

Inclination of the plane (θ) = 30°

Distance covered (l) = 22.5 m

Velocity of the ball (v) = ?

g = 10 m/s^2

We have,

$v^2 = u^2 - 2(g \sin\theta)\, l$

or, $v^2 = (25)^2 - 2 \times 10 \times \sin 30° \times 22.5$

or, $v^2 = 625 - 2 \times 10 \times \frac{1}{2} \times 22.5$

or, $v^2 = 400$

∴ v = 20 m/s

53. 2070 Supp. Q.No. 12 b

A particle, projected from the bottom of a smooth inclined plane with a velocity of 19.6 m/s, is just carried to the top in 4 sec; find the inclination of the plane to the horizon and also the length of the plane. [g = 9.8m/s^2]. [2]

Given, u = 19.6 m/s t = 4s

θ = ? l = ?

v = 0

Since the particle is projected up, so we have,

$v = u - g \sin\theta\, t$

or, $0 = 19.6 - 9.8 \times \sin\theta \times 4$

or, $-19.6 = -39.2 \sin\theta$

or, $\sin\theta = \frac{19.6}{39.2} = \frac{1}{2} = \sin 30°$

∴ θ = 30°

Again, we have

$v^2 = u^2 - 2(g \sin\theta)\, l$

or, $0 = (19.6)^2 - 2 \times 9.8 \times \sin 30° \times l$

or, $0 = 384.16 - 2 \times 9.8 \times \frac{1}{2} \times l$

or, $-384.16 = -9.8l$

or $l = \frac{384.16}{9.8}$

∴ $l = 39.2$ m

54. 2071 Set C Q.No. 12 c

A ball is thrown up an inclined plane with a velocity of 14.7 m/s. Where will the velocity of the ball be 4.9m/s? Assume that the inclination of the plane to the horizon is 30°. (g = 9.8m/s^2) [2]

SOLUTION

Here, u = 14.7 m/s v = 4.9 m/s

θ = 30° g = 9.8 m/s^2

t = ?

Since the ball is thrown up, we have,

$v^2 = u^2 - 2(g \sin\theta)\, l$

or, $(4.9)^2 = (14.7)^2 - 2 \times 9.8 \times \sin 30° \times l$

or, $24.01 = 216.09 - 2 \times 9.8 \times \frac{1}{2} \times l$

or, $24.01 - 216.09 = -9.8l$

or, $-192.08 = -9.8l$

or, $l = \frac{192.08}{9.8}$

∴ l = 19.6 m

55. 2071 Supp. Q.No. 12b

A particle slides from the rest $39.2\sqrt{3}$ m in 4 seconds down a smooth inclined plane. Calculate the angle of inclination of the plane [g = 9.8 m/s^2] [2]

SOLUTION

Since the particle slides from the rest, so u = 0

$l = 39.2\sqrt{3}$

t = 4s

g = 9.8 m/s^2

Angle of inclination (θ) = ?

We have,

$l = ut + \frac{1}{2}(g \sin\theta)\, t^2$

or, $39.2\sqrt{3} = 0 \times t + \frac{1}{2} \times 9.8 \times \sin\theta \times 4^2$

or, $39.2\sqrt{3} = 78.4 \sin\theta$

or, $\sin\theta = \frac{39.2\sqrt{3}}{78.4} = \frac{\sqrt{3}}{2} = \sin 60°$

∴ θ = 60°

56. 2072 Set D Q.No. 12c

A particle slides down a smooth inclined plane 10 m long and acquires a velocity $10\sqrt{2}$ ms^{-1}. Find the inclination of the plane. (g = 10ms^{-2}) [2]

SOLUTION

Here, final velocity (v) = $10\sqrt{2}$ m/s

Length of inclined plane (l) = 10 m

Initial velocity (u) = 0 g = 10 ms^{-2}

Inclination of the plane (θ) = ?

We have,

$v^2 = u^2 + 2(g \sin\theta)\, l$

or, $(10\sqrt{2})^2 = 0 + 2 \times 10 \times \sin\theta \times 10$

or, $200 = 200 \sin\theta$

or, $\sin\theta = 1 = \sin 90°$

∴ θ = 90°

57. 2072 Set E Q.No. 12b

A ball is projected up a smooth plane with velocity 25m/s. If the inclination of the plane to the horizon be 30°, find the velocity of the ball when it travels a distance of 22.5m. (g = 10 m/s^2) [2]

SOLUTION

Given, u = 25 m/s θ = 30

l = 22.5 m g = 10 m/s^2 v = ?

We have,

$v^2 = u^2 - 2(g \sin\theta)\, l$

or, $v^2 = 25^2 - 2 \times 10 \times \sin 30° \times 22.5$

or, $v^2 = 625 - 2 \times 10 \times \frac{1}{2} \times 22.5$

or, $v^2 = 400$

v = 20 m/s

58. 2073 Set C Q.No. 12b

A particle slides down a smooth inclined plane 10 m long and acquires a velocity of 10 $\sqrt{2}$ ms^{-1}. Find the inclination of the plane. (g = 10ms^{-2}) [2]

➥ Please refer to 2072 Set D Q.No. 12c

59. 2075 Set A Q.No. 12c

A body is projected up an inclined plane with a velocity of 25 m/s. If the inclination of the plane to the horizon be 30°, what length of the plane will it cover after 4 sec? (g = 10 m/s^2) [2]

SOLUTION

Here,

u = 25 m/s

θ = 30

l = 22.5 m

v = ?

g = 10 ms^{-2}

We have,

$v^2 = u^2 - 2g \sin\theta.\, l$

$= 25^2 - 2 \times 10 \times \sin 30° \times 22.5$

$= 625 - 2 \times 10 \times \frac{1}{2} \times 22.5$

$= 625 - 225 = 400$

∴ v = 20m/sec

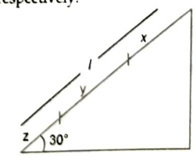
4 MARKS QUESTIONS

60. 2066 Q.No. 13 b

A particle slides down from rest from the top of a smooth plane of height 1962 cms and inclination 30° with the horizon. Divide the plane into three parts so that a particle at the top of the plane may describe each part in equal times. (g = 981cm/sec^2) [4]

SOLUTION

Let l be the length of inclined plane and t be the time taken for each part. Given h = 1,962 cms and θ = 30°. Also, u = 0 m/s. Let x, y, z be the length of 1st, 2nd and 3rd part respectively.



Now,

$\sin\theta = \frac{h}{l}$

or, $\sin 30° = \frac{1962}{l}$

or, $\frac{1}{2} = \frac{1962}{l}$

∴ l = 3924.

Again,

$l = ut + \frac{1}{2}(g \sin\theta\, t^2$

or, $3924 = 0(3t) + \frac{1}{2} \times 981 \times \sin 30° \times (3t)^2$

or, $3924 = \frac{1}{2} \times 981 \times \frac{1}{2} \times 9t^2$

or, $t^2 = \frac{3924 \times 4}{981 \times 9} = \frac{16}{9}$

$\therefore\ t = \frac{4}{3}$ s

Length of first part (x)

$= \frac{1}{2} g \sin\theta\, t^2$

$= \frac{1}{2} \times 981 \times \sin 30° \times \left(\frac{4}{3}\right)^2$

$= \frac{1}{2} \times 981 \times \frac{1}{2} \times \frac{16}{9}$

$= 436$ cms

Length of first two parts $(x + y)$

$= \frac{1}{2} g \sin\alpha\, (2t)^2$

$= \frac{1}{2} \times 981 \times \sin 30° \times 4 \times \left(\frac{4}{3}\right)^2$

$= 1744$ cms

$\therefore$ Length of second part (y)

$= 1744 - 436 = 1308$ cms

$\therefore$ Length of third part (z)

$= 3924 - (x + y)$

$= 3924 - 1744 =$ ~~1808~~ 2180 cms

6 MARKS QUESTIONS

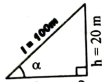
61. 2074 Set A Q.No. 15 OR

A body slides down a smooth plane whose length is 100m and height 20m. Find (i) the velocity of the body when it reaches the bottom of the plane, (ii) time taken by it to reach the bottom of the plane (iii) velocity of the body after 4 seconds. [g = 10m/s²] [6]

SOLUTION

Let α be the inclination of the plane. Then,

$\tan\alpha = \frac{20}{100} = \frac{1}{5}$



i. u = 0, v = ?

We have, $v^2 = u^2 + 2g\sin\alpha.l$

or, $v^2 = 0 + 2 \times 10 \times \frac{1}{5} \times 100 = 400$

$\therefore\ v = 20$ m/sec

ii. Let the required time be t sec.

We have, $v = u + g \sin\alpha.\ t$

$20 = 0 + 10 \times \frac{1}{5} \times t$

or, $20 = 2t$

$\therefore\ t = 10$ sec.

iii. Let v be the velocity of the body after 4 seconds

Then,

$v = u + g \sin\alpha.t$

or, $v = 0 + 10 \times \frac{1}{5} \times 4$

$\therefore\ v = 8$ m/sec.

UNIT 15

DYNAMICS (CONTINUED)

A. NEWTON'S LAW OF MOTION, IMPULSE

2 MARKS QUESTIONS

1. 2062 Q.No. 6 a

A constant force of 10N acting on an object reduces its velocity from 15ms⁻¹ to 5ms⁻¹ in 2 seconds. Find the mass of the object. [2]

SOLUTION

Given, u = 15 m/s, v = 5 m/s

t = 2 s, F = 10 N

m = ?

We know that

$v = u + at$

or, $5 = 15 + a \cdot 2$

or, $-10 = 2a$

or, $a = -5$

Retardation = 5 m/s²

We know that

F = Mass × Retardation

or, 10 = Mass × 5

$\therefore$ Mass $= \frac{10}{5} = 2$ kg.

2. 2064 Q.No. 6 a

Show that Newton's second law of motion gives the measurement of a force. [2]

SOLUTION

Second law of motion

The rate of change of momentum of a body is proportional to the impressed force and takes place in the direction in which the force acts.

Suppose a force F acts on a body of mass 'm' for time 't', and changes its velocity form u to v. The change in momentum of the body in time 't' is mv − mu.

The rate of change of momentum is $\frac{mv - mu}{t}$

By Newton's second law of motion,

$F \propto \frac{mv - mu}{t}$

or, $F \propto \frac{m(v-u)}{t}$

But $\frac{v-u}{t} = \frac{\text{change in velocity}}{\text{time}} =$ acceleration = a (say)

Therefore $F \propto ma$

or, F = kma, where k is a constant.

If we take m = 1 kg and a = 1 m/s², the numerical value of F = 1, if we take k = 1. Therefore

F = ma

This is the equation from which we define an absolute unit of force. The unit of force is the force which produces an acceleration of 1 m/s² when it acts on a mass of 1 kg.

3. 2065 Q.No 6 a

The pull of the earth on a body is 49 N. If the acceleration due to gravity is g = 9.8 m/sec². Find the mass of the body. [2]

SOLUTION

Given, F = 49 N

g = 9.8 m/s²

m = ?

We have,

F = mg

or, 49 = m × 9.8

or, $m = \frac{49}{9.8} = 5$ kg

$\therefore$ Mass(m) = 5 kg

4. 2066 C Q.No. 6 a

Find the velocity of a 4 kg shot that will just penetrate through a wall 16 cms thick, the resistance being 4 metric tonnes weight. [2]

SOLUTION

Resistance force = −4 metric tones wt.

= −4 × 1,000 × g N

Let 'a' be the retardation. Then,

F = −ma

or, −4 × 1,000 g = −4 · a

or, a = 9,800 m/s²

Let 'u' be the velocity of the shot which can just penetrate a wall of thickness 16 cm = 0.16 m. Then

$0 = u^2 - 2as$

or, $0 = u^2 - 2 \times 9800 \times 0.16$

or, $u^2 = 2 \times 9800 \times 0.16$

$\therefore\ u = 56$ m/s

5. 2066 Q.No. 6 a

A body of mass 1 kg is falling under gravity at the rate of 28 ms⁻¹. What uniform force will stop it in 0.1 second? (g = 9.8ms⁻²) [2]

SOLUTION

Given, m = 1kg

u = 28 m/s

t = 0.1 s

v = 0

F = ?

We have,

v = u − at

or, 0 = 28 − a × 0 · 1

or, $a = \frac{28}{0.1} = 280$

If 'F' be the force retarding the motion, then by Newton's second law of motion,

$F - mg = ma$

or, $F = mg + ma = 1 \times 9.8 + 1 \times 280 = 289.8$ N

$= \frac{289.8}{10}$ kg wt $= 28.98$ kg wt

$= 29$ kg wt (approx)

6. 2067 Q.No. 5b

A bullet fired into a target loses half its velocity after penetrating 6 cms. How much further will it penetrate? [2]

SOLUTION

Let a be the retardation. Then,

$\left(\frac{u}{2}\right)^2 = u^2 - 2a \cdot 6$

or, $\frac{u^2}{4} = u^2 - 12a$

or, $12a = \frac{3u^2}{4}$

or, $u^2 = 16a$ ...(i)

Let $(6 + x)$ cms be the thickness penetrated before coming to rest. Then,

$0 = u^2 - 2a(6 + x)$

or, $0 = 16a - 2a(6 + x)$

or, $2a(6 + x) = 16a$

or, $6 + x = 8$

∴ $x = 2$ cm

∴ It will penetrate 2 cm further.

7. 2067 Q.No. 6b

State Newton's second law of motion hence define a force. [2]

➥ Please refer to 2064 Q.No. 6a

8. 2068 Q.No. 5b

A constant force of 10N acting on an object reduces its velocity from 15m/s to 5m/s in 2 seconds. Find the mass of the object. [2]

➥ Please refer to 2062 Q.No. 6 a

9. 2069 (Set A) Old Q.No. 5b

A bullet of mass 2 kg is fired from a gun of mass 100 kg with a velocity 250 m/sec, find the recoil velocity of the gun. [2]

SOLUTION

We know that,

Mass of bullet × muzzle velocity = Mass of the gun × recoil velocity

or, $2 \times 250 = 100 \times$ recoil velocity

∴ Recoil velocity $= \frac{500}{100} = 5$ m/s.

10. 2069 Old (Set B) Q.No. 6c

A car of mass 1000kg is brought to rest by applying a braking forces of 2500N. Find the average retardation. [2]

SOLUTION

Given, $m = 1000$ kg

$F = 2500$ N

$a = ?$

We have,

$F = ma$

$2500 = 1000 \times a$

∴ $a = \frac{2500}{1000} = 2.5$ m/s²

11. 2070 (Old) Q.No. 6 b

A car of mass 1000 kg is brought to rest by applying a breaking force of 2500N. Find the average retardation. [2]

➥ Please refer to 2069 Old (Set B) Q.No. 6c

12. 2070 Set C Q.No. 12 b

A cart is pushed on a frictionless smooth plane with an average force of 20N for 5 seconds. If the cart with mass 50kg is at rest in the beginning find the velocity acquired by the cart. [2]

SOLUTION

Given, $F = 20$ N, $m = 50$ kg, $v = ?$, $t = 5$ s, $u = 0$ m/s

We have,

$F = \frac{m(v - u)}{t}$

or, $20 = \frac{50(v - 0)}{5}$

or, $50v = 100$

∴ $v = 2$ m/s

13. 2070 Supp. Q.No. 12 c

A bullet of mass 25 gm moving 250 m/s penetrates into a tree trunk and is then brought to rest in 0.02 seconds. Find the distance of penetration of the tree-trunk. [2]

SOLUTION

Given, $m = 25$ gm $= \frac{25}{1,000}$ kg $= 0.025$ kg

$u = 0$ m/s

$v = 250$ m/s

$t = 0.02$ s

We have,

$F \times t = m(v - u)$

or, $F \times 0.02 = 0.025(250 - 0)$

or, $F = \frac{6.25}{0.02}$

∴ $F = 312.5$

Again, we have, $F = ma$

or, $312.5 = 0.025 \times a$

∴ $a = \frac{312.5}{0.025} = 12500$

Also, we have

$v^2 = u^2 + 2as$

or, $(250) = 0^2 + 2 \times 12500 \times s$

∴ $s = \frac{62500}{25000} = 2.5$ m

∴ The distance of penetration of the tree trunk is 2.5 m.

14. 2071 Set C Q.No. 12 b

A body of mass 50 kg is falling from a certain height is brought to rest after striking the ground with a speed of 5m/s. If the resistance force of the ground is 500N, find the duration of contact. [2]

SOLUTION

Mass of a body (m) = 50 kg

Initial velocity (u) = 5 m/s

Final velocity (v) = 0 m/s

Resistance force (F) = –500 N

Duration of contact (t) = ?

We have,

$F = \frac{m(v - u)}{t}$

or, $-500 = \frac{50(0 - 5)}{t}$

or, $-500t = -250$

∴ $t = 0.5$ sec

15. 2071 Supp. Q.No. 12c

A bullet of mass 25gm moving 250m/s penetrating into a tree trunk and is then brought to rest in 0.02 seconds. Find impulse of the force on the bullet. [2]

SOLUTION

Given, $m = 25$ gm $= \frac{25}{1,000}$ kg $= 0.025$ kg

$u = 0$ m/s

$v = 250$ m/s

Impulse of the force (I) = ?

We have,

Impulse $= m(v - u) = 0.025(250 - 0) = 6.25$ kg m/s

16. 2073 Supp Q.No. 12c

Find the mass of an object which on earth weights 98N. (g = 9.8ms⁻²) [2]

SOLUTION

Weight of object on earth (F) = 98N

$g = 9.8$ ms^{-2}

Mass (m) = ?

We have,

$F = mg$

or, $98 = m \times 9.8$

or, $m = \frac{98}{9.8} = 10$

$m = 10$ kg

Thus, the mass of object = 10kg.

17. 2074 Set A Q.No. 12c

A uniform force of 150N change the velocity of a body moving in a straight line from 300 to 350 meters per second in 2 minutes. Find the mass of the body. [2]

SOLUTION

Here, $F = 150$N, $u = 300$ m/sec,

$v = 350$ m/sec, $t = 2$ minutes

$= 2 \times 60$ sec $= 120$ sec

$m = ?$

We have, $F \times t = m(v - u)$

or, $150 \times 120 = m(350 - 300)$

or, $18000 = m \times 50$

∴ $m = \frac{18000}{50} = 3600$ kg.

18. 2075 Set A Q.No. 12b

A cart is pushed on a frictionless smooth plane with an average force of 20N for 5 seconds. If the cart with mass 50 kg is at rest in the beginning, find the velocity acquired by the cart. [2]

SOLUTION

Here,

$F = 20$N

$t = 5$ seconds.

$m = 50$ kg

$u = 0$

$v = ?$

We have,

$F = \frac{m}{t}(v - u)$

or, $20 = \frac{50}{5}(v - 0)$

or, $20 = 10v$

∴ $v = 2$ m/sec

4 MARKS QUESTIONS

19. 2059 Q.No. 14 b

A body of mass 1 kg is falling under gravity at the rate of 28 ms⁻¹. What is the uniform force that will stop it in (i) 0.1 sec (ii) 20 cm (g = 10 ms⁻²). Instead of falling under gravity if the body is moving at the rate of 28 ms⁻¹ along a horizontal line, what will be the force required in above two cases? [4]

SOLUTION

Given, $m = 1$ kg, $t = 0.1$ s, $u = 28$ m/s, $v = 0$

We have,

$v = u + at$

or, $0 = 28 + a \times (0.1)$

or, $0.1a = -28$

or, $a = \frac{-28}{0.1} = -280$

Retardation = 280 m/s²

If F be the uniform force applied in upward direction to stop the body. Then,

$F - mg = ma$

or, $F = m(g + a) = 1(10 + 280) = 290$ N

(ii) $u = 28$ ms^{-1}, $s = 20$ cms $= 0.2$ m. $v = 0$, $a = ?$

We have,

$v^2 = u^2 + 2as$

or, $0 = (28)^2 + 2 \times a \times 0.2$

or, $a = \frac{-(28)^2}{0.4} = -1960.$

Retardation = 1960.

Again,

$F = m(g + a) = 1(10 + 1960) = 1970$ N

If the body is moving in a horizontal line, there is no component of mg.

(i) F = ma = 1 × 280 = 280 N

(ii) F = ma = 1 × 1960 = 1960 N

20. 2060 Q.No. 14 b OR

A shot whose mass is 40kg is discharged from a 700kg gun with a velocity of 140ms^{-1}. Find the constant force which acts on the gun would stop it after a recoil of 6.4m. **[4]**

SOLUTION

Momentum of the shot = mv = 40 × 140 = 5600

Momentum of the gun = MV = 700V

We have,

momentum of the shot = momentum of the gun

or, 5600 = 700V

or, V = 8 ...(i)

Let a be the retardation. Then,

$0 = V^2 - 2as$

or, $V^2 = 2as$

or, $8^2 = 2 \times a \times 6.4$ [using (i)]

or, 64 = 12.8a

or, $a = \frac{64}{12.8} = 5$

Now,

F = ma = 700 × 5 = 3,500 N

21. 2063 Q.No. 14 b OR

State Newton's laws of motion. Show that the Newton's second law of motion gives the measurement of a force. **[4]**

SOLUTION

1st Part

First law of motion: Every body continues in its state of rest or of uniform motion in a straight line unless compelled by some external forces to change that state.

Second law of motion: The rate of change of momentum of a body is proportional to the impressed force and takes place in the direction in which the force acts.

Third law of motion: To every action, there is an equal and opposite reaction.

2nd Part

Suppose a force F acts on a body of mass 'm' for time 't', and changes its velocity form u to v. The change in momentum of the body in time 't' is mv – mu.

The rate of change of momentum is $\frac{mv - mu}{t}$

By Newton's second law of motion,

$F \propto \frac{mv - mu}{t}$

or, $F \propto \frac{m(v - u)}{t}$

But $\frac{v - u}{t} = \frac{\text{change in velocity}}{\text{time}}$ = acceleration = a (say)

Therefore F ∝ ma

or, F = kma, where k is a constant.

If we take m = 1 kg and a = 1 m/s^2, the numerical value of F = 1, if we take k = 1. Therefore

F = ma

This is the equation from which we define an absolute unit of force. The unit of force is the force which produces an acceleration of 1 m/s^2 when it acts on a mass of 1 kg.

22. 2069 (Set A) Q.No. 13b

A mass of 5 kg falls 300 cm from rest and is then brought to rest by penetrating 30 cm into some sand, find the average thrust of the sand. **[4]**

SOLUTION

Suppose v is the velocity of the body when it falls 300 cm from rest. Then u = 0, h = 3m.

We have,

$v^2 = u^2 + 2gh$

or, $v^2 = 2 \times g \times 3$

∴ $v^2 = 6g$ ...(i)

The velocity given by (i) is reduced to 0 when the body goes to 30 cm = 0.3 m into the sand.

If a is the retardation, then

$0^2 = v^2 - 2 \times a \times 0.3$

∴ $a = \frac{v^2}{2 \times 0.3} = \frac{6g}{0.6} = 10g$

Let T be the average thrust of the sand on the body. When the body is penetrating into the sand, then the forces acting on the body are:

(i) the weight 5 kg of the body acting downwards.

(ii) a force T N of the sand acting upward.

Resultant upward thrust = (T – 5g)

By Newton's second law of motion, we have

T – mg = ma

or, T – 5g = 5 × 10g

∴ T = 55 kg-wt

23. 2069 (Set B) Q.No. 13b

State Newton's laws of motion. Show that Newton's second law gives the measurement of a force. **[4]**

➥ Please refer to 2063 Q.No. 14 b OR

24. 2069 Old (Set B) Q.No. 14a

A balloon is raising with an acceleration f. Prove that the fraction of the weight of the balloon which be emptied out of the balloon in order to double the acceleration is $\frac{f}{g + 2f}$. **[4]**

SOLUTION

If 'm' be the mass of the balloon, then its weight = mg and R be the upward lifting force, then

R – mg = mf

∴ R = mg + mf ...(i)

If x be the mass to be taken out from the balloon, so that it may now move up with acceleration 2f. Then,

$R - (m - x)g = (m - x)2f$

or, $mg + mf - mg + xg = 2mf - 2xf$ [using (i)]

or, $xg + 2xf = 2mf - mf$

or, $x(g + 2f) = mf$

∴ $\frac{x}{m} = \frac{f}{g + 2f}$

25. 2071 Set D Q.No. 13 b

A gun of mass 400 kg fires a shot of mass 3kg with a velocity of 200m/s, find the constant force which acting on the gun would stop it after a recoil of 2.5 meters. **[4]**

SOLUTION

Momentum of the shot = mv = 3 × 200 = 600

Momentum of the gun = MV = 400 × V

We have,

Momentum of the shot = Momentum of the gun

or, 600 = 400V

∴ $V = \frac{3}{2}$

Again, let a be the retardation. Then,

$0 = V^2 - 2as$

or, $V^2 = 2as$

or, $\left(\frac{3}{2}\right)^2 = 2 \times a \times 2.5$

or, $a = \frac{9}{20}$ m/s^2

Now,

F = ma

or, $F = 400 \times \frac{9}{20} = 180$

∴ F = 180 N

26. 2072 Set C Q.No. 13b

State laws of motion. A body of mass 50kg falling from a certain height is brought to rest after striking the ground with a speed of 5 ms^{-1}. If the resistance force of ground is 500N, find the duration of the contact. **[4]**

➥ **First Part:** Please refer to 2063 Q.No. 14 b OR

Second Part: Please refer to 2071 Set C Q.No. 12 b

27. 2072 Set D Q.No. 13b

State laws of motion. Use Newton's Law to define an absolute unit of force. **[4]**

➥ Please refer to 2063 Q.No. 14 b OR

28. 2072 Set E Q.No. 13b

State Newton's laws of motion. Prove that Newton's second law provides the measurement of the force. **[4]**

➥ Please refer to 2063 Q.No. 14 b OR

29. 2072 Supp Q.No. 13b

A balloon is rising with a acceleration f. Prove that the fraction of the weight of the balloon which must be emptied out of the balloon in order to double the acceleration is $\frac{f}{g + 2f}$. **[4]**

➥ Please refer to 2069 Old (Set B) Q.No. 14a

30. 2073 Set D Q.No. 13b

A mass of 5 kg falls 3 m from rest and is then brought to rest by penetrating 30 cm into some sand. Find the average thrust of the sand on it. **[4]**

➥ Please refer to 2069 (Set A) Q.No. 13b

31. 2074 Supp Q.No. 13b

A shot of 2 kg is discharged by a gun of mass 400 kg with a velocity of 800 m/s. Find the constant force which would be required to stop the recoil of the gun in $1\frac{1}{4}$ sec. **[4]**

SOLUTION

Let v be the recoil velocity of the gun. Then,

Momentum of the shot = 2 × 800

Momentum of gun = 400 × v

We know that,

Momentum of the shot = Momentum of the gun

or, 2 × 800 = 400 × v

∴ v = 4 m/sec

Given, $t = 1\frac{1}{4}$ seconds $= \frac{5}{4}$ seconds

m = 400 kg

Let a be the retardation. Then,

0 = v – at

or, $a = \frac{v}{t} = \frac{4}{\frac{5}{4}} = \frac{16}{5}$

If F is the required constant force to be applied, then

$F = m \times a = 400 \times \frac{16}{5} = 80 \times 16 = 1280$ N

6 MARKS QUESTIONS

32. 2073 Set C Q.No. 14

State Newton's Laws of Motion. A bullet of mass 10g is fired from a gun of mass 3kg with a velocity 300 kmh^{-1}. Find the velocity of recoil of the gun. **[6]**

SOLUTION

First Part: Please refer to 2063 Q.No. 14 b OR

Second part:

Here, mass of bullet = 10 g

Mass of gun = 3 kg = 300 g

Muzzle velocity = 300 kmh^{-1}

Recoil velocity = ?

We have,

Mass of bullet × Muzzle velocity = Mass of the gun × Recoil velocity

or, 10 × 300 = 3000 × recoil velocity

∴ Recoil velocity = 1 kmh^{-1}

B. PROJECTILES

2 MARKS QUESTIONS

33. 2059 Q.No. 6 a

If u and α be the velocity and angle of projection of a projectile, then find the time of flight. [2]

SOLUTION

The time taken by the projectile to reach the horizontal plane through the point of projection again can be obtained from

$h = u \sin\alpha \cdot t - \frac{1}{2}gt^2$

When the particle strikes the horizon again, h = 0, then

$0 = u \sin\alpha \cdot t - \frac{1}{2}gt^2$

or, $t\left(u \sin\alpha - \frac{1}{2}gt\right) = 0$

$t = 0$ or $t = \frac{2u\sin\alpha}{g}$

But t = 0 means projectile is at the point of projection.

Hence, the time of flight $= \frac{2u\sin\alpha}{g}$

34. 2071 Old Q.No. 6 c

A particle is projected at an angle 75° to the horizon with a velocity of 2943 cm/sec. Find the range on a horizontal plane. [2]

SOLUTION

Given, $\alpha = 75°$

$u = 2943$ cm/s

$R = ?$

We have,

$R = \frac{u^2 \sin 2\alpha}{g}$

$= \frac{(2943)^2 \sin 150°}{10} = 433062.45$ cm

$= \frac{433062.45}{100}$ m $= 4330.62$ m

4 MARKS QUESTIONS

35. 2057 Q.No. 14 b

If R be the horizontal range of a projectile and h is greatest height, prove that its initial velocity is $\sqrt{2g\left(h + \frac{R^2}{16h}\right)}$. [4]

SOLUTION

Let α be the angle of projection and u be the initial velocity of the projectile.

We have, greatest height (h) $= \frac{u^2\sin^2\alpha}{2g}$

horizontal range (R) $= \frac{u^2 \sin 2\alpha}{g}$

Now,

$\sqrt{2g\left(h + \frac{R^2}{16h}\right)}$

$= \sqrt{2g\left\{\frac{u^2\sin^2\alpha}{2g} + \frac{\left(\frac{u^2\sin 2\alpha}{g}\right)^2}{16\cdot\frac{u^2\sin^2\alpha}{2g}}\right\}}$

$= \sqrt{2g\left(\frac{u^2\sin^2\alpha}{2g} + \frac{u^4\cdot 4\sin^2\alpha\cos^2\alpha}{g^2} \times \frac{g}{8u^2\sin^2\alpha}\right)}$

$= \sqrt{2g\left(\frac{u^2\sin^2\alpha}{2g} + \frac{u^2\cos^2\alpha}{2g}\right)}$

$= \sqrt{u^2(\sin^2\alpha + \cos^2\alpha)} = u$

Hence, $u = \sqrt{2g\left(h + \frac{R^2}{16h}\right)}$

36. 2058 Q.No. 14 b

If R be the horizontal range and T, the time of flight of a projectile, show that

$\tan\alpha = \frac{gT^2}{2R}$, where α is the angle of projection. [4]

SOLUTION

If u be the velocity of projection, then

horizontal range (R) $= \frac{u^2\sin 2\alpha}{g}$ and

time of flight (T) $= \frac{2u\sin\alpha}{g}$,

Now,

$\text{L.H.S.} = \frac{gT^2}{2R} = \frac{g\left(\frac{2u\sin\alpha}{g}\right)^2}{2\cdot\frac{u^2\sin 2\alpha}{g}} = \frac{g\cdot\frac{4u^2\sin^2\alpha}{g^2}}{\frac{2u^2\cdot 2\sin\alpha\cos\alpha}{g}}$

$= \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{R.H.S.}$

37. 2059 Q.No. 14 b OR

A particle is projected with a velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is

$R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$. [4]

SOLUTION

Let α be the angle of projection.

We have,

Greatest height (H) $= \frac{u^2\sin^2\alpha}{2g}$ and

Horizontal range (R) $= \frac{u^2\sin 2\alpha}{g}$

Now,

$\text{R.H.S.} = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$

$= 4\sqrt{\frac{u^2\sin^2\alpha}{2g}\left(\frac{u^2}{2g} - \frac{u^2\sin^2\alpha}{2g}\right)}$

$= 4\sqrt{\frac{u^2\sin^2\alpha}{2g}\left(\frac{u^2 - u^2\sin^2\alpha}{2g}\right)}$

$= 4\sqrt{\frac{u^2\sin^2\alpha\cdot u^2\cos^2\alpha}{(2g)^2}}$

$= \frac{4u^2\sin\alpha\cos\alpha}{2g}$

$= \frac{u^2\, 2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g} = R = \text{L.H.S.}$

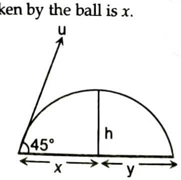
38. 2060 Q.No. 14 b

From a point on the ground at a distance 'x' from the foot of a vertical wall, a ball is thrown at an angle of 45° which just clears the top of the wall and afterwards strikes the ground at a distance 'y' on the other side. Prove that the height of the wall is $\frac{xy}{x+y}$. [4]

SOLUTION

Let u be the velocity of projection. Given α = 45°. Then the horizontal and vertical component of u are u cos 45° and u sin 45° i.e. $\frac{u}{\sqrt{2}}$ and $\frac{u}{\sqrt{2}}$ respectively.

Let t be the time taken by the ball to reach the top of the wall. The horizontal distance in this time taken by the ball is x.



Then, $x = u\cos 45°\cdot t = 4\cdot\frac{1}{\sqrt{2}}\cdot t$

$\therefore t = \frac{x\sqrt{2}}{u}$

Now, if h be the height of the wall, then

$h = (u\sin 45°)\,t - \frac{1}{2}gt^2$

$= u \times \frac{1}{\sqrt{2}} \times \frac{x\sqrt{2}}{u} - \frac{1}{2}g \times \frac{2x^2}{u^2}$

$h = x - \frac{x^2 g}{u^2}$ ...(i)

Again, the horizontal range $= \frac{u^2}{g}$

i.e. $x + y = \frac{u^2}{g}$

or, $u^2 = g(x + y)$ ...(ii)

Substituting the value of u^2 in (i), we have

$h = x - \frac{x^2 g}{g(x+y)}$

or, $h = x - \frac{x^2}{x+y} = \frac{x^2 + xy - x^2}{x+y}$

$\therefore h = \frac{xy}{x+y}$

39. 2061 Q.No. 14 b

If R be the horizontal range of a projectile and h its greatest height. Prove that its initial velocity is $\sqrt{2g\left(h + \frac{R^2}{16h}\right)}$. [4]

➥ Please refer to 2057 Q.No. 14 b

40. 2062 Q.No. 14 b OR

If R be the horizontal range and T be the time of flight of a projectile, show that

$\tan\alpha = \frac{gT^2}{2R}$ where α is the angle of projection

➥ Please refer to 2058 Q.No. 14 b

41. 2063 Q.No. 14 b

A projectile thrown from a point in a horizontal plane comes back to the plane in 4 sec. at a distance of 60m in front of the point of projection. Find the velocity of projection. (g = 10 m/s²). [4]

SOLUTION

Let u be the velocity of projection and α be the angle of projection.

Given, Time of flight (T) = 4s

Horizontal range (R) = 60 m

We have,

$T = \frac{2u\sin\alpha}{g}$

or, $4 = \frac{2\times u\times\sin\alpha}{10}$

$\therefore u\sin\alpha = 20$ ...(i)

Again, we have

$R = \frac{u^2\sin 2\alpha}{g}$

or, $60 = \frac{u^2\cdot 2\sin\alpha\cos\alpha}{g}$

or, $60 = \frac{(u\sin\alpha)\cdot 2\cdot u\cos\alpha}{g}$

or, $60 = \frac{20\times 2\times u\cos\alpha}{10}$ [using (i)]

or, $u\cos\alpha = 15$ ...(ii)

Squaring and adding (i) and (ii), we get

$u^2\sin^2\alpha + u^2\cos^2\alpha = 20^2 + 15^2$

or, $u^2 = 625$

$\therefore u = 25$ m/s

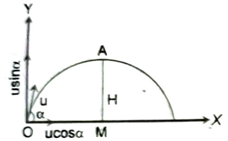
Hence, the velocity of projection is 25 m/s.

42. 2064 Q.No. 14 b

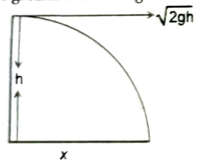
A stone is thrown horizontally with velocity $\sqrt{2gh}$ from the top of a tower of height h. Find where it will strike the level ground through the foot of the tower. What will be its striking velocity? [4]

SOLUTION

A projectile is any object with a given initial velocity and moving along a path determined by gravitational force acting on it and frictional resistance of the air. The path along which the projectile moves is called the trajectory. The point from which the particle is projected is called point of projection. The angle α between the horizontal plane through the point of projection and direction of the projection is called angle of projection. The time taken by the particle to come back to the horizontal plane again is called the time of flight, and the distance between the point of projection and the point where the particle strikes the horizontal plane again is called the horizontal range.



Next Part:
Let t be the time taken by the stone when it strikes the ground after falling from a height h.



Then, $h = 0 + \frac{1}{2}gt^2$

$\therefore \quad t = \sqrt{\frac{2h}{g}}$

Also, $x = ut = \sqrt{2gh} \times \sqrt{\frac{2h}{g}} = 2h$

Let v be the striking velocity of the stone, when it strikes after falling a height h.

Then,

$v^2 = u^2 + 2gh$

$= 2gh + 2gh = 4gh = (2\sqrt{gh})^2$

$\therefore \quad v = 2\sqrt{gh}$

43. 2065 Q.No 14 b

If R be the horizontal range and T the time of flight of a projectile, show that:

$\tan\alpha = \frac{gT^2}{2R}$, where α is the angle of projection. [4]

➥ Please refer to 2058 Q.No. 14 b

44. 2066 C Q.No. 14 b

A particle is projected with a velocity u. If the greatest height attained by the particle be H, prove that the range of R on the horizontal plane through the point of projection is:

$R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$ [4]

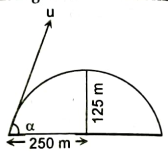
➥ Please refer to 2059 Q.No. 14 b OR

45. 2066 Q.No.14 b

Find the velocity and direction of projection of a shot which passes in a horizontal direction just over the top of a wall 250 m off and 125 m high. ($g = 9.8ms^{-2}$) [4]

SOLUTION

Let u be the velocity of projection of a shot making an angle α with the horizon.



Maximum height (H) = 125 m

Horizontal range (R) = 2 × 250 = 500 m

We have,

$H = \frac{u^2 \sin^2\alpha}{2g}$

or, $125 = \frac{u^2 \sin^2\alpha}{2g}$ ...(i)

and $R = \frac{u^2 \sin 2\alpha}{g}$

or, $500 = \frac{u^2 \sin 2\alpha}{g}$ ...(ii)

Dividing (i) by (ii), we get

$\frac{125}{500} = \frac{\frac{u^2 \sin^2\alpha}{2g}}{\frac{u^2 \sin 2\alpha}{g}}$

or, $\frac{1}{4} = \frac{u^2 \sin^2\alpha}{2g} \times \frac{g}{u^2 \cdot 2\sin\alpha\cos\alpha}$

or, $1 = \frac{\sin\alpha}{\cos\alpha}$

or, $\tan\alpha = 1 = \tan 45°$

$\therefore \quad \alpha = 45°$

Substituting the value of α in (i), we have

$125 = \frac{u^2 \sin^2 45°}{2 \times 9.8}$

or, $125 = \frac{u^2 \times 1/2}{19.6}$

or, $125 \times 19.6 = \frac{u^2}{2}$

or, $u^2 = 4{,}900$

$\therefore \quad u = 70$ m/s

46. 2067 Q.No. 14b

Find the velocity and direction of projection of a shot which passes in a horizontal direction just over the top of a wall which is 250 m off and 125 m high. ($g = 9.8 ms^{-2}$) [4]

➥ Please refer to 2066 Q.No.14 b

47. 2068 Q.No. 14b

The horizontal and vertical components of the initial velocity of a projectile are U and V respectively. If R be the range and the H the greatest height attained, prove that: $\frac{4H}{R} = \frac{V}{U}$ [4]

➥ Please refer to Model Set II, Q.No. 14

48. 2069 (Set A) Old Q.No. 14b

A projectile thrown from a point in a horizontal plane comes back to the plane in 4 secs at a distance of 60m in front of the point of projection, find the velocity of projection. ($g = 10 ms^{-2}$) [4]

SOLUTION

Let u be the velocity of projection and α be the angle of projection.

Given, Time of flight (T) = 4s

Horizontal range (R) = 60 m

We have,

$T = \frac{2u\sin\alpha}{g}$

or, $4 = \frac{2 \times u \times \sin\alpha}{10}$

$\therefore$ $u\sin\alpha = 20$ ...(i)

Again, we have

$R = \frac{u^2 \sin 2\alpha}{g}$

or, $60 = \frac{u^2 \cdot 2\sin\alpha\cos\alpha}{g}$

or, $60 = \frac{(u\sin\alpha) \cdot 2 \cdot u\cos\alpha}{g}$

or, $60 = \frac{20 \times 2 \times u\cos\alpha}{10}$ [using (i)]

or, $u\cos\alpha = 15$ ...(ii)

Squaring and adding (i) and (ii), we get

$u^2\sin^2\alpha + u^2\cos^2\alpha = 20^2 + 15^2$

or, $u^2 = 625$

$\therefore$ $u = 25$ m/s

Hence, the velocity of projection is 25 m/s.

49. 2069 Old (Set B) Q.No. 14b

A ball is thrown from the top of a building towards a tall building $50\sqrt{3}$ m away. The initial velocity of the ball is $20ms^{-1}$ at 30° above the horizontal. How far above or below its original level will the ball strike the opposite wall? [4]

SOLUTION

Let the ball thrown from the top A of a building strike another tall building $50\sqrt{3}$ m away at B after t seconds.

Let y be the distance of B from the original level.

By question, u = 20 m/s, α = 30°

Horizontal component of u = u cos α = 20 cos 30° = $20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$

Vertical component of u = u sin α = 20 sin 30°

$= 20 \times \frac{1}{2} = 10$

For the horizontal motion, we have

$50\sqrt{3} = 10\sqrt{3}\,t$

t = 5s

The vertical distance covered

$y = ut - \frac{1}{2}gt^2 = 10 \times 5 - \frac{1}{2} \times 10 \times 5^2 = -75$ m

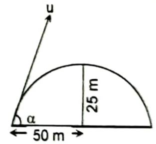
Since y is negative, so it strikes the opposite wall 75m below the original level.

50. 2070 (Old) Q.No. 14 b

Find the velocity and direction of projection of a shot which passes in a horizontal direction just over the top of a wall which is 50 meter off and 25 meter high ($g = 9.8\ mr^{-2}$). [4]

SOLUTION

Let u be the velocity of projection of a shot making an angle α with the horizon.



Maximum height (H) = 25 m

Horizontal range (R) = 2 × 50 = 100 m

We have,

$H = \frac{u^2 \sin^2\alpha}{2g}$

or, $25 = \frac{u^2 \sin^2\alpha}{2g}$ ...(i)

and $R = \frac{u^2 \sin 2\alpha}{2g}$

or, $100 = \frac{u^2 \sin 2\alpha}{g}$ ...(ii)

Dividing (i) by (ii), we get

$\frac{25}{100} = \frac{\frac{u^2 \sin^2\alpha}{2g}}{\frac{u^2 \sin 2\alpha}{g}}$

or, $\frac{1}{4} = \frac{u^2 \sin^2\alpha}{2g} \times \frac{g}{u^2 \times 2\sin\alpha\cos\alpha}$

or, $\frac{\sin\alpha}{\cos\alpha} = 1$

or, $\tan\alpha = 1 = \tan 45°$

$\therefore$ $\alpha = 45°$

Substituting the value of α in (i), we have

$25 = \dfrac{u^2 \sin^2 45^\circ}{2 \times 9.8}$

or, $25 = \dfrac{u^2 \times 1/2}{19.6}$

or, $u^2 = 25 \times 19.6 \times 2$

or, $u = 980$

$\therefore$ $u = 14\sqrt{5}$ m/s

$\therefore$ The velocity of projection = $14\sqrt{5}$ m/s.

51. 2070 Set C Q.No. 13 b

Find the velocity and the direction of projection of a shot which passes in a horizontal direction just over the top of wall which is 250 m off and 125 m high.($g = 9.8 m/s^2$) [4]

➥ Please refer to 2066 Q.No.14 b

52. 2070 Supp. Q.No. 13 b

A ball is projected at an angle of 30° to the horizon and land on the surface of height 10m which is $20\sqrt{3}$ m away from the point of projection. Find the velocity of projection and its striking velocity on the surface. ($g = 10m/s^2$) [4]

SOLUTION

Given, angle of projection (α) = 30°

Greatest height (H) = 10 m

Horizontal range (R) = $20\sqrt{3}$

Velocity of projection (u) = ?

Striking velocity (v) = ?

We have,

$H = \dfrac{u^2 \sin^2\alpha}{2g}$

or, $10 = \dfrac{u^2 \sin^2 30^\circ}{2 \times 10}$

or, $200 = u^2 \times \left(\dfrac{1}{2}\right)^2$

$\therefore$ $u = 20\sqrt{2}$ m/s.

We have,

$R = u \cos\alpha \, T$, where T be the time of flight.

or, $20\sqrt{3} = 20\sqrt{2} \cdot \cos 30^\circ \cdot T$

or, $20\sqrt{3} = 20\sqrt{2} \cdot \dfrac{\sqrt{3}}{2} \cdot T$

$\therefore$ $T = \sqrt{2}$

Let v be the velocity with which it strikes the ground at an angle θ with the horizontal, then

$v \cos\theta = u \cos\alpha = 20\sqrt{2} \cdot \cos 30^\circ$

$= 20\sqrt{2} \cdot \dfrac{\sqrt{3}}{2} = 10\sqrt{6}$

and

$v \sin\theta = u \sin\alpha - g \cdot T$

$= 20\sqrt{2} \cdot \sin 30^\circ - 10\sqrt{2}$

$= 20\sqrt{2} \cdot \dfrac{1}{2} - 10\sqrt{2} = 0$

Squaring and adding, we have

$v^2 = (10\sqrt{6})^2 + 0^2$

$\therefore$ $v = 10\sqrt{6}$ m/s

53. 2071 Old Q.No. 14 b

A ball is thrown by a player from a height of 2 meters, at an angle of 30° with the horizon with a velocity of $18ms^{-1}$, is caught by another player at the height of 0.4 meter from the ground. How far apart were the two players? ($g = 9.8 ms^{-2}$) [4]

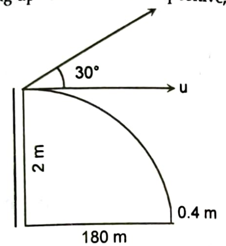
SOLUTION

Given, $\alpha = 30^\circ$

So, the horizontal and the vertical components of velocity of projection are u cos 30° and u sin 30° respectively.

A ball thrown from a height of 2m is to be caught by a player at a height of 0.4m. So the ball has to descend 2 - 0.4 = 1.6 m.

Taking upward direction as positive, we have



$-h = u \sin 30^\circ \, t - \dfrac{1}{2} gt^2$

or, $-1.6 = 18 \times \dfrac{1}{2} \times t - \dfrac{1}{2} \times 9.8 \times t^2$

or, $-1.6 = 9t - 4.9t^2$

or, $49t^2 - 90t - 16 = 0$

or, $(t - 2)(49t + 8) = 0$

$\therefore$ $t = 2, \dfrac{-8}{49}$

Rejecting the -ve value of t, we have t = 2s

If x be the horizontal distance apart between the two players, then

$x = u \cos 30^\circ t = 18 \times \dfrac{\sqrt{3}}{2} \times 2 = 18\sqrt{3}$ m

$\therefore$ Distance between the two players is $18\sqrt{3}$ m.

54. 2071 Set C Q.No. 13 b

A stone is thrown horizontally with velocity $\sqrt{2gh}$ from the top of a tower of height h. Find where it will strike the level ground through the foot of the tower. What will be its striking velocity? [4]

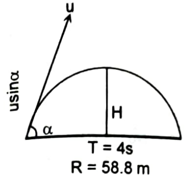
➥ Please refer to 2064 Q.No. 14 b

55. 2071 Supp. Q.No. 13b

A body thrown from a point in a horizontal plane comes back to the plane in 4 sec at a distance of 58.8m from the point of projection. Find the velocity of the projection. [$g=9.8m/s^2$] [4]

SOLUTION

Let u and α be the velocity and angle of projection respectively.



By given,

Time of flight (T) = 4s

Horizontal range (R) = 58.8 m

We have,

$T = \dfrac{2u \sin\alpha}{g}$

or, $u \sin\alpha = \dfrac{1}{2} Tg = \dfrac{1}{2} \times 4 \times 9.8 = 19.6$

$\therefore$ $u \sin\alpha = 19.6$ ...(i)

Again, we have

$R = \dfrac{u^2 \sin 2\alpha}{g}$

or, $u^2 \sin 2\alpha = Rg$

or, $u^2 \cdot 2 \sin\alpha \cos\alpha = 58.8 \times 9.8$

or, $u^2 \sin\alpha \cos\alpha = \dfrac{58.8 \times 9.8}{2}$

or, $u^2 \sin\alpha \cos\alpha = 288.12$ ...(ii)

Dividing (ii) by (i), we get

$u \cos\alpha = \dfrac{288.12}{19.6} = 14.7$ ...(iii)

Squaring and adding (i) and (iii), we get

$u^2 \sin^2\alpha + u^2 \cos^2\alpha = 19.6^2 + 14.7^2$

or, $u^2 = 600.25$

or, $u = 24.5$ m/s

Hence, the velocity of projection is 24.5 m/s.

56. 2072 Set E Q.No. 13b OR

With what velocity must a body be projected at an angle of 45° from the top of a tower 65 m high, if it is to reach a point on the ground 180 m from the base of the tower. [4]

SOLUTION

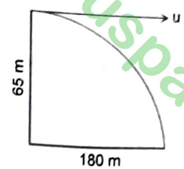
Let u be the velocity with which a body be projected and t be time taken by the body to reach the ground. Now, taking upward direction positive, we have

$-h = u \sin\alpha \cdot t - \dfrac{1}{2} gt^2$

or, $-65 = u \sin 45^\circ \cdot t - \dfrac{1}{2} \times 9.8 \, t^2$

or, $-65 = \dfrac{ut}{\sqrt{2}} - 4.9t^2$ ...(i)

The particle hits at a distance of 180 m from the base of the tower, so



$s = u \cos\alpha \cdot t$

or, $180 = u \cos 45^\circ \cdot t$

or, $180 = \dfrac{ut}{\sqrt{2}}$ ...(ii)

From (i) and (ii), we have

$-65 = 180 - 4.9t^2$

or, $4.9 t^2 = 180 + 65$

or, $4.9 t^2 = 245$

or, $t^2 = 50$

$\therefore$ $t = \sqrt{50}$ s

Again, from (ii), we get

$u \cdot \dfrac{\sqrt{50}}{\sqrt{2}} = 180$

or, $u = \dfrac{180\sqrt{2}}{\sqrt{50}} = \dfrac{180\sqrt{2}}{5\sqrt{2}} = 36$

$\therefore$ $u = 36$ m/s

57. 2073 Supp Q.No. 13b

Find the velocity and the direction of the projection of a shot which passes in a horizontal direction just over the top of a wall which is 250 m off and 125 m high. ($g = 9.8 ms^{-2}$) [4]

➥ Please refer 2066 Q.No. 14b

58. 2074 Set A Q.No. 13b OR

Find the velocity and the direction of the projection of a shot which passes in a horizontal direction just over the top of a wall which is 250 m off and 125 m high. ($g = 9.8m/s^2$) [4]

➥ Please refer to 2066 Q.No. 14b

59. 2075 Set A Q.No. 13b

If R be the horizontal range and T, the time of flight of a projectile, show that $\tan\alpha = \dfrac{gT^2}{2R}$ where α is the angle of projection. [4]

➥ Please refer to 2058 Q.No. 14b

6 MARKS QUESTIONS

60. 2069 (Set A) Q.No. 14

The horizontal and the vertical components of the initial velocity of a projectile are U and V. If R be the range and H, the greatest height attained, prove that: [6]

(a) $\dfrac{4H}{R} = \dfrac{V}{U}$ (b) $\left(\dfrac{R}{U}\right)^2 = \dfrac{8H}{g}$

➥ Please refer to Model Set II, Q.No. 14

61. 2069 (Set B) Q.No. 14a

If R be the horizontal range of a projectile and h is greatest height, prove that its initial velocity is $\sqrt{2g\left(h+\frac{R^2}{16h}\right)}$. [6]

➥ Please refer to 2057 Q.No. 14 b

62. 2070 Set D Q.No. 14

A projectile thrown form a point in a horizontal plane comes back to the plane in 4 sec. at a distance of 60 m in front of the point of projection. Find the velocity of projection. (g = 10m/s²) [6]

➥ Please refer to 2069 (Set A) Old Q.No. 14b

63. 2071 Set D Q.No. 14 OR

A cannon ball has the same range R on a horizontal plane for two different angles of projection. If H and H¹ are the greatest heights in two paths for which this is possible, prove that: $R^2 = 16HH^1$. [6]

SOLUTION

Let α and β be two different angle of projections having the same range R. Then, R

$= \frac{u^2 \sin 2\alpha}{g}$ and $R = \frac{u^2 \sin 2\beta}{g}$

$\therefore \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 \sin 2\beta}{g}$

or, $\sin 2\alpha = \sin 2\beta$

Since $\alpha \neq \beta$, we must have

$2\beta = 180° - 2\alpha$

or, $\beta = 90° - \alpha$

We have, $H = \frac{u^2 \sin^2\alpha}{2g}$,

and $H_1 = \frac{u^2 \sin^2\beta}{2g} = \frac{u^2 \sin^2(90° - \alpha)}{2g}$

$= \frac{u^2 \cos^2\alpha}{2g}$

Now, $R^2 = \left(\frac{u^2 \sin 2\alpha}{g}\right)^2 = \frac{u^4 \cdot 4 \sin^2\alpha \cos^2\alpha}{g^2}$

$= 4 \times 4 \times \frac{u^2 \sin^2\alpha}{2g} \times \frac{u^2 \cos^2\alpha}{2g} = 16\,HH'$.

64. 2072 Set C Q.No. 15

If R be the horizontal range of a projectile and h its greatest height, prove that its initial velocity is $\sqrt{2g\left(h+\frac{R^2}{16h}\right)}$. [6]

➥ Please refer to 2057 Q.No. 14 b

65. 2072 Set D Q.No. 15 OR

Describe motion of a projectile. A stone is thrown horizontally with velocity $\sqrt{2gh}$ from the top of a tower of height h. Find where it will strike the level ground through the foot of the tower and also find the striking velocity.

➥ Please refer to 2064 Q.No. 14 b

66. 2072 Supp Q.No. 14

From a point on the ground at a distance x from the foot of a vertical wall, a ball is thrown at an angle of 45° which just clears the top of the wall and afterwards strikes the ground at a distance y on the other side. Prove that the height of the wall is $\frac{xy}{x+y}$. [6]

➥ Please refer to 2060 Q.No. 14 b

67. 2073 Set C Q.No. 14 OR

If R be the horizontal range of a projectile and h is greatest height, prove that its velocity is $\sqrt{2g\left(h+\frac{R^2}{16h}\right)}$.

➥ Please refer to 2057 Q.No. 14 b

68. 2073 Set D Q.No. 14

A projectile thrown from a point in horizontal plane comes back to the plane in 4 secs at a distance of 60 m from the point of projection. Find the velocity of the projection. (g = 10m/s²) [6]

➥ Please refer to 2069 (Set A) Old Q.No. 14b

69. 2074 Supp Q.No. 14

With what velocity must a body be projected at an angle of 45° from the top of a tower 65 m high, if it is to reach a point on the ground 180 m from the base of the tower. [6]

➥ Please refer to 2072 Set E Q.N. 13b OR

70. 2074 Set B Q.No. 15

From a point on the ground at a distance x from the foot of a vertical wall a ball is thrown at an angle of 45° which just clears the top of the wall and afterwards strikes the ground at a distance y on the other side.

Prove that the height of the wall is $\frac{xy}{x+y}$. [6]

➥ Please refer to 2072 Supp. Q. No. 14

71. 2075 Set C Q.No. 14OR

A particle is projected with a velocity u. If the greatest height attained by the particle be H, prove that the range R on the horizontal plane through the point of projection is $R = 4\sqrt{H\left(\frac{u^2}{2g} - H\right)}$. A body is projected with a velocity of 9.8 m/sec and rises upto the height 2.45 m. Find the horizontal range. [6]

SOLUTION

First part: Please refer to 2059 Q.No. 14b OR

Second part:

Velocity of projection (u) = 9.8 m/sec.

Greatest height (H) = 2.45m.

Horizontal range (R) = ?

We have,

$H = \frac{u^2 \sin^2\alpha}{2g}$

or, $2.45 = \frac{(9.8)^2 \sin^2\alpha}{2 \times 9.8}$

or, $\sin^2\alpha = \frac{2.45 \times 2}{9.8}$

or, $\sin^2\alpha = \frac{1}{2}$

$\therefore \sin\alpha = \frac{1}{\sqrt{2}}$

$\cos\alpha = \sqrt{1 - \sin^2\alpha}$

$= \sqrt{1 - \left(\frac{1}{\sqrt{2}}\right)^2}$

$= \sqrt{1 - \frac{1}{2}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

We have,

$R = \frac{u^2 \sin 2\alpha}{g} = \frac{u^2 . 2 \sin\alpha \cos\alpha}{g}$

$= \frac{(9.8)^2 \times 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}}{9.8} = 9.8$ m

C. WORK, ENERGY AND POWER

2 MARKS QUESTIONS

72. 2057 Q.No. 5 b

Calculate the power of a pump which can lift 300 kgs of waters through a vertical height of 4m in 10 secs. [g = 10m s⁻²] [2]

SOLUTION

Mass of water (m) = 300 kg

Time (t) = 10s

Height (h) = 4 m

We have, Power of the pump

$= \frac{\text{Work done}}{\text{Time}} = \frac{\text{Weight} \times \text{Height}}{\text{Time}} = \frac{mgh}{t}$

$= \frac{300 \times 10 \times 4}{10} = 1{,}200$ W

73. 2059 Q.No. 6 c

A car is moving at 36 kmh⁻¹. What velocity will double its kinetic energy? [2]

SOLUTION

Here, $v = 36$ km/hr $= \frac{36 \times 1000}{60 \times 60} = 10$ m/s

If the mass of the car is m kg, then its

$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot (10)^2 = 50m$J

If its K.E is doubled, then new K.E = 2 × 50 mJ = 1,000 mJ and let its new velocity be v_1.

Then,

$\frac{1}{2}mv_1^2 = 100$ m

or, $v_1^2 = 200$

$\therefore v_1 = \sqrt{200} = 10\sqrt{2}$

Required velocity = $10\sqrt{2}$ m/s

74. 2060 Q.No. 5 b

A car covers a distance of 50m in 5 secs against a frictional force. If the power of the engine is 4000 watts, find the frictional force. [2]

SOLUTION

Given,

Distance covered = 50 m

Time taken = 5 s

Power = 4,000 W

Frictional force (F) = ?

We know that,

$\text{Power} = \frac{\text{Work done}}{\text{time}} = \frac{\text{Force} \times \text{distance}}{\text{time}}$

or, $4{,}000 = \frac{F \times 50}{5}$

$\therefore F = 400$ N

75. 2061 Q.No. 5 b

A car of mass 1000 kg moves up an incline of 30° at a constant speed of 20 m/sec. If the frictional force is 2000N, calculate the power developed by the engine. (g = 10m/sec²). [2]

SOLUTION

Here, mass (m) = 1,000 kg

Angle of inclination (α) = 30°

Velocity (v) = 20 m/s

Friction al force (F) = 2,000 N

Force acting on car (F)

= Component of weight down the plane + frictional force

$= mg \sin\alpha + 2{,}000$

$= 1{,}000 \times 10 \times \sin 30° + 2{,}000$

$= 1{,}000 \times 10 \times \frac{1}{2} + 2{,}000 = 7{,}000$ N

We have,

Power = Force × Velocity = 7,000 × 20

= 140,000 W = 140 KW

76. 2063 Q.No. 6 a

A pump having a power of 294 w pumps water at the rate of 90 liters per minute. Find the height to which the water is raised. (g = 9.8 m/s², 1 litre of water = 1 kg) [2]

SOLUTION

Power = 294 w

Mass of water ejected per second $\left(\frac{m}{t}\right)$

$= \frac{90 \times 1}{60}$ kg $= \frac{3}{2}$ kg

Height (h) = ?

We have,

$\text{Power} = \frac{\text{Work done}}{\text{Time taken}} = \frac{mgh}{t}$

or, $294 = \left(\frac{m}{t}\right) \times g \times h$

or, $294 = \frac{3}{2} \times 9.8 \times h$

or, $h = \frac{294 \times 2}{9.8 \times 3}$

$\therefore$ h = 20 m

77. 2071 Old Q.No. 6 b

A car is moving at 36km/h. What velocity will double its kinetic energy? [2]

➥ Please refer to 2059 Q.No. 6 c

78. 2072 Set E Q.No. 12c

Calculate the power of a pump which can lift 300 kg of water through a vertical height of 4m in 10 secs. (g = 10 m/s²) [2]

SOLUTION

Given, m = 300 kg h = 4 m t = 10s

$\therefore$ Weight (force due to gravity) = mg
= 300 × 100 = 3000 N

We have,

$\text{Power} = \frac{\text{Weight}\times\text{Height}}{\text{Time}} = \frac{3000\times4}{10} = 1200$ watt.

79. 2073 Set C Q.No. 12c

How large a force is required to cover a distance of 80m if the total work done is 800J? [2]

SOLUTION

Here, distance covered (d) = 80 m
Work done (W) = 800 J

Force (F) = ?

We have,

W = F × d

800 = F × 80

$F = \frac{800}{80} = 10$ N

$\therefore$ Required force = 10 N

4 MARKS QUESTIONS

80. 2057 Q.No. 14 b OR

Define work, power and energy. Prove that the sum of the kinetic and potential energies of a freely falling body remains constant throughout the motion. [4]

SOLUTION

First Part:

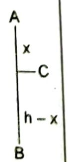
Work: The work done by a force is defined as the product of the force and the distance moved in the direction of the force.

Power: The time rate of work done is called power. So, $\text{power} = \frac{\text{Work done}}{\text{Time taken}}$.

Energy: Energy of a body is defined to be its capacity to do work.

Second Part:

Suppose that a body of mass m is initially at the point A which is at a height h from the ground B. Let the body start falling from A and C be the position of the body at any instant such that AC = x. Then BC = h−x



At the point A

K.E. = 0

P.E. = mgh

$\therefore$ K.E + P.E = 0 + mgh = mgh

At the point C

Let v_1 be the initial velocity of the body at the point C. Then. $v_1^2 = 2gx$.

$K.E. = \frac{1}{2}mv_1^2 = \frac{1}{2}m \cdot 2gx = mgx$

P.E. = mg(h − x)

$\therefore$ K.E. + P.E = mg$x$ + mg(h − x) = mgh

At the point B

Let v be the velocity of the body when it reaches the ground. Then,

$v^2 = 2gh$

$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot 2gh = mgh$

P.E = 0

$\therefore$ K.E + P.E = mgh + 0 = mgh

The sum of K.E and P.E of the freely falling body at any instant is same (i.e. mgh) and hence it is constant.

81. 2058 Q.No. 14 b OR

"The change in kinetic energy of a body is equal to the work done by the acting force". Prove this statement. [4]

SOLUTION

Let F be a force applied on a body of mass m. Let a be the acceleration. Then,

F = ma

Let s be the distance covered by the body when it changes its velocity from u to v. Then,

$v^2 = u^2 + 2as$

or, $v^2 - u^2 = 2as$

Initial K.E of the body $= \frac{1}{2}mu^2$

Final K.E of the body $= \frac{1}{2}mv^2$

Change in K.E of the body $= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}m \cdot 2as = m \cdot as = (ma)s$

= F × S = work done by the force

82. 2061 Q.No. 14 b OR

State the principle of conservation of energy. Illustrate it with the consideration of a body sliding down a smooth inclined plane. [4]

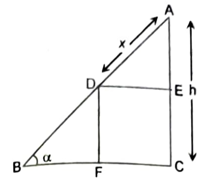
SOLUTION

Suppose a particle of mass m slides down a smooth inclined plane AB with AB = l and height h which is inclined at an angle α to the horizon.

Then, $\sin\alpha = \frac{h}{l} \Rightarrow h = l\sin\alpha$.

Let the particle start from A and get the point D at any instant such that AD = x.

Then BD = $l - x$



At the point A

$K.E. = \frac{1}{2}mu^2 = \frac{1}{2}m \cdot 0^2 = 0$

P.E. = m(g sinα) l = mg (l sin α) = mgh

$\therefore$ K.E + P.E = 0 + mgh = mgh

At the point D

Let v_1 be the velocity of the particle at D. The acceleration down the inclined plane is g sin α. Also, $v^2 = 2\,g\sin\alpha \cdot x = 2gx\sin\alpha$

$K.E = \frac{1}{2}mv^2 = \frac{1}{2}m \times 2gx\sin\alpha = mgx\sin\alpha$

P.E. = mg ($l - x$) sin α

$\therefore$ K.E + P.E. = mg$x$ sin α + mg ($l - x$) sin α
= mg (l sin α) = mgh

At the point B

Let v be the velocity at B. Then

$v^2 = 0^2 + 2(g\sin\alpha)\cdot l = 2gl\sin\alpha$

$\therefore$ $K.E = \frac{1}{2}mv^2 = \frac{1}{2}m \times 2gl\sin\alpha = mg(l\sin\alpha) =$ mgh

P.E = mg × 0 = 0

$\therefore$ K.E + P.E = mgh + 0 = mgh

Hence, the sum of K.E and P.E is constant throughout the motion.

83. 2062 Q.No. 14 b

An engine pumps 746 liters of water per minute from a well through an average height of 60m. Find the horse power of the engine if 50% of the power is wasted. (1 liter of water = 1 kg., g = 10 m/s²) [4]

SOLUTION

Mass of water (m) = 746 litres = 746 kg
[$\because$ 1 litre = 1 kg (given)]

Time (t) = 1 minute = 60s

Height (h) = 60 m

Horse power (HP) = ?

We have,

$\text{Power} = \frac{\text{Work done}}{\text{time taken}} = \frac{mgh}{t}$

$= \frac{746 \times 10 \times 60}{60}$

= 7460 watts

Horse power $= \frac{7460}{746}$ HP = 10 HP

Since 50% of the power is wasted. So the efficiency of engine is (100 − 50)% = 50%.

Let x be the required HP of the engine. Then,

50% of x = 10

or, $\frac{50}{100} \times x = 10$

$\therefore$ x = 20 HP

Required HP of the engine = 20 HP

84. 2064 Q.No. 14 b OR

If a force acts on a body, prove that the change in kinetic energy of a body is equal to the work done by the force. [4]

➥ Please refer to 2058 Q.No. 14 b OR

85. 2065 Q.No 14 b OR

State and prove the Principle of Conservation of Energy. [4]

➥ Please refer to 2057 Q.No. 14 b OR

86. 2066 Q.No.14 b OR

A bullet loses $1/20^{th}$ of its velocity in passing through a plank. Find how many such uniform planks it would pass through before coming to rest assuming the retardation to be uniform. [4]

SOLUTION

Let u be the velocity of penetration. Let x be the thickness of each plank. The velocity of plank after penetrating the first plank

$= u - \frac{u}{20} = \frac{19u}{20}$

Change in K.E $= \frac{1}{2}mu^2 - \frac{1}{2}m\left(\frac{19u}{20}\right)^2$

$= \frac{39}{800}mu^2$

We have, work done by the resistance force F = F × x

$\therefore$ $F \times x = \frac{39}{800}mu^2$ …(i)

Suppose that the bullet passes n planks before coming to rest. So, the total distance passed by the bullet is nx.

Change in K.E. $= \frac{1}{2}mu^2 - 0 = \frac{1}{2}mu^2$

Work done = F × nx

or, $F \times nx = \frac{1}{2}mu^2$

$\therefore$ $F \times x = \frac{1}{2n}mu^2$ …(ii)

From (i) and (ii), we have

$\frac{1}{2n}mu^2 = \frac{39}{800}mu^2$

or, 78n = 800

or, $n = \frac{800}{78} = 10\frac{10}{39}$

87. 2066 C Q.No. 14 b OR

An engine pumps 746 litres of water per minute from a well through an average height of 60 m. Find the horse power of the engine if 50% of the power is wasted. (1 litre of water = 1 kg, g = 10m /sec²) [4]

➥ Please refer to 2062 Q.No. 14 b

88. 2067 Q.No. 14b OR

A bullet of mass 200 gm is fired into a target with a velocity of 500 ms^{-1}. If the man of the target is 4.8 kg and in free to move, find the loss of kinetic energy by the impact. [4]

SOLUTION

Here, mass of bullet (m) = 200 gm = 0.2 kg

Velocity of bullet (v) = 500 m/s

Mass of the target (M) = 4.8 kg

Let V be the velocity of the combination immediate after the impact.

Using the principle of conservation of linear momentum,

$mv = (m + M)V$

or, $0.2 \times 500 = (0.2 + 4.8)V$

or, $V = \frac{100}{5} = 20$ m/s

K.E before impact $= \frac{1}{2}mv^2 = \frac{1}{2} \times 0.2 \times (500)^2$

$= 25000$ J

K.E after impact $= \frac{1}{2}(M + m)V^2$

$= \frac{1}{2}(4.8 + 0.2) \times 20^2$

$= \frac{1}{2} \times 5 \times (20)^2 = 1000$ J

Loss of K.E = 25000 J – 1000 J = 24000 J

89. 2068 Q.No. 14 b Or

If a force be applied on the body, prove that the change in kinetic energy of a body is equal to the work done by the force. [4]

➡ Please refer to 2058 Q.No. 14 b OR

90. 2069 (Set A) Q.No. 13b or

Define work done by a force. Prove that the change in kinetic energy of a body is equal to the work done by the force. [4]

➡ First Part: Please refer to 2057 Q.No. 14 b OR
Second Part: Please refer to 2058 Q.No. 14 b OR

91. 2069 (Set A) Old Q.No. 14b or

Define work and energy. Prove that the sum of the kinetic and potential energies of a freely falling body at any instant is constant. [4]

➡ Please refer to 2057 Q.No. 14 b OR

92. 2069 Old (Set B) Q.No. 14b Or

What do you mean by the principle of conservation of energy? Verify its validity for a body falling under gravity. [4]

➡ Please refer to 2057 Q.No. 14 b OR

93. 2070 (Old) Q.No. 14 b Or

A shot of mass m is projected from a gun of mass M by an explosion which generates a kinetic energy E. Show that the gun recoils with a velocity $\sqrt{\frac{2mE}{M(M + m)}}$. [4]

SOLUTION

Let v and V be the velocity of the shot and gun respectively.

Momentum of the shot = mv

Momentum of the gun = MV

We know,

Momentum of the shot = momentum of gun

or, mv = MV

or, $v = \frac{MV}{m}$ ...(i)

Again, we have

E = total kinetic energy

or, $E = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$

or, $E = \frac{1}{2}m\frac{M^2V^2}{m^2} + \frac{1}{2}MV^2$ [using (i)]

or, $E = \frac{1}{2}MV^2\left(\frac{M}{m} + 1\right)$

or, $E = \frac{1}{2}MV^2\left(\frac{M + m}{m}\right)$

or, $2mE = MV^2(M + m)$

or, $V^2 = \frac{2mE}{M(M + m)}$

∴ $V = \sqrt{\frac{2mE}{M(M + m)}}$

Hence, gun recoils with a velocity $\sqrt{\frac{2mE}{M(M + m)}}$.

94. 2070 Set C Q.No. 13 a or

A bullet passes through two planks in succession whose initial velocity is 1200 m/s and loses a velocity of 200m/s in penetrating each plank. Find the ratio of the thickness of the planks assuming that they offer the same average resistance. [4]

SOLUTION

Let s_1 be the thickness of first plane and s_2 be the thickness of second plank.

Here, u = 1200 m/s, v = 1000 m/s

Change in K.E $\frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2}m(v^2 - u^2)$

$= \frac{1}{2}m(1000^2 - 1200^2) = -220000$ m

Also, work done = $F \times s_1$

We know that

work done = change in K.E

$F \times s_1 = -2200000$ ...(i)

Again,

u = 1000 m/s, v = 800 m/s

Change in K.E $= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$= \frac{1}{2}m(v^2 - u^2) = \frac{1}{2}m(800^2 - 1000^2)$

$= -180000$ m

wordone = $F \times s_2$

We have,

work done = change in K.E

$F \times s_2 = -180000$ m ...(ii)

Dividing (i) by (ii), we get

$\frac{F \times s_1}{F \times s_2} = \frac{-220000\text{ m}}{-180000\text{ m}}$

$\frac{s_1}{s_2} = \frac{11}{9}$

∴ $s_1 : s_2 = 11 : 9$

95. 2070 Supp. Q.No. 13 b OR

A particle is slide down a smooth inclined plane. Show that the sum of its kinetic and potential energies always constant throughout its motion. [4]

➡ Please refer to 2061 Q.No. 14 b OR

96. 2071 Old Q.No. 14 b OR

Define work, power and energy. Prove that the sum of the kinetic and potential energy of a freely falling body remains constant throughout the motion. [4]

➡ Please refer to 2057 Q.No. 14 b OR

97. 2071 Set C Q.No. 13 b OR

Find the H.P. of an engine which can travel at the rate of 144km/hr up an incline of 1 in 200, the mass of the engine and load being 15 metric tons and the resistance due to friction etc. being 15 kg weight per metric ton. (g = 10m/sec²). [4]

SOLUTION

Let α be the angle of inclination of the inclined plane with the horizon.

Then $\sin\alpha = \frac{1}{200}$

Again, 144 km/hr $= \frac{144 \times 1000}{60 \times 60} = 40$ m/s

As the engine moves up an inclined plane, the resolved part of the weight of the engine and the resistance force act down an inclined plane.

Mass of the engine (m) = 15 metric tons = 15 × 1000 = 15000 kg

The forces acting on the engine are:

(a) **resolved part of the weight of the engine down an inclined plane**

$= mg\sin\alpha = 15{,}000 \times 10 \times \frac{1}{200} = 750$ N

(b) **resistance force due to friction**

= 15 kg wt per metric ton

= 15 × 10 × 15 N = 2250 N

Resultant force acting on the engine down an inclined plane = (750 + 2250) N = 3000 N.

We have,

Power = Force × velocity = 3000 × 40 = 120000

Required H.P of an engine $= \frac{120000}{746} = 160.86$

98. 2071 Supp. Q.No. 13b OR

A particle is allowed to slide down a smooth inclined plane. Show that the sum of its kinetic and potential energies is always constant throughout its motion. [4]

➡ Please refer to 2061 Q.No. 14 b OR

99. 2072 Set C Q.No. 13b OR

State the principle of conservation of energy. Also prove that the sum of the kinetic and potential energies of a moving body remains constant throughout the motion. [4]

SOLUTION

Principle of Conservation of Energy: Under the action of a conservative system of forces, the sum of the kinetic and potential energies of a moving body remains constant throughout the motion.

Second Part: Please refer to 2057 Q.No. 14 b OR

100. 2074 Set A Q.No. 13b

A particle is allowed to slide down a smooth inclined plane. Show that the sum of its kinetic and potential energies is always constant throught out its motion. [4]

➡ Please refer to 2061 supp Q.No. 14 b OR

101. 2075 Set A Q.No. 13b OR

Define K.E. and P.E. of a body. Prove that the change in the K.E. of a body is equal to the work done by the force. [4]

SOLUTION

First Part

Potential Energy: Potential energy of a body is its capacity of doing work by virtue of its state or position and is measured by the amount of work which it can do in changing from its actual position to some standard position. It is denoted by P.E and given by P.E. = mgh, where m = mass of the body

g = acceleration due to gravity

h = height

Kinetic Energy: Kinetic energy of a body is its capacity to do work by virtue of its motion.

If m be the mass of the body and v be its velocity, then

K.E. $= \frac{1}{2}mv^2$

Second Part: 2058 Q.No. 14 b OR

102. 2075 Set C Q.No. 13b

A bullet of mass 100 g is fired into a target with a velocity of 500 ms^{-1}. The mass of the target is 4.9 kg and is free to move; find the loss of kinetic energy by the impact. [4]

SOLUTION

Here,

m = 100g = 0.1 kg

M = 4.9 kg, v = 500 ms^{-1}

Let V be the velocity of the combination immediate after the impact.

Then, by using the principle of conservation of linear momentum,

we have,

$mv = (m + M)V$

or, $0.1 \times 500 = (0.1 + 4.9)$ V

∴ $V = 10\ ms^{-1}$

KE before the impact $= \frac{1}{2}mv^2$

$= \frac{1}{2} \times 0.1 \times 500^2$

$= 12500$ J

KE after the impact $= \frac{1}{2}(M + m)V^2$

$= \frac{1}{2}(4.9 + 0.1) \times 10^2$

$= \frac{1}{2} \times 5 \times 100$

$= 250$ J.

Loss of KE by the impact $= 12500 - 250$

$= 12250$ J.

6 MARKS QUESTIONS

103. 2069 (Set B) Q.No. 14a Or

A car of mass 2000 kg moves up an inclined plane at an angle 30° to the horizon at a constant speed of 20m/s. If the frictional force is 2000N, calculate the power developed by the engine (g = 10 m/s²). [6]

SOLUTION

Here, mass (m) = 2000 kg

Angle of inclination (α) = 30°

velocity (v) = 20 m/s

Frictional force (F) = 2000 N

Force acting on car (F) = Component of weight down the plane + frictional force

$= mg \sin \alpha + 2000$

$= 2000 \times 10 \times \sin 30° + 2000$

$= 2000 \times 10 \times \frac{1}{2} + 2000 = 12000$ N

We have,

Power = Force × Velocity = 12,000 × 20

= 240000 W = 240 kW

104. 2070 Set D Q.No. 14 Or

A bullet of mass 20g is fired horizontally into a suspended stationary wooden block of mass 380 g with a velocity of 200 m/s. What is the common velocity of the bullet and the block if the bullet is embedded into the block? Find the loss of K.E. by the impact. (g = 10m/s²) [6]

SOLUTION

Here, mass of the bullet (m) = 20g $= \frac{20}{1000}$ kg

Mass of the wooden block (M) = 380 g $= \frac{380}{1000}$ kg

Velocity (v) = 200 m/s

We know that,

Momentum of the shot = momentum of the combination

or, $mv = (m + M)V$

or, $\frac{20}{1000} \times 200 = \left(\frac{20}{1000} + \frac{380}{1000}\right) v$

or, $4 = \frac{400}{1000} v$

∴ $V = 10$ m/s

Initial K.E $= \frac{1}{2}mv^2 = \frac{1}{2} \times \frac{20}{1000} \times (200)^2 = 400$ J

Again,

Final K.E $= \frac{1}{2}(m + M)V^2 = \frac{1}{2} \times \frac{400}{1000} \times (10)^2 = 20$ J

Loss of K.E by the impact = 400 J – 20 J = 380 J

105. 2071 Set D Q.No. 14

Define potential energy and kinetic energy of a body. Prove that the sum of the K.E. and P.E. of a freely falling body at any instant is constant. [6]

SOLUTION

First Part

Potential Energy: Potential energy of a body is its capacity of doing work by virtue of its state or position and is measured by the amount of work which it can do in changing from its actual position to some standard position. It is denoted by P.E and given by

P.E. = mgh, where m = mass of the body

g = acceleration due to gravity

h = height

Kinetic Energy: Kinetic energy of a body is its capacity to do work by virtue of its motion. If m be the mass of the body and v be its velocity, then

$K.E. = \frac{1}{2}mv^2$

Second Part: Please refer to 2057 Q.No. 14 b OR

106. 2072 Set D Q.No. 15

Define energy. State principle of conservation of energy. Also prove that the sum of the kinetic and potential energy of a moving body remains constant throughout the motion. [6]

SOLUTION

Energy: Energy of a body is its capacity of doing work.

Principle of Conservation of Energy: Under the action of a conservative system of forces, the sum of the kinetic and potential energies of a moving body remains constant throughout the motion.

Next part: Please refer to 2057 Q.No. 14 b

107. 2072 Supp Q.No. 14 OR

Define kinetic and potential energies of a body. Prove that the sum of the kinetic and potential energies of a freely falling body remains constant throughout the motion. [6]

SOLUTION

First Part:

Kinetic Energy: Kinetic energy of a body is its capacity to do work by virtue of its motion. If m be the mass of the body and v be its velocity, then

$K.E. = \frac{1}{2}mv^2$

Potential Energy: Potential energy of a body is its capacity of doing work by virtue of its state or position and is measured by the amount of work which it can do in changing from its actual position to some standard position. It is denoted by P.E and given by

P.E. = mgh, where m = mass of the body

g = acceleration due to gravity

h = height

Second Part: Please refer to 2057 Q.No. 14 b OR

108. 2073 Set D Q.No. 14 OR

800 kg of air, moving at 20m/s, imping on the vanes of a windmill every second. At what rate in kilowatt is the energy arriving at the windmill? What is the maximum mass of water that could be pumped each second through a vertical height of 2.5 m?(g = 10 m/s²) [6]

SOLUTION

Mass (m) = 800 kg

Velocity (v) = $20\ ms^{-1}$

We have,

$K.E. = \frac{1}{2}mv^2 = \frac{1}{2} \times 800 \times (20)^2 = 160000$ J

∴ Work done = 160000 watt $= \frac{160000}{1000}$ kw = 160 kw

Again, let M be the maximum mass of water that could be pumped each second through the vertical height of 2.5 m.

Then, work done = mgh

or, $160000\ J = M \times 10 \times 2.5$

∴ $M = \frac{160000}{10 \times 2.5} = 6400\ kgs^{-1}$

109. 2073 Supp Q.No. 16

State the principle of conservation of energy. Also prove that the sum of the Kinetic and Potential energies of a moving body remains constant throughout the motion. [6]

➥ **First Part:** Please refer 2072 Set D Q.No. 15

Second Part: Please refer 2057 Q.No. 14 OR

110. 2074 Set B Q.No. 15 OR

Define energy. State principle of conservation of energy. Also prove that the sum of the kinetic and potential energies of a moving body remains constant throughout the vertical motion. [6]

➥ **First Part:** Please refer 2072 Set D Q.No. 15

Second Part: Please refer 2057 Q.No. 14 OR

111. 2074 Supp Q.No. 14 OR

A bullet passes through two planks in succession. Its initial velocity is 1200 m/s and it loses a velocity of 200 m/s in penetrating through each plank. Find the ratio of the thickness of the planks, assuming that they offer the same average resistance. [6]

➥ Please refer to 2070 Set C Q.No. 13a OR

112. 2075 Set B Q.No. 15a

a. A rocket expels gas at the rate of 0.4 kg/s. If the velocity of the gas is 400 m/s, what is the force produced by the rocket?

b. A particle is projected with a velocity of 49 m/s, at an angle of 30° to the horizon, find the time of flight and the range.

c. A pump having a power of 392 W pumps water at the rate of 100 litres per minute. Find the height to which water is raised. (g = 9.8 m/s², 1 litre = 1kg) [6]

SOLUTION

a. Here,

Expel rate $\left(\frac{m}{t}\right) = 0.4$ kg/s

Initial velocity (u) = 400 m/s

Final velocity (v) = 0

Force produced by a rocket (F) = ?

We have,

$-F = \frac{m}{t}(v - u)$

$= 0.4 (0 - 400) = -160$

∴ F = 160 N

b. Velocity of projection (u) = 49 m/s

Angle of projection (α) = 30°

Time of flight (T) = ?

Range (R) = ?

We have,

$T = \frac{2u \sin\alpha}{g} = \frac{2 \times 49 \times \sin 30}{9.8}$

$= \frac{2 \times 49 \times \frac{1}{2}}{9.8} = 5$ sec

Again, we have

$R = \frac{u^2 \sin 2\alpha}{g} = \frac{(49)^2 \sin 60°}{9.8}$

$= \frac{2401 \times \frac{\sqrt{3}}{2}}{9.8} = 212.18$m

c. Please refer to Model Set 1 Q.No. 12c

113. 2075 Set C Q.No. 15

Define kinetic energy and potential energy with examples. A shot of mass 'm' is projected from a gun of mass 'M' by an explosion, which generates a kinetic energy E. Find (i) the initial velocity of the shot (ii) the velocity of the gun. [6]

SOLUTION

First part: Please refer to 2071 Set D Q.No. 14

Second part:

Let v and V be the initial velocities of the shot and gun respectively.

Momentum of the shot = mv

Momentum of the gun = MV

We know,

Momentum of the shot = momentum of gun

or, $mv = MV$

or, $v = \frac{MV}{m}$ ...(i)

Again, we have

E = total kinetic energy

or, $E = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$

or, $E = \frac{1}{2}m\frac{M^2V^2}{m^2} + \frac{1}{2}MV^2$ [using (i)]

or, $E = \frac{1}{2}MV^2\left(\frac{M}{m} + 1\right)$

or, $E = \frac{1}{2}MV^2\left(\frac{M+m}{m}\right)$

or, $2mE = MV^2(M + m)$

or, $V^2 = \frac{2mE}{M(M+m)}$

$\therefore V = \sqrt{\frac{2mE}{M(M+m)}}$

Hence, gun recoils with a velocity $\sqrt{\frac{2mE}{M(M+m)}}$.

And, initial velocities of the shot,

$v = \frac{MV}{m}$

$= \frac{M}{m}\sqrt{\frac{2mE}{M(M+m)}}$

$= \sqrt{\frac{2ME}{m(M+m)}}$

UNIT 16

LINEAR PROGRAMMING

A. GRAPHICAL METHOD

2 MARKS QUESTIONS

1. 2056 Q.No. 3c

Graph the half plane given by: $y - x \geq 1$ [2]

SOLUTION

The given inequality is $y - x \geq 1$

The boundary line is $y - x = 1$

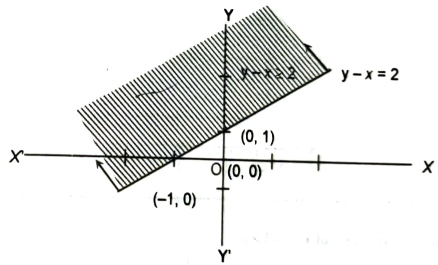
x	0	-1
y	1	0

Taking (0, 0) as a testing point, we get

$0 - 0 \geq 1$ (False)

Hence, the half plane determined by the given inequality does not contain origin.

Now, We draw the graph.



2. 2057 Q.No. 3c

Find the solution set of $4x + 3 \geq 2x - 1$. [2]

SOLUTION

We have,

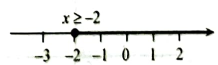
$4x + 3 \geq 2x - 1$

or, $4x - 2x \geq -1 - 3$

or, $2x \geq -4$

or, $\frac{2x}{2} \geq \frac{-4}{2}$

or, $x \geq -2$



$\therefore$ The required solution set = $\{x: x \geq -2\}$

3. 2058 Q.No. 3c

Determine the half plane given by the inequality $2x - y < 2$. [2]

SOLUTION

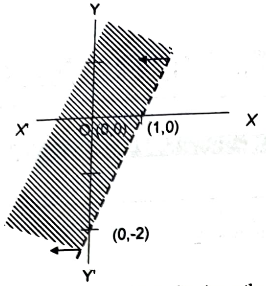
The given inequality is $2x - y < 2$

The corresponding boundary line is $2x - y = 2$

x	0	1
y	- 2	0

Take (0, 0) as a testing point, we get

$2 \times 0 - 0 < 2$ (True)



Hence, the half plane determined by the given inequality is on the same side of testing point (0, 0). But it does not contain the boundary line. So on the graph we draw the dotted boundary line.

4. 2059 Q.No. 3c

Determine the solution set of $2x - 1 > 4x + 3$ [2]

SOLUTION

We have,

$2x - 1 > 4x + 3$

$\Rightarrow 2x - 1 + 1 > 4x + 3 + 1$

$\Rightarrow 2x > 4x + 4$

$\Rightarrow 2x - 4x > 4x - 4x + 4$

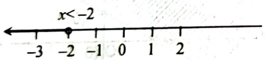
$\Rightarrow -2x > 4$

$\Rightarrow 2x < -4$

$\Rightarrow \frac{2x}{2} < -\frac{4}{2}$

$\Rightarrow x < -2$

$\therefore$ Solution set = $\{x: x < -2, x \in \Re\}$



5. 2060 Q.No. 3c

Graphically show the solution of $x - y - 3 > 0$. [2]

SOLUTION

The corresponding boundary line is

$x - y - 3 = 0$ i.e. $x - y = 3$

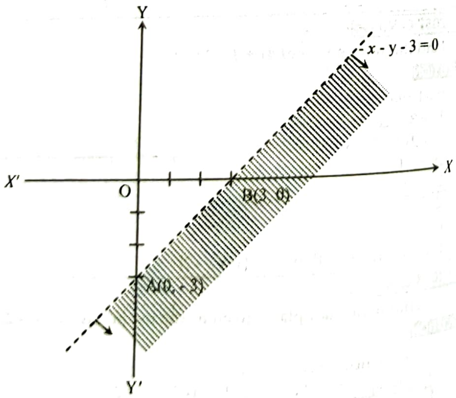
x	0	3
y	-3	0

Taking (0, 0) as a testing point,

$0 - 0 - 3 > 0$

i.e. $-3 > 0$ (false)

So, the half plane determined by the given inequality does not contain the testing point (0, 0). The solution of given inequality is shaded as shown in the graph below:

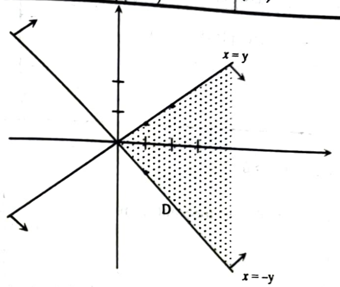


6. 2061 Q.No. 3c

Solve graphically: $x \geq y$ and $x \geq -y$ [2]

SOLUTION

Linear inequalities	Boundary line	Points	Testing point	Result
$x \geq y$	$x = y$	(1, 1) (2, 2)	(1, 0)	$1 \geq 0$ (true)
$x \geq -y$	$x = -y$	(1, -1) (2, -2)	(1, 0)	$1 \geq 0$ (true)



Hence, the solution set is shaded on the graph.

7. 2062 Q.No. 3c

Determine the half plane given by the inequality. $2x - y < 2$, graphically. [2]

SOLUTION

Please See 2058 Q.No. 3c

8. 2065 Q. No. 3 c

Solve graphically: $y \geq 2x - 1$ [2]

SOLUTION

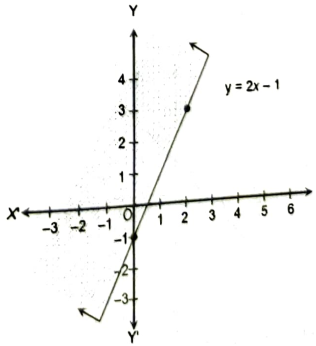
The corresponding equations of boundary line is:

$y = 2x - 1$

x	0	2
y	-1	3

Taking testing point (0, 0) in $y \geq 2x - 1$, we have

$0 \geq 2 \times 0 - 1$ (true)



9. 2066 Q.No. 3c

Determine the half plane given by the inequality $y \geq -x$. [2]

SOLUTION

The corresponding equation of given inequality is:

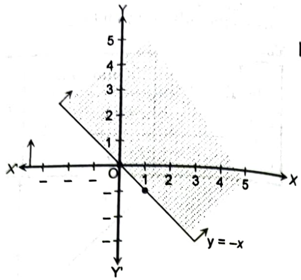
$y = -x$

x	0	1
y	0	–1

Taking testing point (1, 0) in $y \geq -x$, we have

$0 \geq -1$ (true)

So, the graph of $y \geq -x$ is the plane region containing the point (1, 0)



10. 2067 Q.No. 3 c

Solve graphically: $x - y \leq 0$ and $x \geq 0$ [2]

SOLUTION

The equations of boundary lines of given inequalities are:

$x - y = 0$ ...(i)

$x = 0$ ...(ii)

From (i)

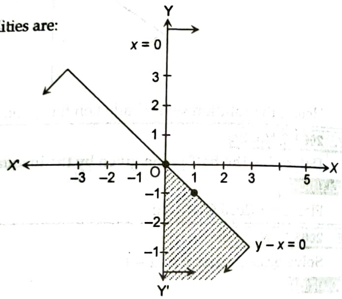
x	0	1
y	0	–1

Taking testing point (1, 0) in $x - y \leq 0$, we have

$1 - 0 \leq 0$ (false)

So, the graph of $x - y \leq 0$ is the plane region not containing the point (1, 0)

From (ii) $x = 0$ is y-axis and $x \geq 0$ gives the right half plane including y-axis.



11. 2068 Q.No. 3c

Determine the half plane represented b y the inequality $y - x \geq 1$ [2]

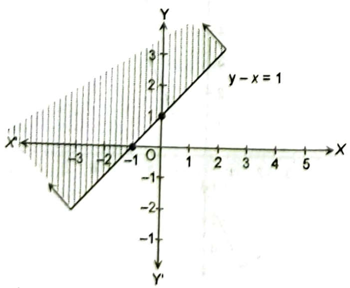
SOLUTION

The boundary line of given inequality is $y - x = 1$

x	0	–1
y	1	0

Taking testing point (0,0) in $y - x \geq 1$, we get

$0 - 0 \geq 1$ (false)



So, the graph $y - x \geq 1$ is the plane region without containing the origin.

12. 2069 (Set A) Q.No. 16a

Shade the feasible region determined by the following inequalities: $3x + 2y \leq 12, x + y \leq 5, x, y \geq 0$ [2]

SOLUTION

The corresponding equations of boundary lines are:

$3x + 2y = 12$ ... (i)

$x + y = 5$ ... (ii)

$x = 0$ ... (iii)

$y = 0$ ...(iv)

From (i) $3x + 2y = 12$

x	0	4
y	6	0

Taking testing point (0,0) in $3x + 2y \leq 12$, we get

$3 \times 0 + 2 \times 0 \leq 12$ (true)

From (ii) $x + y = 5$

x	0	5
y	5	0

Taking testing point (0,0) in $x + y \leq 5$, we get

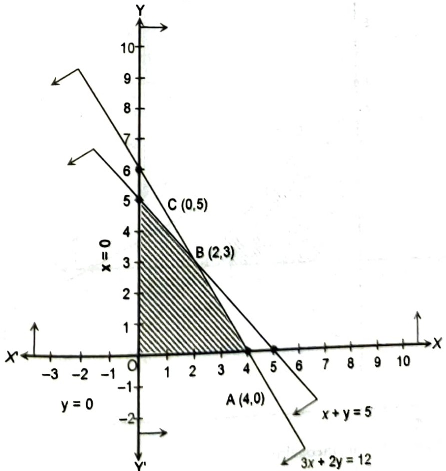
$0 + 0 \leq 12$ (true)

From (iii) $x = 0$ which is y-axis.

$x \geq 0$ gives the right half plane including y-axis.

From (iv) $y = 0$ which is x-axis.

$y \geq 0$ gives the upper half plane including x-axis.



13. 2069 (Set B) Q.No. 16a

Shade the feasible region determined by the inequalities $x + 2y \leq 10, x + y \leq 6, x, y \geq 0$. [2]

SOLUTION

The corresponding equations of boundary lines are

$x + 2y = 10$ ... (i)

$x + y = 6$ ... (ii)

$x = 0$ ... (iii)

$y = 0$ ... (iv)

From (i) $x + 2y = 10$

x	0	10
y	5	0

Taking testing point (0,0) in $x + 2y \leq 10$, we get

$0 + 2 \times 0 \leq 10$ (true)

From (ii) $x + y = 6$

x	0	6
y	6	0

Taking testing point (0,0) in $x + y \leq 6$, we get

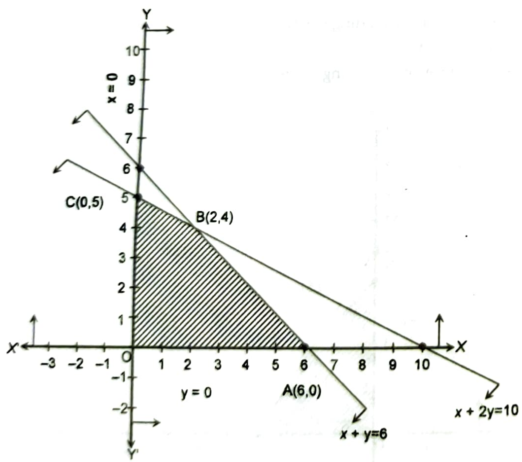
$0 + 0 \leq 6$ (true)

From (iii) $x = 0$ which is y-axis.

$x \geq 0$ gives the right half plane including y-axis.

From (iv) $y = 0$ which is x-axis.

$y \geq 0$ gives the upper half plane including x-axis.



14. 2070 Set C Q.No. 16 a

Draw the graph of the following inequalities: [2]

$3x + 4y \leq 24$, $0 \leq y \leq 4$, $0 \leq x \leq 7$.

SOLUTION

The boundary equations of given inequalities are

$3x + 4y = 24$ ...(i)

$y = 0$ ...(ii)

$y = 4$ ...(iii)

$x = 0$ ... (iv)

$x = 7$ ...(v)

From (i) $3x + 4y = 24$

x	0	8
y	6	0

Taking testing point (0,0) in $3x + 4y \leq 24$, we get

$3 \times 0 + 4 \times 0 \leq 24$ (true)

The graph of $3x + 4y \leq 24$ contains origin.

From (ii) $y = 0$ which is x-axis.

$y \geq 0$ gives the upper half plane containing x-axis.

From (iii) $y = 4$ is the line parallel to x-axis and through the point (0, 4)

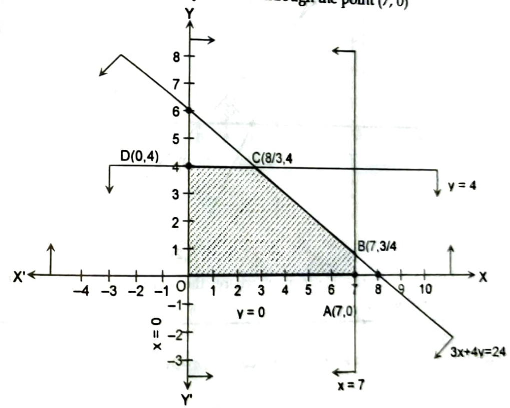
Taking testing point (0,0) in $y \leq 4$, we get

$0 \leq 4$ (true)

From (iv) $x = 0$ which is y-axis.

$x \geq 0$ gives the right half plane containing the y-axis.

From (v) $x = 7$ is the line parallel to y-axis and through the point (7, 0)



The required solution is shaded in the figure.

15. 2070 Set D Q.No. 16 a

Draw the graph of the following inequalities. [2]

$x + y \leq 6$, $2x + y \geq 8$, $y \geq 0$.

SOLUTION

The corresponding equations of boundary lines are

$x + y = 6$ ...(i)

$2x + y = 8$ ...(ii)

$y = 0$ ... (iii)

From (i) $x + y = 6$

x	0	6
y	6	0

Taking testing point (0, 0) in $x + y \leq 6$, we get

$0 + 0 \leq 6$ (true)

So, the graph of $x + y \le 6$ is the plane region containing the origin.

Again, from (ii) $2x + y = 8$

x	0	4
y	8	0

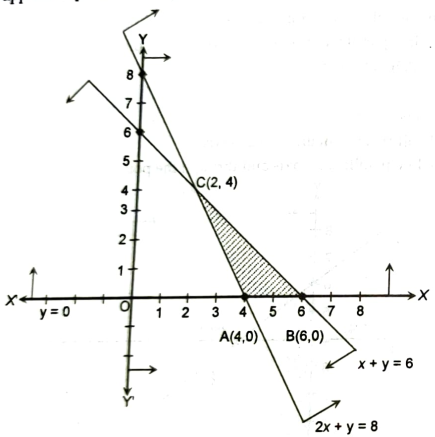
Taking testing point (0, 0) in $2x + y \ge 8$, we get

$2 \times 0 + 0 \ge 8$ (false)

So, the graph of $2x + y \ge 8$ is the plane region without containing the origin.

From (iii) $y = 0$ which is x-axis.

$y \ge 0$ gives the upper half plane including x-axis.



From the figure ABC is the feasible region where the coordinates of A, B and C are (4, 0), (6, 0) and (2, 4) respectively.

16. 2070 Supp. Q.No. 16 a

Determine graphically the solution set of the inequality: $2x - 3y \le 6$. [2]

SOLUTION

The corresponding equation of boundary line is

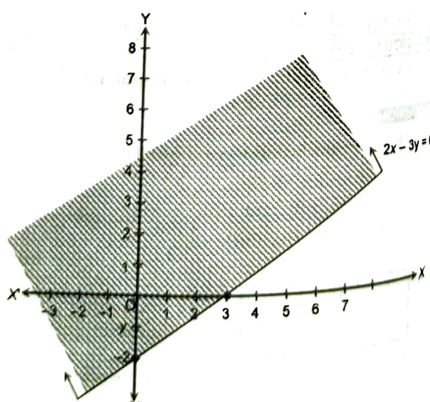
$2x - 3y = 6$

x	0	3
y	-2	0

Taking testing point (0,0) in $2x - 3y \le 6$, we get

$2 \times 0 - 3 \times 0 \le 6$ (true)

So, the graph of $2x - 3y \le 6$ is the plane region containing the origin.



17. 2071 Set C Q.No. 16 a

Determine the feasible region of the following system of inequalities: $2x + y \le 8$, $x + 2y \le 10$, $x, y \ge 0$ [2]

SOLUTION

The corresponding equations of boundary lines are

$2x + y = 8$ …(i)

$x + 2y = 10$ … (ii)

$x = 0$ … (iii)

$y = 0$ … (iv)

From (i) $2x + y = 8$

x	4	0
y	0	8

Taking testing point (0, 0) in $2x + y \le 8$, we get

$2 \times 0 + 0 \le 8$ (true)

Again, from (i) $x + 2y = 10$

x	0	10
y	5	0

Taking testing point (0, 0) in $x + 2y \le 10$, we get

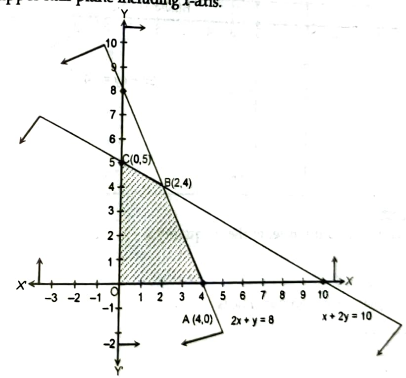
$0 + 2 \times 0 \le 10$ (true)

From (ii) $x = 0$ is y-axis

$x \ge 0$ gives the right half plane including y-axis.

And, from (iv) $y = 0$ which is x-axis.

$y \ge 0$ gives the upper half plane including x-axis.



The required feasible region is shaded in the graph.

18. 2071 Set D Q.No. 16 a

Determine graphically the feasible region determined by the following inequalities: $3x + 4y \le 24$, $x \ge 2$, $y \ge 1$ [2]

SOLUTION

The corresponding equations of boundary lines are

$3x + 4y = 24$ …(i)

$x = 2$ … (ii)

$y = 1$ …(iii)

From (i) $3x + 4y = 24$

x	0	8
y	6	0

Taking testing point (0, 0) in $3x + 4y \le 24$, we get

$3 \times 0 + 4 \times 0 \le 24$ (true)

From (ii) $x = 2$

x	2	2
y	0	1

Taking testing point (0, 0) in $x \geq 2$, we get

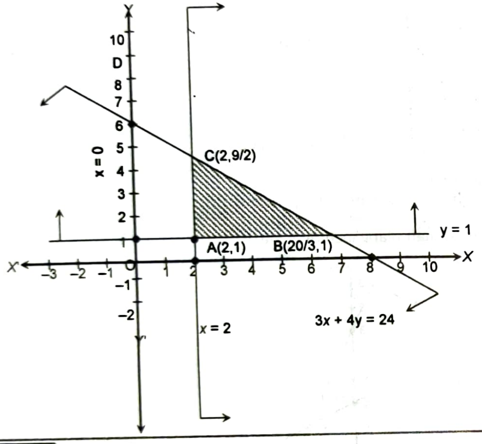
$0 \geq 2$ (false)

From (iii) $y = 1$

x	0	3
y	1	1

Taking testing point (0, 0) in $y \geq 1$, we get

$0 \geq 1$ (false)



19. 2071 Supp. Q.No. 16a

Determine graphically the solution set of the inequality $x - 5y \leq 5$. [2]

SOLUTION

Given inequality is $x - 5y \leq 5$

The corresponding equation of boundary line is

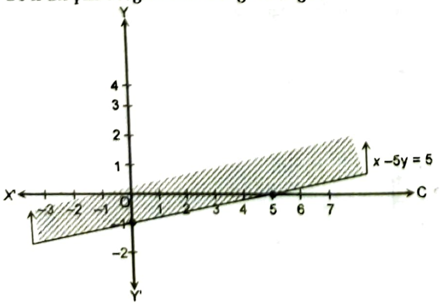
$x - 5y = 5$

x	5	0
y	0	–1

Taking testing point (0, 0) in $x - 5y \leq 5$, we get

$0 - 5 \times 0 \leq 5$ (true)

The graph of $x - 5y \leq 5$ is the plane region containing the origin.



20. 2072 Set C Q.No. 16a

Shade the feasible region for the constraints $x + 2y \leq 7, x, y \geq 0$ [2]

SOLUTION

The corresponding equation of boundary lines are

$x + 2y = 7$ ... (i)

$x = 0$ ...(ii)

$y = 0$ ...(iii)

(i) $x + 2y = 7$

x	7	1
y	0	3

Taking testing point (0, 0) in $x + 2y \leq 7$, we have

$0 + 2 \times 0 \leq 7$(true)

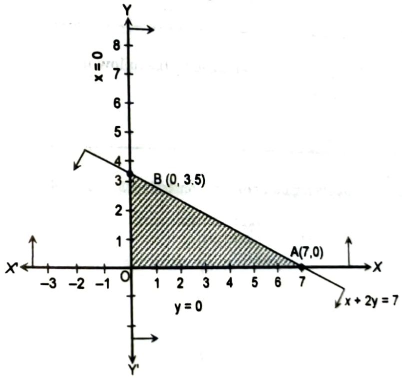
So, the plane region determined by $x + 2y \leq 7$ is the plane region containing the origin.

From (ii) $x = 0$ which is y-axis.

And, $x \geq 0$ gives the right half plane including y-axis.

From (iii) $y = 0$ which is x-axis.

And, $y \geq 0$ gives the upper half plane including x-axis.



21. 2072 Set D Q.No. 16a

Draw the graph of the inequality: $3x - 3 \leq 5x - y$. [2]

SOLUTION

Given inequality is

$3x - 3 \leq 5x - y$

or, $-3 \leq 5x - 3x - y$

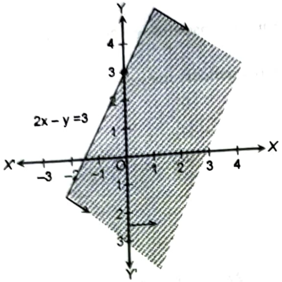
or, $2x - y \geq -3$

The corresponding equation of boundary lines is $2x - y = -3$

x	0	–1
y	3	1

Taking testing point (0, 0) in $2x - y \geq -3$, we get

$2 \times 0 - 0 \geq -3$ (true)



So, the graph of given inequality is the plane region containing the origin.

22. 2072 Set E Q.No. 16a

Determine the feasible region bounded by the following system of inequalities:
$x + y \le 6$, $2x + y \ge 8$, $y \ge 0$. [2]

SOLUTION

Please see 2070 Set D Q.No. 16 a

23. 2072 Supp Q.No. 16a

Find the vertices of the feasible region determined by the following inequalities:
$2x + y \le 8$, $x + 2y \le 10$ and $x, y \ge 0$. [2]

SOLUTION

Please see 2071 Set C Q.No. 16 a

24. 2073 Set C Q.No. 16a

Find the vertices of the feasible region under the constraints $3x + 2y \le 48$, $x + y \le 20$; $x, y \ge 0$. [2]

SOLUTION

The corresponding equations of boundary lines are:

$3x + 2y + 48$ ...(i)

$x + y = 20$ ...(ii)

$x = 0$ ...(iii)

$y = 0$ ...(iv)

From (i), $3x + 2y = 48$

x	0	16
y	24	0

Taking testing point (0, 0) in $3x + 2y \le 48$, we get

$3 \times 0 + 2 \times 0 \le 48$ (true)

From (ii) $x + y = 20$

x	0	20
y	20	0

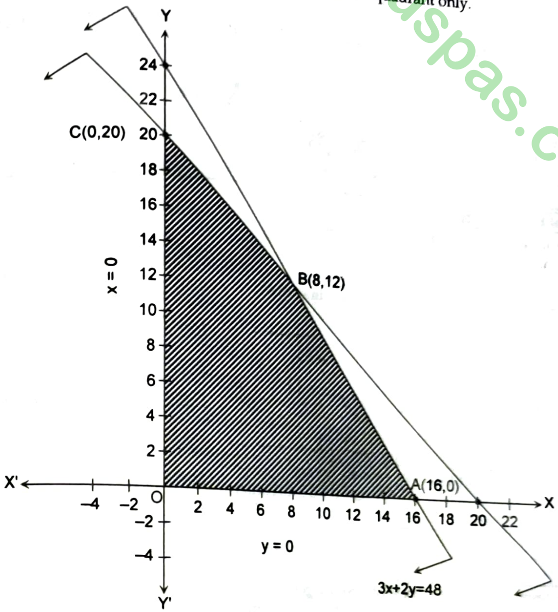
Taking testing point (0, 0) in $x + y \le 20$, we get

$0 + 0 \le 20$ (true)

From (iii) $x = 0$, which is the equation of y-axis.

From (iv) $y = 0$, which is the equation of x-axis.

$x \ge 0$ and $y \ge 0$ means we have to consider the region in first quadrant only.



From the figure OABC is the feasible region where the coordinates of O, A, B and C are (0, 0), (16, 0), (8, 12) and (0, 20) respectively.

25. 2073 Set D Q.No. 16a

Find the feasible region determined by the inequalities
$2x + y \le 8$, $x + 2y \le 10$, $x, y \ge 0$. [2]

SOLUTION

Please see 2071 Set C Q.No. 16 a

26. 2073 Supp Q.No. 16a

Write the procedure of solving a linear programming problem by the graphical method. [2]

SOLUTION

The procedure of solving a linear programming problems by the graphical method are as follows:

i. Formulate the given linear programming problem into the mathematical form if necessary.
ii. Express the constraints (inequalities) into the corresponding equations.
iii. Find the feasible region from the graphs of the equations obtained in (ii).
iv. Determine the vertices of feasible region.
v. Evaluate the value of the objective function at each of the vertices of the feasible region.
vi. Find the optimal solution

27. 2074 Set A Q.No. 16a

Determine graphically the solution set of the inequality $x - 3y \le 3$. [2]

SOLUTION

The corresponding equation of boundary line is $x - 3y = 3$

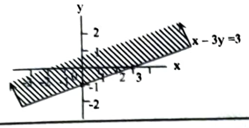
x	0	3
y	-1	0

So, the boundary line passes through the points (0, -1) and (3, 0).

Taking testing point (0, 0) in $x - 3y \le 3$, we get

$0 - 3 \times 0 \le 3$ (true)

So, the solution set of given inequality is the closed half plane region containing the origin as shown below.



28. 2074 Set B Q.No. 16a

Shade the feasible region under the constraints

$2x + y \le 40$, $x + 2y \le 50$, $x, y \ge 0$ [2]

SOLUTION

The corresponding equations of boundary lines are

$2x + y = 40$ ...(i)

$x + 2y = 50$ ...(ii)

$x = 0$ ...(iii)

$y = 0$ ...(iv)

From (i) $2x + y = 40$

x	0	20
y	40	0

Taking testing point (0, 0) in $2x + y \le 40$, we get,

$2 \times 0 + 0 \le 40$ (True)

From (ii) $x + 2y = 50$

x	0	50
y	25	0

Taking testing point (0, 0) in $x + 2y \le 50$, we get

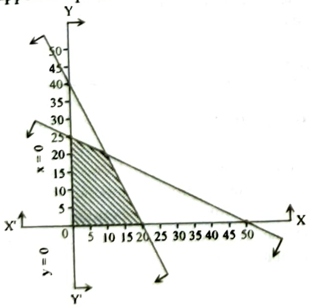
$0 + 2 \times 0 \le 50$ (True)

From (iii) $x = 0$ which is Y-axis

And $x \ge 0$ gives the closed right half plane.

From (iv) $y = 0$ which is X-axis

And $y \ge 0$ gives the closed upper half plane.



The Feasible region is shaded.

29. 2074 Supp Q.No. 16a [2]

Find the feasible region determined by the inequalities $3x + 4y \le 24$, $0 \le y \le 4$, $0 \le x \le 7$.

➡ Please refer to 2070 Set C Q.N. 16a

30. 2075 Set A Q.No. 16a [2]

Find the feasible region determined by the inequalities $x + y \le 6$, $x - y \ge -2$, $x \ge 0$, $y \ge 0$.

SOLUTION

The corresponding equations of given inequalities are

$x + y = 6$ ... (i)

$x - y = -2$ ... (ii)

$x = 0$ ... (iii)

$y = 0$ ... (iv)

From (i), $x + y - 6$

x	0	6
y	6	0

Taking testing point (0, 0) in $x + y \le 6$, we get

$0 + 0 \le 6$ (true)

So, it contains origin.

From (ii), $x - y = -2$

x	0	-2
y	2	0

Taking testing point (0, 0) in $x - y \ge -2$, we get,

$0 - 0 \ge -2$

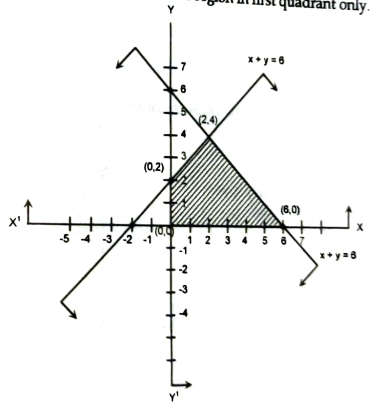
$0 \ge -2$ (true)

So, it contains origin

From (iii) $x = 0$ which is y – axis

From (iv) $y = 0$ which is x – axis

And, $x \ge 0$, $y \ge 0$ means we have to consider the region in first quadrant only.



31. 2075 Set B Q.No. 16a

Determine graphically the solution set of $2x + y \ge 2$, $x \ge 0$, $y \ge 0$. [2]

SOLUTION

The corresponding equations of boundary lines are

$2x + y = 2$ ... (i)

$x = 0$ ... (ii)

$y = 0$ ... (iii)

From (i) $2x + y = 2$

x	0	1
y	2	0

Taking testing point (0, 0) in $2x + y \ge 2$, we get,

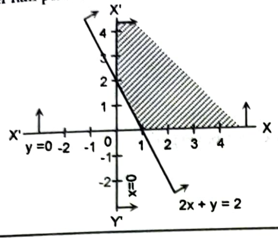
$2 \times 0 + 0 \ge 2$

$0 \ge 2$ (false)

The solution set of $2x + y \ge 2$ is the closed plane region without containing the origin.

From (ii) $x = 0$ which is y - axis.

and, $x \geq 0$ is the closed right half plane.
From (iii) $y = 0$ which is the x - axis.
and, $y \geq 0$ is the closed upper half plane.



32. 2075 Set C Q.No. 16a

Shade the feasible region bounded by $x + y \leq 6$, $x, y \geq 0$. [2]

SOLUTION

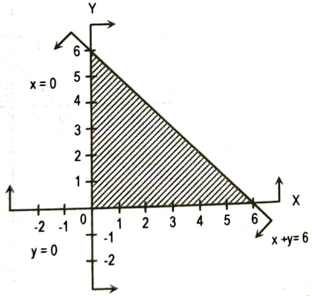
The corresponding equations of boundary lines are

$x + y = 6$ ... (i)
$x = 0$ ... (ii)
$y = 0$ ... (iii)

From (i), $x + y = 6$

x	0	6
y	6	0

Taking testing point (0, 0) in $x + y \leq 6$, we get
$0 + 0 \leq 6$ (true)
So, the region determined by $x + y \leq 6$ is the closed half plane containing the origin.
From (ii) $x = 0$ which is y–axis.
And, $x \geq 0$ gives the closed right half plane.
From (iii) $y = 0$ which is x-axis.
And $y \geq 0$ gives the closed upper half plane.



4 MARKS QUESTIONS

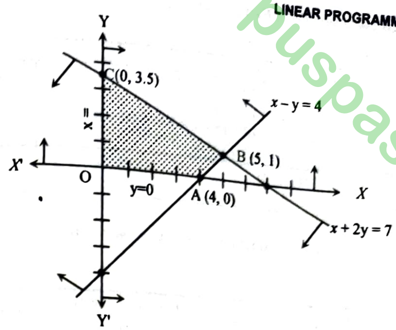
33. 2056 Q.No. 14b

Maximize and minimize the function $F(x, y) = 9x + 7y$.
Subject to constraints $x + 2y \leq 7; x - y \leq 4; x \geq 0; y \geq 0$. [4]

SOLUTION

Linear inequalities	Boundary lines	Points	Testing points	Result
$x + 2y \leq 7$	$x + 2y = 7$	(0, 3.5) (7, 0)	(0, 0)	$0 \leq 7$ (True)
$x - y \leq 4$	$x - y = 4$	(0, – 4) (4, 0)	(0, 0)	$0 \leq 4$ (True)

The inequalities $x \geq 0$, $y \geq 0$ indicate that the common region lies on the first quadrant.



Hence the common region is quadrilateral AOBC, which is shaded on the graph.
Further, in order to maximize and minimize the given function
$F(x, y) = 9x + 7y$;

Vertices	F (x, y) = 9x + 7y	Remarks
A (0, 3.5)	$F = 9 \times 0 + 7 \times 3.5 = 24.5$	
O (0, 0)	$F = 0$	Minimum
B (4, 0)	$F = 9 \times 4 + 7 \times 0 = 36$	
C (5, 1)	$F = 9 \times 5 + 7 \times 1 = 52$	Maximum

Maximum value = 52 at E (5, 1)
Minimum value = 0 at O (0, 0)

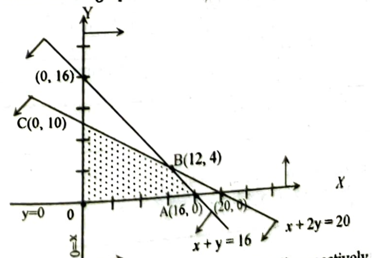
34. 2057 Q.No. 14b

Find the extreme values of the function G(x, y) defined by $G(x, y) = 10x + 15y$ over the convex polygon given by the inequalities. $x + 2y \leq 20; x + y \leq 16; x \geq 0; y \geq 0$ [4]

SOLUTION

Linear inequalities	Boundary line	Points	Testing point	Result
$x + 2y \leq 20$	$x + 2y = 20$	(0, 10) (20, 0)	(0, 0)	$0 \leq 20$ (True)
$x + y \leq 16$	$x + y = 16$	(0, 16) (16, 0)	(0, 0)	$0 \leq 16$ (True)

The inequalities $x \geq 0$; $y \geq 0$ indicate that the common region (i.e. solution set) lies on first quadrant.
Now we draw the graph.
The solution set is shaded on the graph which is quadrilateral OABC.



The coordinates of O, A, B, C are (0, 0), (16, 0), (12, 4) and (0, 10) respectively.

Again,

Vertices	G = 10x + 15y	Remarks
O (0, 0)	G = 10 × 0 + 15 × 0 = 0	Minimum
A (16, 0)	G = 10 × 16 + 15 × 0 = 160	
B (12, 4)	G = 10 × 12 + 15 × 4 = 120 + 60 = 180	Maximum
C (0, 10)	G = 10 × 0 + 15 × 10 = 150	

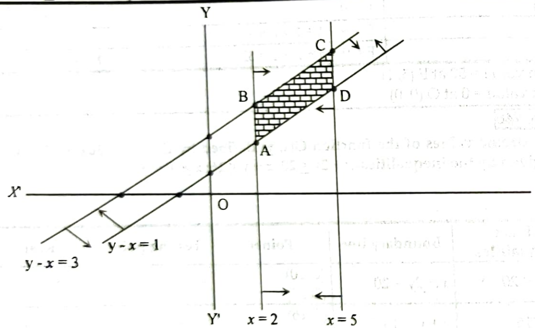
∴ Maximum value of G = 180 at (12, 4)
Minimum value of G = 0 at (0, 0)

35. 2058 Q.No. 14b

Maximize and minimize the function F = 9x + 40y subject to constraints y - x ≥ 1, y - x ≤ 3, 2 ≤ x ≤ 5. [4]

SOLUTION

Linear inequalities	Boundary line	Points	Testing point	Result
y - x ≥ 1	y - x = 1	(0, 1) (-1, 0)	(0, 0)	0 ≥ 1 (False)
y - x ≤ 3	y - x = 3	(0, 3) (-3, 0)	(0, 0)	0 ≤ 3 (True)
x ≥ 2	x = 2	-	(0, 0)	0 ≥ 2 (False)
x ≤ 5	x = 5	-	(0, 0)	0 ≤ 5 (True)



Hence the common region is the parallelogram ABCD which is shaded on the graph.
Here, the coordinates of A, B, C, D are (2, 3), (2, 5), (5, 8) and (5, 6) respectively.
In order to maximize and minimize the function F = 9x + 40y, we have

Vertices	F = 9x + 40y	Remarks
A(2, 3)	F = 9 × 2 + 40 × 3 = 138	Minimum
B(2, 5)	F = 9 × 2 + 40 × 5 = 218	
C(5, 8)	F = 9 × 5 + 40 × 8 = 365	Maximum
D(5, 6)	F = 9 × 5 + 40 × 6 = 285	

Hence the maximum value = 365 at (5, 8) and
Minimum value = 138 at (2, 3).

36. 2059 Q.No. 14b

Maximize and minimize the function F(x, y) = 9x + 7y
Subject to constraints x + 2y ≤ 7, x - y ≤ 4; x ≥ 0; y ≥ 0 [4]

➥ Please see 2056 Q.No. 14b

37. 2060 Q.No. 14b

Maximize and minimize F = 10x + 15y subject to x + 2y ≤ 20, x + y ≤ 16, x ≥ 0, y ≥ 0 [4]

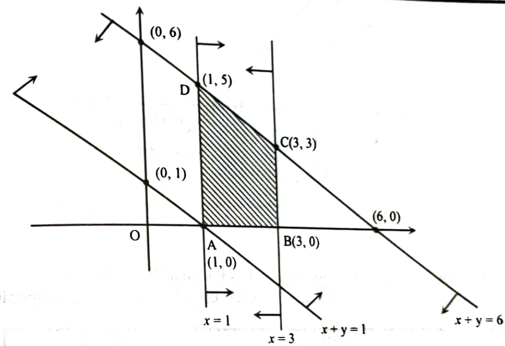
➥ Please see 2057 Q.No. 14b

38. 2061 Q.No. 14b

Maximize and minimize the function F = 34x + 6y subject to the constraints x + y ≥ 1, x + y ≤ 6 and 1 ≤ x ≤ 3 [4]

SOLUTION

Linear inequalities	Boundary lines	Points	Testing point	Results
x + y ≥ 1	x + y = 1	(0, 1) (1, 0)	(0, 0)	0 ≥ 1 (False)
x + y ≤ 6	x + y = 6	(0, 6) (6, 0)	(0, 0)	0 ≤ 6 (True)
x ≥ 1	x = 1	-	(0, 0)	0 ≥ 1 (False)
x ≤ 3	x = 3	-	(0, 0)	0 ≤ 3 (True)



The common region is shaded on the graph, which is quadrilateral ABCD. The coordinates of A, B, C and D are (1, 0), (3, 0), (3, 3) and (1, 5) respectively.

Vertices	F = 34x + 6y	Remarks
A(1, 0)	F = 34 × 1 + 6 × 0 = 34	Minimum
B(3, 0)	F = 34 × 3 + 6 × 0 = 102	
C(3, 3)	F = 34 × 3 + 6 × 3 = 120	Maximum
D(1, 5)	F = 34 × 1 + 6 × 5 = 64	

∴ The maximum value of F = 120 at (3, 3)
The minimum value of F = 34 at (1, 0).

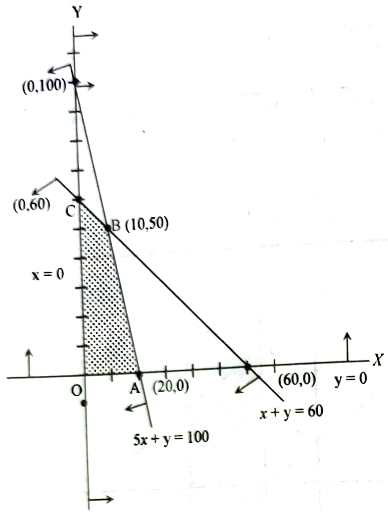
39. 2062 Q.No. 14b

Maximize: F = 50x + 15y, subject to x + y ≤ 60, 5x + y ≤ 100, x ≥ 0, y > 0 [4]

SOLUTION

Linear inequalities	Boundary line	Points	Testing point	Result
x + y ≤ 60	x + y = 60	(0, 60) (60, 0)	(0, 0)	0 ≤ 60 (True)
5x + y ≤ 100	5x + y = 100	(0, 100) (20, 0)	(0, 0)	0 ≤ 100 (True)

The inequalities x ≥ 0, y ≥ 0 indicate that the solution set lies in the first quadrant.



Hence the solution set is shaded on the graph which is the quadrilateral OABC.
The co-ordinates of vertices O, A, B and C are (0, 0), (20, 0), (10, 50) and (0, 60) respectively.
Again,

Vertices	F $= 50x + 15y$	Remarks
O(0, 0)	F $= 50 \times 0 + 15 \times 0 = 0$	
A(20, 0)	F $= 50 \times 20 + 15 \times 0 = 1000$	
B(10, 50)	F $= 50 \times 10 + 15 \times 50 = 1250$	Maximum
C(0, 60)	F $= 50 \times 0 + 15 \times 60 = 900$	

∴ Maximum value of F = 1250 at the point E(10, 50).

40. 2063 Q.No. 14b

Graph the following system of inequalities to find maximum and minimum of the objective function
$F = 16x - 2y + 40$
$3x + 4y \le 24, 0 \le y \le 4, 0 \le x \le 7$ [4]

SOLUTION

The boundary equations of given inequalities are

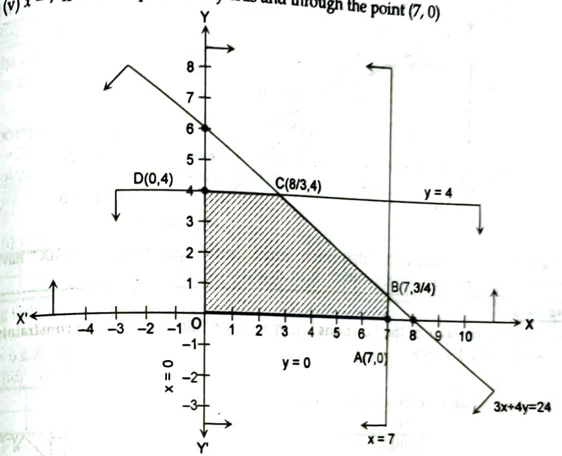
$3x + 4y = 24$ ...(i)
$y = 0$ ...(ii)
$y = 4$ ...(iii)
$x = 0$ ... (iv)
$x = 7$ ...(v)

From (i) $3x + 4y = 24$

x	0	8
y	6	0

Taking testing point (0,0) in $3x + 4y \le 24$, we get
$3 \times 0 + 4 \times 0 \le 24$ (true)
The graph of $3x + 4y \le 24$ contains origin.
From (ii) y = 0 which is x-axis.

$y \ge 0$ gives the upper half plane containing x-axis.
From (iii) y = 4 is the line parallel to x-axis and through the point (0, 4)
Taking testing point (0,0) in $y \le 4$, we get
$0 \le 4$ (true)
From (iv) $x = 0$ which is y-axis.
$x \ge 0$ gives the right half plane containing the y-axis.
From (v) $x = 7$ is the line parallel to y-axis and through the point (7, 0)



The required solution is shaded in the figure.

Vertices	Value of F = 16x – 2y + 40
O(0,0)	$F = 16 \times 0 - 2 \times 0 + 40 = 40$
A(7,0)	$F = 16 \times 7 - 2 \times 0 + 40 = 152$
$B\left(7, \frac{3}{4}\right)$	$F = 16 \times 7 - 2 \times \frac{3}{4} + 40 = 150.5$
$C\left(\frac{8}{3}, 4\right)$	$F = 16 \times \frac{8}{3} - 2 \times 4 + 40 = 74.67$
D(0,4)	$F = 16 \times 0 - 2 \times 4 + 40 = 32$

∴ Max. value of F = 152 at (7, 0)
Min. value of F = 32 at (0, 4)

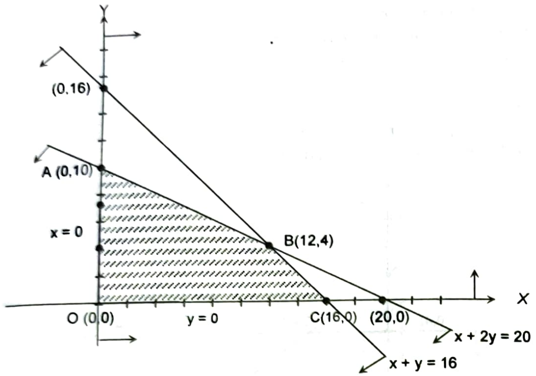
41. 2064 Q.No. 14 b

Graph the following systems of inequalities and find the vertices where they exist: [4]
$x + 2y \le 20, x + y \le 16, x \ge 0$

SOLUTION

The corresponding boundary lines are respectively
$x + 2y = 20$
$x + y = 16$
$x = 0$
$y = 0$
Now,

Linear inequalities	Boundary lines	Points	Testing point	Result
$x + 2y \le 20$	$x + 2y = 20$	(0, 10); (20, 0)	(0, 0)	$0 \le 20$ (true)
$x + y \le 16$	$x + y = 16$	(0, 16); (16, 0)	(0, 0)	$0 \le 16$ (true)
$x \ge 0$	$x = 0$ i.e. y-axis	–		
$y \ge 0$	y = 0 i.e. x-axis	–		



The common region is shaded on the graph which is the quadrilateral AOBC having the coordinates with the vertices A (0, 10), O(0, 0). B(16, 0) and C (12, 4).

42. 2065 Q. No. 14 b

Determine the extreme value of the functions F(x, y) = x + y + 100 subject to the constraints: $y - x \geq 1$; $y - x \leq 4$ and $1 \leq x \leq 6$ [4]

SOLUTION

The corresponding equations of boundary lines are

$y - x = 1$ ...(i)

$y - x = 4$ ...(ii)

$x = 1$ ...(iii)

$x = 6$ ...(iv)

From (i)

$y - x = 1$

x	0	−1
y	1	0

Taking testing point (0, 0) in $y - 1 \geq 1$

$0 - 0 \geq 1$ (false)

From (i)

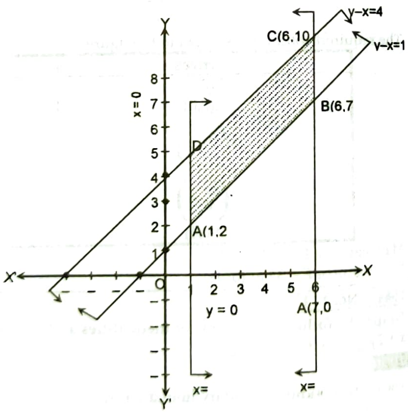
$y - x = 4$

x	0	−4
y	4	0

Taking testing point (0, 0) in $y - x \leq 4$, we get

$0 - 0 \leq 4$ (true)

From (iii) x = 1 which is the line parallel to y-axis and through the point (1, 0)



From (iv) x = 6 which is the line parallel to y-axis and through the point (6, 0)

$0 - 0 \geq 1$ (false)

Vertices	Value of F = 10x + 15y
A(1,2)	F = 1 + 2 + 100 = 103
B(6,7)	F = 6 + 7 + 100 = 113
C(6,10)	F = 6 + 10 + 100 = 116
D(1,5)	F = 1 + 5 + 100 = 106

Max. value of F = 116 at (3, 7)

Min. value of F = 103 at (1, 2)

43. 2066 Q.No. 14 b

How does a linear inequality differ from the linear equation? Determine the maximum and minimum value of the function Φ(x, y) = 16x − 2y + 40 subject to $3x + 4y \leq 24$, $0 \leq x \leq 7$, $0 \leq y \leq 4$. [4]

SOLUTION

First Part: An equation of the form ax + by = c ...(i) where a, b, c are constants and x and y are variables is called a linear equation. If '=' is replaced by < or ≤ or > or ≥ in (i), then it is called linear inequality.

Next Part: Please see 2063 Q.No. 14b

44. 2067 Q.No. 14 b

Find the extreme values of the objective function 10x + 15y subject to constraints: $x + 2y \leq 25$, $2x + y \leq 20$, $x \geq 3$, $y \geq 4$ [4]

SOLUTION

The corresponding equations of boundary lines are:

$x + 2y = 25$ ...(i)

$2x + y = 20$ ...(ii)

$x = 3$ ... (iii)

$y = 4$ ...(iv)

From (i) $x + 2y = 25$

x	0	25
y	12.5	0

Taking testing point (0, 0) in $x + 2y \leq 25$, we have

$0 + 2 \times 0 \leq 25$ (true)

From (ii) $2x + y = 20$

x	0	10
y	20	0

Taking testing point (0, 0) in $2x + y \leq 20$, we have

$2 \times 0 + 0 \leq 0$ (true)

From (iii) $x = 3$

x	3	3
y	0	2

Taking testing point (0, 0) in $x \geq 3$, we have

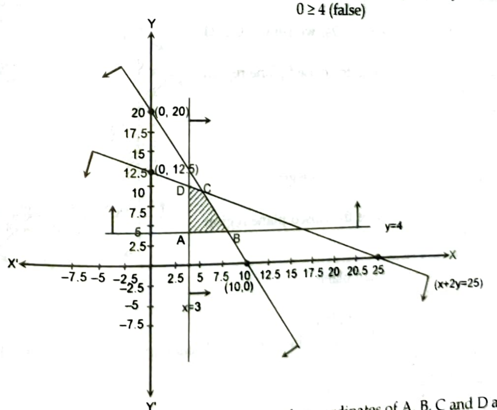
$0 \geq 3$ (false)

From (iv) $y = 4$

x	0	2
y	4	4

Taking testing point (0, 0) in $y \geq 4$, we have

$0 \geq 4$ (false)



From the figure ABCD is the feasible region where the coordinates of A, B, C and D are A(3, 4), B(8, 4), C(5, 10) and D(3, 11)

Vertices	Value of F = 10x + 15y
A(3, 4)	F = 10 × 3 + 15 × 4 = 90
B(8, 4)	F = 10 × 8 + 15 × 4 = 140
C(5,10)	F = 10 × 5 + 15 × 10 = 200
D(3, 11)	F = 10 × 3 + 15 × 11 = 195

Max. value = 200 at (5, 10)

Min. value = 90 at (3, 4)

45. 2068 Q.No. 14 b

Find the maximum and the minimum values of the objective function. [4]

$F = 16x - 2y + 40$ subjected to $3x + 4y \le 24$, $0 \le y \le 4$, $0 \le x \le 7$.

➥ Please see 2063 Q.No. 14b

6 MARKS QUESTIONS

46. 2074 Set A Q.No. 18

A small industry manufactures necklaces and bracelets. The combined number of necklace and bracelets that it can handle per day is not more than 24. Each bracelet takes 1 hour of labour to make and each necklace takes a half hour. The total number of hours of labour available does not exceed 16. If the profit on the necklace is Rs. 80 and the profit on the bracelet is Rs. 50. How many of each product should be produced daily to maximize profit? [6]

SOLUTION

Let no of necklaces = x

& no. of bracelets = y

If P be the total profit to be maximized, then given LPP can be written as

Maximize $P = 80x + 50y$ subject to the constraints

$x + y \le 24$

$x + \frac{1}{2}y \le 16$ i.e. $2x + y \le 32$

$x \ge 0, y \ge 0$.

The corresponding equations of boundary lines are

$x + y = 24$ ... (i)

$2x + y = 32$ ... (ii)

$x = 0$ ... (iii)

$y = 0$ ... (iv)

From (i), $x + y = 24$

x	0	24
y	24	0

Taking testing point (0, 0) in $x + y \le 24$, we get $0 + 0 \le 24$ (true)

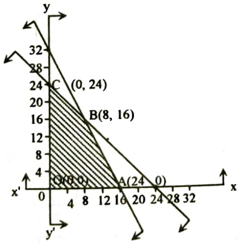
So, the solution set of $x + y \le 24$ is the closed plane region containing the origin.

Again, from (ii), $2x + y = 32$

x	0	16
y	32	0

Taking testing point (0, 0) in $2x + y \le 32$, we get $2 \times 0 + 0 \le 32$ (true)

So the solution set of $2x + y \le 32$ is the closed plane region containing the origin.



From (iii) x = 0 which is y -axis.

And $x \ge 0$ gives the closed right half plane.

From (iv) y = 0, which is x-axis.

And $y \ge 0$ gives the closed upper half plane.

From figure OABC is the feasible region where the coordinates of O, A, B & C are (0, 0), (16, 0), (8, 16) & (0, 24) respectively.

Vertices	Value of P = 80 x + 50y
O(0,0)	$P = 80 \times 0 + 50 \times 0 = 0$
A(16,0)	$P = 80 \times 16 + 50 \times 0 = 1280$
B(8,16)	$P = 80 \times 8 + 50 \times 16 = 1440$
C(0,24)	$P = 80 \times 0 + 50 \times 24 = 1200$

The maximum value of p is Rs. 1440 at (8, 16)

So, the required no. of necklaces & bracelets that should be produced daily to maximize the profit are 8 & 16 respectively.

B. SIMPLEX METHOD

4 MARKS QUESTIONS

47. 2076 Set B Q.No. 17a

Using the simplex method, maximize $p = x + 3y$ subject to constraints $x + y \le 4$, $x - y \le 1$, $x \ge 0$, $y \ge 0$. [4]

SOLUTION

Let r and s be the non-negative slack variables. Adding the slack variables, we can express the given LPP in the following form:

$x + y + r = 4$

$x - y + s = 1$

$x + 3y = p$

$\Rightarrow x + y + r + 0.s + 0.p = 4$

$x - y + 0.r + s + 0.p = 1$

$-x - 3y + 0.r + 0.s + p = 0$

Simplex Tableau

x	y	r	s	p	RHS
1	[1]	1	0	0	4
1	-1	0	1	0	1
-1	-3 ↑	0	0	1	0

Here -3 is the most negative entry. So, column second is the pivot column.

Now, $\frac{4}{1} = 4$ and $\frac{1}{-1} = -1$ (We shouldn't take negative ratio). So, 1 is the pivot element.

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 3R_1$

x	y	r	s	p	RHS
1	1	1	0	0	4
2	0	1	1	0	5
2	0	3	0	1	12

Since all the entries in the last row are non-negative, the optimal solution is obtained.

The optimal solution is

Max. p = 12 at x = 0 and y = 4

6 MARKS QUESTIONS

48. 2069 (Set A) Q.No. 18

Using simplex method, Maximize $Z = 7x_1 + 5x_2$

Subject to $x_1 + 2x_2 \le 6$; $4x_1 + 3x_2 \le 12$; $x_1, x_2 \ge 0$ [6]

SOLUTION

Let x_3 and x_4 be non-negative slack variables. Adding the slack variables, we can write the given LPP in the following form.

$x_1 + 2x_2 + x_3 = 6$

$4x_1 + 3x_2 + x_4 = 12$

$Z = 7x_1 + 5x_2$

$\Rightarrow x_1 + 2x_2 + x_3 + 0 . x_4 + 0 . Z = 6$

$4x_1 + 3x_2 + 0 . x_3 + x_4 + 0 . Z = 12$

$-7x_1 - 5x_2 + 0 . x_3 + 0 . x_4 + Z = 0$

Simplex tableau

Basic Variables	x_1	x_2	x_3	x_4	Z	RHS(b)
x_3	1	2	1	0	0	6
x_4	[4]	3	0	1	0	12
	-7 ↑	-5	0	0	1	0

Here, -7 is the most negative entry in the last row. So, first column is the pivot column. Since $\frac{6}{1} = 6$, $\frac{12}{4} = 3$ and 3 < 6, so 4 is the pivot element.

Applying $R_2 \to \frac{1}{4}R_2$

Basic Variables	x_1	x_2	x_3	x_4	Z	RHS(b)
x_3	1	2	1	0	0	6
x_1	1	3/4	0	1/4	0	3
	-7	-5	0	0	1	0

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 7R_2$

Basic Variables	x_1	x_2	x_3	x_4	Z	RHS(b)
x_3	0	5/4	1	-1/4	0	3
x_1	1	3/4	0	1/4	0	3
	0	1/4	0	7/4	1	21

Since all the entries in the last row are non-negative, so the solution is optimal.
Max. value of $Z = 21$ when $x_1 = 3$ and $x_2 = 0$.

49. 2069 (Set B) Q.No. 18 [6]

Using simplex method, maximize $Z = 7x_1 + 5x_2$
Subject to: $x_1 + 2x_2 \le 6$, $4x_1 + 3x_2 \le 6$, $x_1, x_2 \ge 0$

SOLUTION

Let x_3 and x_4 be the non-negative slack variables. Then the given LPP can be written as

$x_1 + 2x_2 + x_3 = 6$
$4x_1 + 3x_2 + x_4 = 6$
$7x_1 + 5x_2 = Z$
$\Rightarrow$ $x_1 + 2x_2 + x_3 + 0 . x_4 + 0 . Z = 6$
$4x_1 + 3x_2 + 0 . x_3 + x_4 + 0 . Z = 6$
$-7x_1 - 5x_2 + 0 . x_3 + 0 . x_4 + Z = 0$

Simplex tableau

x_1	x_2	x_3	x_4	Z	RHS
1	2	1	0	0	6
[4]	3	0	1	0	6
-7 ↑	-5	0	0	1	0

Here, -7 is the most negative entry. So first column is the pivot column. Since $\frac{6}{1} = 6$, $\frac{6}{4} = 1.5$ and $1.5 < 6$, so 4 is the pivot element.

Applying $R_2 \to \frac{1}{4}R_2$

x_1	x_2	x_3	x_4	Z	RHS
1	2	1	0	0	6
1	3/4	0	1/4	0	3/2
-7	-5	0	0	1	0

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 7R_2$

x_1	x_2	x_3	x_4	Z	RHS
0	5/4	0	-1/4	0	9/2
1	3/4	0	1/4	0	3/2
0	1/4	0	7/4	1	21/2

Since all the entries in the last row are non-negative, so the optimal solution is obtained.

Max $Z = \frac{21}{2}$ at $x_1 = \frac{3}{2}$ and $x_2 = 0$.

50. 2070 Set C Q.No. 18 [6]

Using Simplex method, Max. $z = 5x_1 + 7x_2$
Subject to $2x_1 + 3x_2 \le 13$; $3x_1 + 2x_2 \le 12$; $x_1, x_2 \ge 0$.

SOLUTION

Let x_3 and x_4 be the non-negative slack variables. Then the given LPP can be written as

$2x_1 + 3x_2 + x_3 = 13$
$3x_1 + 2x_2 + x_4 = 12$
$5x_1 + 7x_2 = z$
$\Rightarrow$ $2x_1 + 3x_2 + x_3 + 0 . x_4 + 0 . z = 13$
$3x_1 + 2x_2 + 0 . x_3 + x_4 + 0 . z = 12$
$-5x_1 - 7x_2 + 0 . x_3 + 0 . x_4 + z = 0$

Simplex tableau

x_1	x_2	x_3	x_4	z	RHS
2	[3]	1	0	0	13
3	2	0	1	0	12
-5	-7 ↑	0	0	1	0

Here, -7 is the most negative entry in the last row. So, second column is the pivot column. Since $\frac{13}{3} = 4.3$, $\frac{12}{2} = 6$ and $4.3 < 6$, so 3 is the pivot element.

Applying $R_1 \to \frac{1}{3}R_1$

x_1	x_2	x_3	x_4	z	RHS
2/3	1	1/3	0	0	13/3
3	2	0	1	0	12
-5	-7	0	0	1	0

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 + 7R_1$

x_1	x_2	x_3	x_4	z	RHS
2/3	1	1/3	0	0	13/3
[5/3]	0	-2/3	1	0	10/3
-1/3 ↑	0	7/3	0	1	91/3

This is not the optimal solution as the last row contains negative entry. Again, first column is the pivot column. Since, $\frac{13/3}{2/3} = 6.5$, $\frac{10/3}{5/3} = 2$ and $2 < 6.5$, so $\frac{5}{3}$ is the pivot element.

Applying $R_2 \to \frac{3}{5}R_2$

x_1	x_2	x_3	x_4	z	RHS
2/3	1	1/3	0	0	13/3
1	0	-2/5	3/5	0	2
-1/3	0	7/3	0	1	91/3

Applying $R_1 \to R_1 - \frac{2}{3}R_2$ and $R_3 \to R_3 + \frac{1}{3}R_2$

x_1	x_2	x_3	x_4	z	RHS
0	1	3/5	-2/5	0	3
1	0	-2/5	3/5	0	2
0	0	11/5	1/5	1	31

Since all the entries in the last row are non-negative, so the optimal solution is obtained.
Max. $z = 31$ at $x_1 = 2$, $x_2 = 3$.

51. 2070 Set D Q.No. 18 [6]

Using Simplex method,
Max. $P = 50x_1 + 80x_2$
Subject to $x_1 + 2x_2 \le 32$; $3x_1 + 4x_2 \le 84$; $x_1, x_2 \ge 0$.

SOLUTION

Let x_3 and x_4 be the non-negative slack variables. Then given LPP can be written as

$x_1 + 2x_2 + x_3 = 32$
$3x_1 + 4x_2 + x_4 = 84$
$5x_1 + 80x_2 = P$
$\Rightarrow$ $x_1 + 2x_2 + x_3 + 0 . x_4 + 0 . P = 32$
$3x_1 + 4x_2 + 0 . x_3 + x_4 + 0 . P = 84$
$-50x_1 - 80x_2 + 0 . x_3 + 0 . x_4 + P = 0$

Simplex tableau

x_1	x_2	x_3	x_4	P	RHS
1	[2]	1	0	0	32
3	4	0	1	0	84
-50	-80 ↑	0	0	1	0

Here, –80 is the most negative entry in the last row, so second row is the pivot column. Since 32/ 2 = 16, 84/4 = 21 and 16 < 21, so 2 is the pivot element.

Applying $R_1 \rightarrow \frac{1}{2} R_1$

x_1	x_2	x_3	x_4	P	RHS
1/2	1	1/2	0	0	16
3	4	0	1	0	84
-50	-80	0	0	1	0

Applying $R_2 \rightarrow R_2 - 4R_1$ and $R_3 \rightarrow R_3 + 80R_1$

x_1	x_2	x_3	x_4	p	RHS
1/2	1	1/2	0	0	16
[1]	0	-2	1	0	20
-10 ↑	0	40	0	1	1280

This is not the optimal solution as the last row contains negative entry. Again, first column is the pivot point column. Since $\frac{20}{1} = 20, \frac{16}{1/2} = 32$ and 20 < 32, so 1 is the pivot element.

Applying $R_1 \rightarrow R_1 - \frac{1}{2} R_2$ and $R_3 \rightarrow R_3 + 10R_2$

x_1	x_2	x_3	x_4	p	RHS
0	1	3/2	-1/2	0	6
1	0	-2	1	0	20
0	0	20	10	1	1480

Since all the entries in the last row are non-negative, so the optimal solution is obtained.
Max. P = 1480 at $x_1 = 20$, $x_2 = 6$.

52. 2070 Supp. Q.No. 18

Using the simplex method, maximize
p = 20 x + 30 y subject to constraints
2x + 5y ≤ 20, 2x + y ≤ 12, x ≥ 0, y ≥ 0 [6]

SOLUTION

Let r and s be the non-negative slack variables. Then the given LPP can be written as

$2x + 5y + r = 20$
$2x + y + s = 12$
$20x + 30y = p$
$\Rightarrow 2x + 5y + r + 0.s + 0.p = 20$
$2x + y + 0.r + s + 0.p = 12$
$-20x - 30y + 0.r + 0.s + p = 0$

Simplex tableau

x	y	r	s	p	RHS
2	[5]	1	0	0	20
2	1	0	1	0	12
-20	-30 ↑	0	0	1	0

Here, –30 is the most negative entry. So, second column is the pivot column.
Since $\frac{20}{5} = 4, \frac{12}{1} = 12$ and 4 < 12, so 5 is the pivot element.

Applying $R_1 \rightarrow \frac{1}{5} R_1$

x	y	r	s	p	RHS
2/5	1	1/5	0	0	4
2	1	0	1	0	12
-20	-30	0	0	1	0

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 + 30R_1$

x	y	r	s	p	RHS
2/5	1	1/5	0	0	4
[8/5]	0	-1/5	1	0	8
-8 ↑	0	6	0	1	120

This is not the optimal solution as last row contains the negative entry. Again first column is the pivot column. Since $\frac{4}{2/5} = 10, \frac{8}{8/5} = 5$ and 5 < 10, so 8/5 is the pivot element.

Applying $R_2 \rightarrow \frac{5}{8} R_2$

x	y	r	s	p	RHS
2/5	1	1/5	0	0	4
1	0	-1/8	5/8	0	5
-8	0	6	0	1	120

Applying $R_1 \rightarrow R_1 - \frac{2}{5} R_2$ and $R_3 \rightarrow R_3 + 8R_2$

x	y	r	s	p	RHS
0	1	7/25	-1/4	0	2
1	0	-1/8	5/8	0	5
0	0	5	5	1	160

Since the last row contains all non-negative entries, so the optimal solution is obtained.
Max. p = 160 at x = 5 and y = 2.

53. 2071 Set C Q.No. 18

Using simplex method
Maximize $f = 15x_1 + 10x_2$
subject to $2x_1 + x_2 \le 10$ [6]
$x_1 + 3x_2 \le 10$,
$x_1, x_2 \ge 0$

SOLUTION

Let x_3 and x_4 be the non-negative slack variables. Adding the slack variables, we can write the given LPP in the following form.

$2x_1 + x_2 + x_3 = 10$
$x_1 + 3x_2 + x_4 = 10$
$15x_1 + 10x_2 = f$
$\Rightarrow 2x_1 + x_2 + x_3 + 0.x_4 + 0.f = 10$
$x_1 + 3x_2 + 0.x_3 + x_4 + 0.f = 10$
$-15x_1 - 10x_2 + 0.x_3 + 0.x_4 + f = 0$

Simplex tableau

x_1	x_2	x_3	x_4	f	RHS
[2]	1	1	0	0	10
1	3	0	1	0	10
-15 ↑	-10	0	0	1	0

Here, –15 is the most negative entry, so first column is the pivot column. Since $\frac{10}{2} = 5, \frac{10}{1} = 10$ and 5 < 10. So 2 is the pivot element.

Applying $R_1 \rightarrow \frac{1}{2} R_1$

x_1	x_2	x_3	x_4	f	RHS
1	1/2	1/2	0	0	5
1	3	0	1	0	10
-15	-10	0	0	1	0

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 + 15R_1$

x_1	x_2	x_3	x_4	f	RHS
1	1/2	1/2	0	0	5
0	[5/2]	-1/2	1	0	5
0	-5/2 ↑	15/2	0	1	75

This is not the optimal solution as last row contains negative entry.
Again, second column is the pivot column. Since $\frac{5}{1/2} = 10, \frac{5}{5/2} = 2$ and 2 < 10, so 5/2 is the pivot element.

Applying $R_2 \to \frac{2}{5} R_2$

x_1	x_2	x_3	x_4	f	RHS
1	1/2	1/2	0	0	5
0	1	-1/5	2/5	0	2
0	-5/2	15/2	0	1	75
	↑				

Again, applying $R_1 \to R_1 - \frac{1}{2} R_2$ and $R_3 \to R_3 + 5/2\, R_2$

x_1	x_2	x_3	x_4	f	RHS
1	0	3/5	-1/5	0	4
0	1	-1/5	2/5	0	2
0	0	7	1	1	80

Since all the entries in the last row are non-negative, so optimal solution is obtained.

Max. f = 80 at $x_1 = 4$, $x_2 = 2$.

54. 2071 Set D Q.No. 18

Using simplex method,
Maximize U = 25 x + 45 y
subject to $x + 3y \le 21$
$2x + 3y \le 24$
$x, y \ge 0$ [6]

SOLUTION

Let r and s be the non-negative slack variables. Adding the slack variables we can write the given LPP in the following standard form:

$x + 3y + r = 21$
$2x + 3y + s = 24$
$25x + 45y = U$
$\Rightarrow x + 3y + r + 0 . s + 0 . U = 21$
$2x + 3y + 0 . r + s + 0 . U = 24$
$-25x - 45y + 0 . r + 0 . s + U = 0$

Simplex table

x	y	r	s	U	RHS
1	[3]	1	0	0	21
2	3	0	1	0	24
-25	-45	0	0	1	0
	↑				

Here, -45 is the most negative entry in the last row. So, second column is the pivot column. Since $\frac{21}{3} = 7, \frac{24}{3} = 8$ and 7 < 8 so, marked 3 of first row is the pivot element.

Applying $R_1 \to \frac{1}{3} R_1$

x	y	r	s	U	RHS
1/3	1	1/3	0	0	7
2	3	0	1	0	24
-25	-45	0	0	1	0

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 + 45R_1$

x	y	r	s	U	RHS
1/3	1	1/3	0	0	7
[1]	0	-1	1	0	3
-10	0	15	0	1	315
↑					

This is not the optimal solution as the last row contains negative entry. Again, first column is the pivot column. Since $\frac{7}{1/3} = 21, \frac{3}{1} = 3$ and 3 < 21, so 1 is the pivot element.

Applying $R_1 \to R_1 - \frac{1}{3} R_2$ and $R_3 \to R_3 + 10R_2$

x	y	r	s	U	RHS
0	1	2/3	-1/3	0	6
1	0	-1	1	0	3
0	0	1	10	1	345

Since all the entries in the last row are non negative. So the optimal solution is obtained.

∴ Max U = 345 at x = 3 and y = 6.

55. 2071 Supp. Q.No. 18

Using the simplex method, maximize
p = 4x + 5y subject to
$2x + 5y \le 25$, $6x + 5y \le 45$, $x \ge 0$, $y \ge 0$. [6]

SOLUTION

Let r and s be the non negative slack variables. Adding the slack variables we can write the given LPP in the following form:

$2x + 5y + r = 25$
$6x + 5y + s = 45$
$4x + 5y = p$
$\Rightarrow 2x + 5y + r + 0. s + 0. p = 25$
$6x + 5y + 0 . r + s + 0 . p = 45$
$-4x - 5y + 0 . r + 0 . s + p = 0$

Simplex tableau

x	y	r	s	p	RHS
2	[5]	1	0	0	25
6	5	0	1	0	45
-4	-5	0	0	1	0
	↑				

Here, -5 is the most negative entry in the last row. So second column is the pivot column. Since $\frac{25}{5} = 5, \frac{45}{5} = 9$ and 5 < 9, so 5 is the pivot element.

Applying $R_1 \to \frac{1}{5} R_1$

x	y	r	s	p	RHS
2/5	[1]	1/5	0	0	5
6	5	0	1	0	45
-4	-5	0	0	1	0

Applying $R_2 \to R_2 - 5R_1$ and $R_3 \to R_3 + 5R_1$

x	y	r	s	p	RHS
2/5	1	1/5	0	0	5
[4]	0	-1	1	0	20
-2	0	1	0	1	25

Again, first column is the pivot column. Since $\frac{5}{2/5} = \frac{25}{2}, \frac{20}{4} = 5$ and $5 < \frac{25}{2}$, so 4 is the pivot element.

Applying $R_2 \to \frac{1}{4} R_2$

x	y	r	s	p	RHS
2/5	1	1/5	0	0	5
1	0	-1/4	1/4	0	5
-2	0	1	0	1	25

Applying $R_1 \to R_1 - \frac{2}{5} R_2$ and $R_3 \to R_3 + 2R_2$

x	y	r	s	p	RHS
0	1	3/10	-1/10	0	3
1	0	-1/4	1/4	0	5
0	0	7/5	0	1	35

Since all the entries in the last row are non-negative, so the solution is optimal.

Max p = 35 at x = 5, y = 3

56. 2072 Set C Q.No. 18

Solve by Simplex method, the LP problem to maximize $z = 7x + 5y$ subject to $x + 2y \le 6$, $4x + 3y \le 12$, $x, y \ge 0$. [6]

SOLUTION

Please see 2069 (Set A) Q.No. 18

57. 2072 Set D Q.No. 18

Using Simplex method, find the optimal solution of $z = 7x_1 + 5x_2$ subject to $x_1 + 2x_2 \le 6$, $4x_1 + 3x_2 \le 12$, $x_1, x_2 \ge 0$. [6]

SOLUTION

Please see 2069 (Set A) Q.No. 18

58. 2072 Set E Q.No. 18

Using Simplex method,
Maximize $F = 5x - 3y$, subject to
$3x + 2y \le 6$, $-x + 3y \ge -4$, $x, y \ge 0$ [6]

SOLUTION

Given inequalities are

$3x + 2y \le 6$

$-x + 3y \ge 4 \Rightarrow x - 3y \le 4$

$x, y \ge 0$

Let r and s be the non-negative slack variables. Adding the slack variables, we can write the given LPP in the following form

$3x + 2y + r = 6$

$x - 3y + s = 4,$ and

$5x - 3y = F$

$\Rightarrow 3x + 2y + r + 0.s + 0.F = 6$

$x - 3y + 0.r + s + 0.F = 4$

$-5x + 3y + 0.r + 0.s + F = 0$

Simplex tableau

x	y	r	s	F	RHS
3	2	1	0	0	6
1	-3	0	1	0	4
-5	3	0	0	1	0
↑					

Here, -5 is the negative entry in the last row. So, first column is the pivot column. Since $\frac{6}{3} = 2, \frac{4}{1} = 4$ and $2 < 4$, so 3 is the pivot element.

Applying $R_1 \to \frac{1}{3}R_1$

x	y	r	s	F	RHS
1	2/3	1/3	0	0	2
1	-3	0	1	0	4
-5	3	0	0	1	0

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 5R_1$

x	y	r	s	F	RHS
1	2/3	1/3	0	0	2
0	-11/3	-1/3	1	0	2
0	19/3	5/3	0	1	10

Since all the entries in the last row are non-negative, so the solution is optimal.
Max. value of F = 10 at x = 2, y = 0.

59. 2072 Supp Q.No. 18

Using simplex method, maximize $z = 5x + 3y$ subject to $2x + y \le 40$, $x + 2y \le 50$, $x, y \ge 0$

SOLUTION

Let r and s be the non-negative slack variables. Then given LPP can be written as

$2x + y + r = 40$

$x + 2y + s = 50$

and $z = 5x + 3y$

$\Rightarrow 2x + y + r + 0\cdot s + 0.z = 40$

$x + 2y + 0\cdot r + s + 0\cdot z = 50$

$-5x - 3y + 0\cdot r + 0\cdot s + z = 0$

Simplex tableau

x	y	r	s	z	R.H.S
[2]	1	1	0	0	40
1	2	0	1	0	50
-5	-3	0	0	1	0
↑					

Here, -5 is the most negative entry, so first column is the pivot column. Since $\frac{40}{2} = 20, \frac{50}{1} = 50$ and $20 < 50$, so 2 is the pivot element.

Applying $R_1 \to \frac{1}{2}R_1$

x	y	r	s	z	R.H.S
1	1/2	1	0	0	20
1	2	0/2	1	0	50
-5	-3	0	0	1	0

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 + 5R_1$

x	y	r	s	z	R.H.S
1	1/2	1/2	0	0	20
0	3/2	-1/2	1	0	30
0	-1/2	5/2	0	2	100
	↑				

The solution is not optimal as the third row contains negative entry.

Again, second column is the pivot column. Since $\frac{20}{1/2} = 4, \frac{30}{3/2} = 20$ and $20<40$, so 3/2 is the pivot element.

Applying $R_2 \to \frac{2}{3}R_2$

x	y	r	s	z	R.H.S
1	1/2	1/2	0	0	20
0	1	-1/3	2/3	0	20
0	-1/2	5/2	0	1	100

Applying $R_1 \to R_1 - \frac{1}{2}R_2$, $R_3 \to R_3 + \frac{1}{2}R_2$

x	y	r	s	z	R.H.S
1	0	2/3	-1/3	0	10
0	1	-1/3	2/3	0	20
0	0	7/2	1/3	1	110

Since all entries in the last row are non-negative, so the optimal solution is obtained. Max. z = 110 at x = 10 and y = 20.

60. 2073 Set C Q.No. 18

Maximize $z = 5x_1 + 7x_2$ subject to $2x_1 + 3x_2 \le 13$, $3x_1 + 2x_2 \le 12$; $x_1, x_2 \ge 0$ by Simplex method. [6]

- Please see 2070 Set C Q.No. 18

61. 2073 Set D Q.No. 18

Using the simplex method, Maximum $z = 15x_1 + 10x_2$ [6]
Subject to $2x_1 + x_2 \le 10$, $x_1 + 3x_2 \le 10$, $x_1, x_2 \ge 0$.

- Please see 2071 Set C Q.No. 18

62. 2073 Supp Q.No. 18

Apply Simplex method to maximize $z = 15x_1 + 10x_2$ subject to $2x_1 + x_2 \le 10$, $x_1 + 3x_2 \le 10$, $x_1, x_2 \ge 0$. [6]

- Please refer 2071 Set C Q.No. 18

63. 2074 Set A Q.No. 18 OR

Use the simplex method to maximize $P = x + y$ subject to constraints $x + 2y \le 6$, $3x + 2y \le 12$, $x \ge 0$, $y \ge 0$. [6]

SOLUTION

Let r and s be the non-negative slack variables. Then the given LPP can be written as

$x + 2y + r = 6$

$3x + 2y + s = 12$

$x + y = P$

$\Rightarrow$ $x + 2y + r + 0.s + 0.P = 6$

$3x + 2y + 0.r + s + 0.P = 12$

$-x - y + 0.r + 0.s + P = 0$

Simplex Tableau

x	y	r	s	P	RHS
1	2	1	0	0	6
[3]	2	0	1	0	12
−1 ↑	−1	0	0	1	0

There are two equal negative entries in the last row. So, we may choose any one of 1st or 2nd column as pivot column. Let us choose first column as a pivot column.

Since $\frac{6}{1} = 6$ & $\frac{12}{3} = 4$ and $4 < 6$.

So 3 is the pivot element.

Applying $R_2 \to \frac{1}{3} R_2$

x	y	r	s	P	RHS
1	2	1	0	0	6
1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	4
−1	−1	0	0	1	0

Applying $R_1 \to R_1 - R_2$ & $R_3 \to R_3 + R_2$

x	y	r	s	p	RHS
0	[$\frac{4}{3}$]	1	$-\frac{1}{3}$	0	2
1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	4
0	$-\frac{1}{3}$ ↑	0	$\frac{1}{3}$	1	4

This is not the optimal solution as the last row contains negative entry.

Now, second column is the pivot column. Since $\frac{2}{4/3} = \frac{6}{4} = 1.5$, $\frac{4}{2/3} = 6$ and $1.5 < 6$, so $\frac{4}{3}$ is the pivot element.

Applying $R_1 \to \frac{3}{4} R_1$

x	y	r	s	P	RHS
0	1	3/4	−1/4	0	3/2
1	2/3	0	1/3	0	4
0	−1/3	0	1/3	1	4

Applying $R_2 \to R_2 - \frac{2}{3} R_1$ and $R_3 \to R_3 + \frac{1}{3} R_1$

x	y	r	s	P	RHS
0	1	3/4	−1/4	0	3/2
1	0	−1/2	1/2	0	3
0	0	1/4	1/4	0	9/2

Since all the entries in last row are non-negative, so the optimal solution is obtained.

Max. $P = \frac{9}{2}$ when $x = 3$ and $y = \frac{3}{2}$

64. 2074 Set B Q.No. 18

Apply simplex method to maximize $z = 5x + 3y$ subject to $2x + y \le 40$, $x + 2y \le 50$; $x, y \ge 0$. [6]

➥ Please refer to 2072 Supp. Q. No. 18

65. 2074 Supp Q.No. 18

Using Simplex method, maximize $P = 50x_1 + 80x_2$ subject to $x_1 + 2x_2 \le 32$, $3x_1 + 4x_2 \le 84$, $x_1, x_2 \ge 0$. [6]

➥ Please refer to 2070 Set D Q.N. 18

66. 2075 Set A Q.No. 18

Using simplex method, maximize $U = 25x + 45y$ subject to $x + 3y \le 21$, $2x + 3y \le 24$, $x, y \ge 0$. [6]

➥ Please refer to 2071 Set D Q.No. 18

67. 2075 Set C Q.No. 18

Using the Simplex method, minimize $W = 3x + 2y$ subject to $2x + y \ge 4$, $x + 2y \ge 4$, $x, y \ge 0$. [6]

SOLUTION

A = Augmented matrix formed from the constraints and the objective function

$$= \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 1 & 2 & 4 \\ \hline 3 & 2 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 2 \\ \hline 4 & 4 & 0 \end{array}\right)$$

Now, the corresponding dual problem of the given LPP is

Max. $Z = 4y_1 + 4y_2$

subject to $2y_1 + y_2 \le 3$

$y_1 + 2y_2 \le 2$

$y_1, y_2 \ge 0$

Introducing the slack variables $x_1, x_2 \ge 0$, the problem can be restated as follows:

$2y_1 + y_2 + x_1 + 0.x_2 + 0.z = 3$

$y_1 + 2y_2 + 0.x_1 + x_2 + 0.z = 2$

$-4y_1 - 4y_2 + 0.x_1 + 0.x_2 + z = 0$

Initial Tableau

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	[2]	1	1	0	0	3
x_2	1	2	0	1	0	2
	−4 ↑	−4	0	0	1	0

Since $3/2 = 1.5$, $\frac{2}{1} = 2$ and $1.5 < 2$, So 2 is the pivot element.

Applying $R_1 \to \frac{1}{2} R_1$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
x_2	1	2	0	1	0	2
	−4	−4	0	0	1	0

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 4R_1$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
x_2	0	[$\frac{3}{2}$]	$-\frac{1}{2}$	1	0	$\frac{1}{2}$
	0	−2 ↑	2	0	1	6

Since $\frac{3/2}{1/2} = 3$, $\frac{1/2}{3/2} = \frac{1}{3}$ and $\frac{1}{3} < 3$, $\frac{3}{2}$ is pivot.

Applying $R_2 \to \frac{2}{3} R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{3}{2}$
y_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
	0	-2	2	0	1	6

Applying $R_1 \to R_1 - \frac{1}{2}R_2$, $R_3 \to R_3 + 2R_2$

Basic variable	y_1	y_2	x_1	x_2	z	RHS
y_1	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{4}{3}$
y_2	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
	0	0	$\frac{4}{3}$	$\frac{4}{3}$	1	$\frac{20}{3}$

This is the optimal solution as all the entries in last row are non-negative.

$\therefore$ Max. $Z = \frac{20}{3}$ at $y_1 = \frac{4}{3}$ & $y_2 = \frac{1}{3}$

i.e. min. $W = \frac{20}{3}$ at $x_1 = \frac{4}{3}$, $x_2 = \frac{4}{3}$

UNIT 17

COMPUTATIONAL METHOD

A. NUMBER SYSTEM

2 MARKS QUESTIONS

1. 2069 (Set A) Q.No. 16b

Convert the octal numeral 3733_8 into decimal form. [2]

SOLUTION

Position	3	2	1	0
Octal point	3	7	3	3

$3733_8 = 8^3 \times 3 + 8^2 \times 7 + 8^1 \times 3 + 8^0 \times 3$
$= 1536 + 448 + 24 + 3$
$= 2011_{10}$

2. 2069 (Set B) Q.No. 16b

Convert the hexadecimal numeral 2E4B into decimal form. [2]

SOLUTION

Position	3	2	1	0
Hexadecimal point	2	E	4	B

$2E4B_{16} = 16^3 \times 2 + 16^2 \times E + 16^1 \times 4 + 16^0 \times B$
$= 8192 + 3584 + 64 + 11$
$= 11851_{10}$

3. 2070 Set C Q.No. 16 b

Convert the decimal numeral 1503 to hexadecimal form. [2]

SOLUTION

		Remainder
16	1503	
16	93	F
16	5	D
	0	5

$\therefore 1503_{10} = 5DF_{16}$

4. 2070 Set D Q.No. 16 b

Convert the decimal number 3058 to hexadecimal form. [2]

SOLUTION

		Remainder
16	3058	
16	191	2
16	11	F
	0	B

$\therefore 3058_{10} = BF2_{16}$

5. 2070 Supp. Q.No. 16 b

Convert 110011_2 to the decimal number. [2]

SOLUTION

Position	5	4	3	2	1	0
Binary point	1	1	0	0	1	1

$110011_2 = 2^5 \times 1 + 2^4 \times 1 + 2^3 \times 0 + 2^2 \times 0 + 2^1 \times 1 + 2^0 \times 1$
$= 32 + 16 + 0 + 0 + 2 + 1$
$= 51_{10}$

6. 2071 Set C Q.No. 16 b

Convert the decimal number 2567_{10} to octal form [2]

SOLUTION

		Remainder
8	2567	
8	320	7
8	40	0
8	5	0
	0	5

$\therefore 2567_{10} = 5007_8$

7. 2071 Set D Q.No. 16 b

Convert the hexdecimal number $AB5_{16}$ to the decimal number. [2]

SOLUTION

Position	2	1	0
Hexadecimal point	A	B	5

$AB5_{16} = 16^2 \times A + 16^1 \times B + 16^0 \times 5$
$= 256 \times 10 + 16 \times 11 + 1 \times 5$
$= 2560 + 176 + 5$
$= 2741_{10}$

8. 2071 Supp. Q.No. 16b

Convert the binary number 10100011000_2 into the octal number. [2]

SOLUTION

10100011000_2
$= 010\ 100\ 011\ 000$ (group of 3)
$= 2430_8$

$\left[\because \begin{array}{l} 010_2 = 2_8 \\ 100_2 = 4_8 \\ 011_2 = 3_8 \\ 000_2 = 0_8 \end{array}\right]$

9. 2072 Set C Q.No. 16b

Convert the decimal number 3159 into hexadecimal form. [2]

SOLUTION

		Remainder
16	3159	
16	197	7
16	12	5
	0	C

$\therefore 3159_{10} = C57_{16}$

10. 2072 Set D Q.No. 16b

Convert hexadecimal number $70A_{16}$ into binary form. [2]

SOLUTION

We know that

$7_{16} = 0111_2$, $0_{16} = 0000_2$, $A_{16} = 1010_2$

$\therefore (70A)_{16} = 0111\ 0000\ 1010 = 11100001010_2$

11. 2072 Set E Q.No. 16b

Convert the decimal numeral 1503 into hexadecimal form. [2]

➡ Please refer to 2070 Set C Q.No. 16 b

12. 2072 Supp Q.No. 16b

Convert the decimal number 1503 into hexadecimal form. [2]

➡ Please refer to 2070 Set C Q.No. 16 b

13. 2073 Set C Q.No. 16b

Convert the octal number 143_8 into hexadecimal form. [2]

SOLUTION

$143_8 = 001\ 100\ 011$ (a group of 3) $(\because 1_8 = 001_2, 4_8 = 100_2, 3_8 = 011_2)$
$= 1100011$
$= 0110\ 0011$ (a group of 4) $(\because 0110_2 = 6_{16}, 0011_2 = 3_{16})$
$= 63_{16}$

14. 2073 Set D Q.No. 16b

Convert the decimal numeral 3058 to hexadecimal form. [2]

➡ Please refer to 2070 Set D Q.No. 16 b

15. 2073 supp Q.No. 16b

Convert the decimal number 7593 into hexadecimal form. [2]

SOLUTION

		Remainder
16	7593	
16	474	9
16	29	A
16	1	D
	0	1

$\therefore 7593_{10} = 1DA9_{16}$

16. 2074 Set A Q.No. 16b

Convert $2B1_{16}$ into the binary number. [2]

SOLUTION

We know,

$2_{16} = 0010_2$
$B_{16} = 1011_2$
$1_{16} = 0.001_2$
$\therefore 2B1_{16} = 0010\ 1011\ 0001$
$= 1010110001_2$

17. 2074 Set B Q.No. 16b

Convert the hexadecimal number $22F_{16}$ in to binary form. [2]

SOLUTION

We have,

$2_{16} = 0010_2$
$F_{16} = 1111_2$
$\therefore 22F_6 = 0010\ 0010\ 1111$
$= 1000101111_2$

18. 2074 Supp Q.No. 16b

Convert the decimal number 3058_{10} to hexadecimal form. [2]

➡ Please refer to 2070 Set D Q.N. 16b

19. 2075 Set A Q.No. 16b

Convert the hexadecimal numeral $AB5_{16}$ to decimal form. [2]

➡ Please refer to 2071 set D Q.No. 16b

20. 2075 Set C Q.No. 16b

Convert the decimal number 4526_{10} to hexadecimal form. [2]

SOLUTION

			Remainder
16	4526		
16	282	→	E
16	17	→	A
16	1	→	1
	0	→	1

$\therefore 4526_{10} = 11AE_{16}$

B. BISECTION METHOD

2 MARKS QUESTIONS

21. 2075 Set B Q.No. 16b

If f(0) = –1 and f(8) = 1, how many steps of the bisection method will be required to find an approximation to the root of f(x) accurate to 0.25? [2]

SOLUTION

Here,
f(0) = –1 and f(8) = 1
So, the initial interval = [0, 8]
Length of initial interval = |b – a| = |8 – 0| = 8
Here, error (ϵ_i) = 0.25
We have,

$\frac{|b-a|}{2^i} \le \epsilon_i$

$\Rightarrow \frac{8}{2^i} \le 0.25$

$\Rightarrow \frac{8}{0.25} \le 2^i$

$\Rightarrow 32 \le 2^i$

$\Rightarrow 2^5 \le 2^i$

$\Rightarrow 5 \le i$

$\Rightarrow i \ge 5$

Required no. of iterations (steps) = 5 or more.

4 MARKS QUESTIONS

22. 2070 Supp. Q.No. 17 a

Use the Bisection method to find the solution of the equation $x - 2^{-x} = 0$ in the interval [0, 1], accurate to within 10^{-3}. [4]

SOLUTION

Let $f(x) = x - 2^{-x} = x - \frac{1}{2^x}$

$f(0) = 0 - \frac{1}{2^0} = -1$

$f(1) = 1 - \frac{1}{2} = 0.5$

Since f(0) × f(1) = (–1) × 0.5 = –0.5 < 0, a root lies between 0 and 1.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
0	1	0.5	–1	0.5	–0.2071
0.5	1	0.75	–0.2071	0.5	0.1554
0.5	0.75	0.625	–0.2071	0.1554	–0.0234
0.625	0.75	0.6875	–0.0234	0.1554	0.06657
0.625	0.6875	0.65625	–0.0234	0.06657	0.02172
0.625	0.65625	0.640625	–0.0234	0.02172	–0.00081

Here, |f(m)| = |–0.00081| = 0.00081 < 10^{-3} = 0.001
So, required root is 0.640625.

23. 2071 Supp. Q.No. 17a

Find the solution of $x^2 - 10 = 0$ using the bisection method with a = 3, b = 4 and ε = 0.01. [4]

SOLUTION

Let $f(x) = x^2 - 10$
Here, a = 3, b = 4, ε = 0.01
$f(3) = 3^2 - 10 = -1$
$f(4) = 4^2 - 10 = 6$
Since f(3) × f(4) < 0, a real root lies between 3 and 4.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
3	4	3.5	-1	6	2.25
3	3.5	3.25	-1	2.25	0.5625
3	3.25	3.125	-1	0.5625	-0.2343
3.125	3.25	3.1875	-0.2343	0.5625	0.16015
3.125	3.1875	3.15625	-0.2343	0.16015	-0.03809
3.15625	3.1875	3.17188	-0.03809	0.16015	0.06079
3.15625	3.17188	3.16407	-0.03809	0.06079	0.01131
3.15625	3.16407	3.16016	-0.03809	0.01131	-0.01339
3.16016	3.16407	3.16212	-0.01339	0.01131	-0.00103

Here, |f(m)| = |–0.00103| = 0.00103 < 0.01
So, the required root is 3.16212.

24. 2072 Set D Q.No. 17b

Using the bisection method find the root of the equation $x^2 + x - 4 = 0$ in (1, 2) correct to two places of decimals. [4]

SOLUTION

Let $f(x) = x^2 + x - 4$
Here, a = 1, b = 2
$f(1) = 1^2 + 1 - 4 = -2$
$f(2) = 2^2 + 2 - 4 = 2$
Since f(1) and f(2) have opposite signs, so there is a root between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-2	2	-0.25
1.5	2	1.75	-0.25	2	0.8125
1.5	1.75	1.625	-0.25	0.8125	0.2656
1.5	1.625	1.5625	-0.25	0.2656	0.0039
1.5	1.5625	1.5313	-0.25	0.0039	-0.1238
1.5313	1.5625	1.5469	-0.1238	0.0039	-0.0602
1.5469	1.5625	1.5547	-0.0602	0.0039	-0.0282
1.5547	1.5625	1.5586	-0.0282	0.0039	-0.0122
1.5586	1.5625	1.5606	-0.0122	0.0039	-0.0039
1.5606	1.5625		-0.0039	0.0039	

The values of a and b are same to two places of decimal in last row. Hence, the approximate root to two places of decimal is 1.56.

25. 2073 Set C Q.No. 17a

Apply successive bisection method to find the root of the equation $x^3 - 4x - 1 = 0$ lying between 1 and 2 correct to two places of decimal. [4]

SOLUTION

Let $f(x) = x^3 - 4x - 1$
$f(1) = 1^3 - 4 \times 1 - 1 = -4$
$f(2) = 2^3 - 4 \times 2 - 1 = -1$
Since f(1) and f(2) have same sign, so there is no any root lying between 1 and 2.

26. 2074 Set A Q.No. 17a

Use the bisection method to find the solution of $x^3 - x - 1 = 0$ in the interval (1, 2) correct to three places of decimals. [4]

SOLUTION

Let $f(x) = x^3 - x - 1$
Here, $f(1) = 1^3 - 1 - 1 = -1$
$f(2) = 2^3 - 2 - 1 = 5$
Since f(1) & f(2) has opposite sign, a root lies between 1 & 2.

Now,

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-1	5	0.875
1	1.5	1.25	-1	0.875	-0.29688
1.25	1.5	1.375	-0.29688	0.875	0.22461
1.25	1.375	1.3125	-0.29688	0.22461	-0.05151
1.3125	1.375	1.34375	-0.05151	0.22461	0.08261
1.3125	1.34375	1.32813	-0.05151	0.08261	0.01458
1.3125	1.32813	1.32032	-0.05151	0.01458	-0.0187
1.32032	1.32813	1.32422	-0.0187	0.01458	-0.0021
1.32422	1.32813	1.32618	-0.0021	0.01458	0.00622
1.32422	1.32618	1.3252	-0.0021	0.00622	0.00206
1.32422	1.3252	1.32471	-0.0021	0.00206	-0.00003
1.32471	1.3252	1.32496	-0.00003	0.00206	0.00101
1.32471	1.32496		-0.00003	0.00101	

Since a & b are same upto 3 places of decimal, the required root is 1.324.

27. 2074 Set B Q.No. 17a

Show that the equation $x^3 - x - 4 = 0$ has two negative roots and one positive root and find the positive root correct to 3 places of decimal by successive bisection method. [4]

SOLUTION

SOLUTION

Given equation is $x^3 - x - 4 = 0$

Let $f(x) = x^3 - x - 4$

$f(1) = 1^3 - 1 - 4 = -4$

$f(2) = 2^3 - 2 - 4 = 2$

Since f(1) and f(2) have opposite signs, a root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-4	2	-2.125
1.5	2	1.75	-2.125	2	-0.3906
1.75	2	1.875	-0.3906	2	0.7167
1.75	1.875	1.8125	-0.3906	0.7167	0.1418
1.75	1.8125	1.7812	-0.3906	0.1418	-0.1296
1.7812	1.8125	1.7969	-0.1296	0.1418	0.0048
1.7812	1.7969	1.7890	-0.1296	0.0048	-0.0625
1.7890	1.7969	1.7929	-0.0625	0.0048	-0.0292
1.7929	1.7969	1.7949	-0.0292	0.0048	-0.0121
1.7949	1.7969	1.7959	-0.0121	0.0048	-0.0036
1.7959	1.7969	1.7964	-0.0036	0.0048	0.006
1.7959	1.7964	1.7961	-0.0036	0.0006	-0.0014
1.7961	1.7964		-0.0014	0.0006	

Since a and b have same value upto 3 places of decimal, so the required root is 1.796.

6 MARKS QUESTIONS

28. 2069 (Set A) Q.No. 19

Using method of bisection, find the root of the equation $x^3 - x - 4 = 0$ lying between 1 and 2 correct to 3 places of decimals. [6]

SOLUTION

Given equation is $x^3 - x - 4 = 0$

Let $f(x) = x^3 - x - 4$

$f(1) = 1^3 - 1 - 4 = -4$

$f(2) = 2^3 - 2 - 4 = 2$

Since f(1) and f(2) have opposite signs, a root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-4	2	-2.125
1.5	2	1.75	-2.125	2	-0.3906
1.75	2	1.875	-0.3906	2	0.7167
1.75	1.875	1.8125	-0.3906	0.7167	0.1418
1.75	1.8125	1.7812	-0.3906	0.1418	-0.1296
1.7812	1.8125	1.7969	-0.1296	0.1418	0.0048
1.7812	1.7969	1.7890	-0.1296	0.0048	-0.0625
1.7890	1.7969	1.7929	-0.0625	0.0048	-0.0292
1.7929	1.7969	1.7949	-0.0292	0.0048	-0.0121
1.7949	1.7969	1.7959	-0.0121	0.0048	-0.0036
1.7959	1.7969	1.7964	-0.0036	0.0048	0.006
1.7959	1.7964	1.7961	-0.0036	0.0006	-0.0014
1.7961	1.7964		-0.0014	0.0006	

Since a and b have same value upto 3 places of decimal, so the required root is 1.796.

29. 2069 (Set B) Q.No. 19

Show that the equation $f(x) = x^3 - x - 4$ has one positive root and using the method of bisection, find the positive root correct to 3 places of decimals. [6]

- Please refer to 2069 (Set A) Q.No. 19

30. 2070 Set C Q.No. 19

Find the root of the equation $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to three places of decimals by successive bisection method. [6]

SOLUTION

Let $f(x) = x^3 - 2x - 5$

Here, a = 2, b = 3

$f(2) = 2^3 - 2 \times 2 - 5 = -1$

$f(3) = 3^3 - 2 \times 3 - 5 = 16$

Since f(2) and f(3) have opposite sign, so one root lies between 2 and 3.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
2	3	2.5	-1	16	5.625
2	2.5	2.25	-1	5.625	1.8906
2	2.25	2.125	-1	1.8906	0.3457
2	2.125	2.0625	-1	0.3457	-0.3513
2.0625	2.125	2.0938	-0.3513	0.3457	-0.0084
2.0938	2.125	2.1094	-0.0084	0.3457	0.1671
2.0938	2.1094	2.1016	-0.0084	0.1671	0.0790
2.0938	2.1016	2.0977	-0.0084	0.0790	0.0352
2.0938	2.0977	2.0958	-0.0084	0.0352	0.0139
2.0938	2.0958	2.0948	-0.0084	0.0139	0.0028
2.0938	2.0948	2.0943	-0.0084	0.0028	-0.0028
2.0943	2.0948		-0.0028	0.0028	

The values of a and b are same to three places of decimal in last row. Hence, the approximate root to three places of decimal is 2.094.

31. 2070 Set D Q.No. 19

Applying the method of successive bisection, find the root of the equation $x^3 - 4x + 1 = 0$ lying between 1 and 2 correct to 2 places of decimals. [6]

SOLUTION

Let, $f(x) = x^3 - 4x + 1$

$f(1) = 1^3 - 4 \times 1 + 1 = -2$

$f(2) = 2^3 - 4 \times 2 + 1 = 1$

Since f(1) and f(2) have opposite signs, a real root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-2	1	-1.624
1.5	2	1.75	-1.625	1	-0.6406
1.75	2	1.875	-0.6406	1	0.0918
1.75	1.875	1.8125	-0.6406	0.0918	-0.2957
1.8125	1.875	1.8437	-0.2957	0.0918	-0.1073
1.8437	1.875	1.8594	-0.1073	0.0918	-0.0092
1.8594	1.875	1.8672	-0.0092	0.0918	0.0411
1.8594	1.8672	1.8633	-0.0092	0.0411	0.0172
1.8594	1.8633	1.8614	-0.0092	0.0172	0.0038
1.8594	1.8614	1.8604	-0.0092	0.0038	-0.0026
1.8604	1.8614		-0.0026	0.0038	

Since the values of a and b are same for 2 places of decimal, so required root is 1.86.

32. 2071 Set C Q.No. 19

Find a root of an equation $x^3 + x - 4 = 0$ in the interval [1, 4] within an accuracy of 10^{-1} [6]

SOLUTION

Here, $f(x) = x^3 + x - 4$

a = 1, b = 4

$f(1) = 1^3 + 1 - 4 = -2$

$f(4) = 4^3 + 4 - 4 = 64$

Since f(1) and f(4) have opposite signs, a real root lies between 1 and 4.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	4	2.5	-2	64	14.125
1	2.5	1.75	-2	14.125	3.1094
1	1.75	1.375	-2	3.1094	-0.0254

Here, $|f(m)| = |-0.0254| = 0.0254 < 0.1$

So, the required root is 1.375.

33. 2071 Set D Q.No. 19

Using the bisection method, find a root of the equation: [6]

$f(x) = 2x^3 - 5x + 2 = 0$, between 1 and 2 with error less than 10^{-2}

SOLUTION

Let $f(x) = 2x^3 - 5x + 2$

Here, a = 1, b = 2

$f(1) = 2 \times 1^3 - 5 \times 1 + 2 = -1$

$f(2) = 2 \times 2^3 - 5 \times 2 + 2 = 8$

Since $f(1) \times f(2) = -1 \times 8 = -8 < 0$, a root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-1	8	1.25
1	1.5	1.25	-1	1.25	-0.34375
1.25	1.5	1.375	-0.34375	1.25	0.32421
1.25	1.375	1.3125	-0.34375	0.3242	-0.04052
1.3125	1.375	1.34375	-0.04052	0.3242	0.13397
1.3125	1.34375	1.32813	-0.04052	0.13397	0.04478
1.3125	1.32813	1.32032	-0.04052	0.04478	0.00165

Here $|f(m)| = |0.00165| = 0.00165 < 10^{-2}$

So, the required root is 1.32032

34. 2072 Set C Q.No. 19

Apply the method of bisection to find the root of the equation $x^3 - 2x - 5 = 0$ in (2, 3) correct to three places of decimal. [6]

➥ Please refer to 2070 Set C Q.No. 19

35. 2072 Set E Q.No. 19

Using bisection method, find the root of the equation: $2x^3 - 5x + 2 = 0$, $x \in (1, 2)$ with error less than 10^{-2}. [6]

➥ Please refer to 2071 Set D Q.No. 19

36. 2072 Supp Q.No. 19

Using the method of bisection, find the root of the equation $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals. [6]

➥ Please refer to 2070 Set C Q.No. 19

37. 2073 Set D Q.No. 19

Show that the equation $f(x) = x^3 - 3x - 8 = 0$ has only one positive root. Using bisection method, find a root in (2,3) correct to 3 places of decimals. [6]

SOLUTION

Here, $f(x) = x^3 - 3x - 8$

Here, f(x) has 1 change in sign. So, it has only one positive root.

Now,

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
2	3	2.5	-6	10	0.125
2	2.5	2.25	-6	0.125	-3.3594
2.25	2.5	2.125	-3.3594	0.125	-4.7793
2.125	2.5	2.3125	-4.7793	0.125	-2.5710
2.3125	2.5	2.4063	-2.5710	0.125	-1.2857
2.4063	2.5	2.4532	-1.2857	0.125	-0.5958
2.4532	2.5	2.4766	-0.5958	0.125	-0.2395
2.4766	2.5	2.4833	-0.2395	0.125	-0.0582
2.4883	2.5	2.4942	-0.0582	0.125	0.0339
2.4883	2.4942	2.4913	-0.0582	0.0339	-0.0115
2.4913	2.4942	2.4928	-0.0115	0.0339	0.0120
2.4913	2.4928	2.4921	-0.0115	0.0120	0.0010
2.4913	2.4921	2.4917	-0.0115	0.0010	-0.0052
2.4917	2.4921	2.4919	-0.0052	0.0010	-0.0021
2.4919	2.4921	2.4920	-0.0021	0.0010	-0.0005
2.4920	2.4921		-0.0005	0.0010	

Since the values of a and b are same upto 3 places of decimal. So, the required root is 2.492.

38. 2074 Supp Q.No. 19

Applying the method of successive bisection, find the root of the equation $x^3 - 4x + 1 = 0$ lying between 1 and 2 correct to two place of decimals. [6]

➥ Please refer to 2070 Set D Q.N. 19

39. 2075 Set A Q.No. 19

Find the positive root of the equation $x^3 - x - 4 = 0$ correct to 3 places of decimal with error less than 0.005. [6]

➥ Please refer to 2069 Set A Q.No. 19

40. 2075 Set C Q.No. 19 OR

Find the roots of the equation $f(x) = x^3 - 4x - 9$ correct to three decimal places by using bisection method. [6]

SOLUTION

Let, $f(x) = x^3 - 4x - 9$

Here, $f(2) = 2^3 - 4 \times 2 - 9 = -9$

$f(3) = 3^3 - 4 \times 3 - 9 = 6$

Since f(2) and f(3) have opposite sign, one root lies between 2 and 3.

a	b	m	f(a)	f(b)	f(m)
2	3	2.5	-9	6	-3.375
2.5	3	2.75	-3.375	6	0.79688
2.5	2.75	2.625	-3.375	0.79688	-1.41211
2.625	2.75	2.6875	-1.41211	0.79688	-0.33911
2.6875	2.75	2.71875	-0.33911	0.79688	0.22092
2.6875	2.71875	2.70313	-0.33911	0.22092	-0.06099
2.70313	2.71875	2.71094	-0.06099	0.22092	0.07947
2.70313	2.71094	2.70704	-0.06099	0.07947	0.00921
2.70313	2.70704	2.70509	-0.06099	0.00921	-0.02583
2.70509	2.70704	2.70607	-0.02583	0.00921	-0.00823
2.70607	2.70704	2.70656	-0.00823	0.00921	0.00058
2.70607	2.70656		-0.00823	0.00058	

Since a and b are equal upto 3 places of the decimal, the required root is 2.706.

C. NEWTON RAPHSON'S METHOD

4 MARKS QUESTIONS

41. 2070 Supp. Q.No. 17 a OR

Use Newton – Raphson method to find the solution of the equation $x^3 + x - 1 = 0$ in the interval [0, 1], accurate to within 10^{-4}. [4]

SOLUTION

Here, $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

$f(0) = -1$

$f(1) = 1$

Since f(0) and f(1) have opposite signs, a real root lies between 0 and 1. Let us take initial guess $x_0 = 1$.

By Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 0.75$$

$|f(x_1)| = 0.17875 > 10^{-4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{0.171875}{2.6875} = 0.68605$$

$|f(x_2)| = 0.00895 > 10^{-4}$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.68605 - \frac{0.00895}{2.41199} = 0.68234$$

$|f(x_3)| = 0.00003 < 10^{-4}$

Hence, the required root is 0.68234.

42. 2073 Set C Q.No. 17a OR

Find a root of the equation $x^3 - x - 4 = 0$ between 1 and 2 to three places of decimal by Newton Raphson's method. [4]

SOLUTION

Given equation is $x^3 - x - 4 = 0$

Let $f(x) = x^3 - x - 4$

$f(1) = 1^3 - 1 - 4 = -4$

$f(2) = 2^3 - 2 - 4 = 2$

Since $f(1)\ f(2) = (-4) \times 2 = -8 < 0$, a real root lies between 1 and 2.

$f'(x) = 3x^2 - 1$

Let the initial guess $x_0 = 1$.

By Newton Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-4)}{2} = 3$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{20}{26} = 2.2307$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2307 - \frac{4.8693}{13.928} = 1.8811$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8811 - \frac{0.7752}{9.6156} = 1.8004$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.8004 - \frac{0.0354}{8.7243} = 1.7963$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.7963 - \frac{0.00019}{8.68} = 1.7962$$

Comparing the values of x_5 and x_6, we find that the digits in the first three places of decimal are same. Hence, the required root is 1.796.

43. 2074 Set A Q.No. 17a OR

Use Newton-Raphson's method to approximate $\sqrt[3]{2}$ with an error less than 0.00001. [4]

SOLUTION

Let, $x = \sqrt[3]{2}$

or, $x^3 = 2$

or, $x^3 - 2 = 0$

Let $f(x) = x^3 - 2$

$f'(x) = 3x^2$

We have, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Let us take an initial guess $(x_0) = 1$.

Now, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^3 - 2}{3.1^2} = 1.333333$

$|f(x_1)| = |(1.333333)^3 - 2| = 0.370369 > 0.00001.$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.333333 - \frac{(1.333333)^3 - 2}{3.(1.333333)^2} = 1.263889$

$|f(x_2)| = |(1.263889)^3 - 2| = 0.018955 > 0.00001$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.263889 - \frac{(1.263889)^3 - 2}{3.(1.263889)^2} = 1.259933$

$|f(x_3)| = |(1.259933)^3 - 2| = 0.000059 > 0.00001$

$x_4 = x_3 - \frac{(x_3)}{f'(x_3)} = 1.259933 - \frac{(1.259933)^3 - 2}{3(1.259933)^2} = 1.259921$

$|f(x_4)| = |(1.259921)^3 - 2| = 0.00000001 < 0.00001$

Hence the required root is 1.259921

44. 2074 Set B Q.No. 17a OR

Using Newton-Raphson's method, find the positive root of $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to three places of decimal. [4]

SOLUTION

Let, $f(x) = x^3 - 2x - 5$

Then $f'(x) = 3x^2 - 2$

$f(2) = 2^3 - 2 \times 3 - 5 = -1$

$f(3) = 3^3 - 2 \times 3 - 5 = 16$

Since f(2) and f(3) have opposite signs, a root lies between 2 and 3.

Let us take initial guess $(x_0) = 2$

By Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-1)}{6} = 2.1666$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1666 - \frac{0.8371}{12.0824} = 2.0973$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0973 - \frac{0.03072}{11.196} = 2.0945$$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.0945 - \frac{(-0.0005)}{11.1607} = 2.0945$

Comparing the values of x_3 and x_4, we find that the digits in the first three places of decimal are the same. Hence, the required root to three places of decimal is 2.094.

6 MARKS QUESTIONS

45. 2069 (Set A) Q.No. 19 or

Using Newton-Raphson's method, find the square root of 153 correct to 3 places of decimals. [6]

SOLUTION

Let x be the square root of 153.

Then, $x^2 = 153$

$x^2 - 153 = 0$

Let $f(x) = x^2 - 153$

$\therefore$ a = 153

Take, initial guess $(x_0) = 10$

By Newton-Raphson's method, we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$$\therefore x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) = \frac{1}{2}\left(10 + \frac{153}{10}\right) = 12.65$$

$$\therefore x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = \frac{1}{2}\left(12.65 + \frac{153}{12.65}\right) = 12.37243$$

$$\therefore x_3 = \frac{1}{2}\left(x_2 + \frac{a}{x_2}\right) = \frac{1}{2}\left(12.37243 + \frac{153}{12.37243}\right) = 12.36931$$

$$\therefore x_4 = \frac{1}{2}\left(x_3 + \frac{a}{x_3}\right) = \frac{1}{2}\left(12.36931 + \frac{153}{12.36931}\right) = 12.36931$$

Comparing the values of x_3 and x_4, we find that the digits in the first three places of decimal are same. Hence, the required square roots of 153 to three places of decimal is 12.369.

46. 2069 (Set B) Q.No. 19 Or

Using Newton Raphson's method find the positive root of the equation $f(x) = x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals. [6]

SOLUTION

Let, $f(x) = x^3 - 2x - 5$

Then $f'(x) = 3x^2 - 2$

$f(2) = 2^3 - 2 \times 3 - 5 = -1$

$f(3) = 3^3 - 2 \times 3 - 5 = 16$

Since f(2) and f(3) have opposite signs, a root lies between 2 and 3.

Let us take initial guess $(x_0) = 2$

By Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-1)}{6} = 2.1666$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.1666 - \frac{0.8371}{12.0824} = 2.0973$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0973 - \frac{0.03072}{11.196} = 2.0945$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.0945 - \frac{(-0.0005)}{11.1607} = 2.0945$$

Comparing the values of x_3 and x_4, we find that the digits in the first three places of decimal are the same. Hence, the required root to three places of decimal is 2.094.

47. 2070 Set C Q.No. 19 or

Solve $2x^2 - 3x - 1 = 0$ using Newton-Raphson method taking $x_0 = 1$ with error less than 10^{-4}. [6]

SOLUTION

Here, $f(x) = 2x^2 - 3x - 1$

$f(1) = 2 \times 1^2 - 3 \times 1 - 1 = -2$

$f(2) = 2 \times 2^2 - 3 \times 2 - 1 = 1$

Since f(1) and f(2) have opposite signs, so there is a real root between 1 and 2.

$f'(x) = 4x - 3$

Let us take initial guess $x_0 = 1$

By Newton Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-2)}{1} = 3$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{8}{9} = 2.1111$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1111 - \frac{1.5802}{5.4444} = 1.8209$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8209 - \frac{0.1684}{4.2836} = 1.7816$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.7816 - \frac{0.0033}{4.1264} = 1.7808$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.7808 - \frac{0.0001}{4.1232} = 1.7808$$

Here, $|f(1.7808)| = |2(1.7808)^2 - 3(1.7808) - 1| = 0.00009728 < 0.0001$

Hence, the required root is 1.7808.

48. 2070 Set D Q.No. 19 Or

Using Newton-Raphson method, find the positive root of $x^3 - 18 = 0$ in (2, 3) [6]

SOLUTION

Here, $f(x) = x^3 - 18$

$f'(x) = 3x^2$

$f(2) = 2^3 - 18 = -10$

$f(3) = 3^3 - 18 = 9$

Since f(2) and f(3) have opposite signs, a real root lies between 2 and 3.

Let initial guess $x_0 = 2$.

By Newton-Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-10)}{12} = 2.83333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8333 - \frac{4.74529}{24.08328} = 2.63629$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.63629 - \frac{0.32228}{20.85007} = 2.62083$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.62083 - \frac{0.00182}{20.60625} = 2.62074$$

Comparing the values of x_3 and x_4, we find the digits in the first three places of decimal are equal, so the required root is 2.62.

49. 2071 Set C Q.No. 19 OR

Find a root of the equation $x^3 - x - 4 = 0$ between 1 and 2 to three places of decimal by Newton-Raphson method. [6]

SOLUTION

Given equation is $x^3 - x - 4 = 0$

Let $f(x) = x^3 - x - 4$

$f(1) = 1^3 - 1 - 4 = -4$

$f(2) = 2^3 - 2 - 4 = 2$

Since $f(1)\,f(2) = (-4) \times 2 = -8 < 0$, a real root lies between 1 and 2.

$f'(x) = 3x^2 - 1$

Let the initial guess $x_0 = 1$.

By Newton Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-4)}{2} = 3$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{20}{26} = 2.2307$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2307 - \frac{4.8693}{13.928} = 1.8811$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8811 - \frac{0.7752}{9.6156} = 1.8004$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.8004 - \frac{0.0354}{8.7243} = 1.7963$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.7963 - \frac{0.00019}{8.68} = 1.7962$$

Comparing the values of x_5 and x_6, we find that the digits in the first three places of decimal are same. Hence, the required root is 1.796.

50. 2071 Set D Q.No. 19 OR

Derive the formula for Newton-Raphson method. Using Newton Raphson method, find a positive root of $x^3 + 3x - 5 = 0$ lying between 1 and 2 correct to three places of decimals. [6]

SOLUTION

Newton-Raphson Formula

Let x_0 be the initial guess for the function $y = f(x)$. Then for a point (x, y) sufficiently close to it, the function can be approximated by its tangent line

$$\frac{y - f(x_0)}{x - x_0} = f'(x_0)$$

This tangent line crosses the x-axis when $y = 0$. Let us denote this new value of x by x_1. Then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Continuing in this way, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Second Part

Please refer to Model Set II, Q.No. 19 or

51. 2072 Set E Q.No. 19 OR

Find a root of the equation $2x^2 - 3x - 1 = 0$, $x \in (1, 2)$ using Newton Raphson method with error less than 10^{-4}. [6]

➥ Please refer to 2070 Set C Q.No. 19 or

52. 2072 Supp Q.No. 19 OR

Using Newton Raphson's method, find the root of the equation $f(x) = x^3 - x - 4 = 0$ in (1, 2) correct to 3 places of decimals. [6]

➥ Please refer to 2071 Set C Q.No. 19 OR

53. 2073 Set D Q.No. 19 OR

Using Newton -Raphson method, find a root of the equation $f(x) = x^3 - x - 4 = 0$ in (1,2) correct to 3 places of decimals. [6]

➥ Please refer to 2071 Set C Q.No. 19 OR

54. 2073 Supp Q.No. 19 or

Newton Raphson's Method to find a positive root of $\cos x = x^3$. [6]

SOLUTION

Let $f(x) = \cos x - x^3 = 0$

$f'(x) = -\sin x - 3x^2$.

Now, $f(0) = \cos(0) - 0^3 = 1$, & $f(1) = -0.0001523$. Since $f(0)$ & $f(1)$ are of opposite sign, a root lies between 0 & 1.

Let us take an initial guess $x_0 = 0.5$

From Newton Raphson method, we have,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{\cos(0.5) - (0.5)^3}{-\sin(0.5) - 3(0.5)^2} = 1.657284686$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.226971721$$

$$x_3 = x_2 - \frac{(x_2)}{f'(x_2)} = 1.039895734$$

$$x_4 = x_3 - \frac{(x_3)}{f'(x_3)} = 1.001681167$$

$$x_5 = x_4 - \frac{(x_4)}{f'(x_4)} = 0.9999831544$$

$$x_6 = x_5 - \frac{(x_5)}{f'(x_5)} = 0.9999904291$$

$$x_7 = x_6 - \frac{(x_6)}{f'(x_6)} = 0.9999754604$$

$$x_8 = x_7 - \frac{(x_7)}{f'(x_7)} = 0.9999961572$$

$$x_9 = x_8 - \frac{(x_8)}{f'(x_8)} = 0.9999700202$$

$$x_{10} = x_9 - \frac{(x_9)}{f'(x_9)} = 0.9999736755$$

∴ Required root = 0.99997

Note: The angle is in radian measure.

55. 2074 Supp Q.No. 19 OR

Using Newton-Raphson method, find a root of the equation $x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals. [6]

➥ Please refer to 2069 Set B Q.N. 19 OR

56. 2075 Set A Q.No. 19 OR

Using Newton-Raphson method, find a root of the equation $x^3 + 3x - 5 = 0$ between 1 and 2 to three places of decimals. [6]

➥ Please refer to Model Set II, Q.No. 19 OR

57. 2075 Set B Q.No. 18

For $f(x) = x^3 - 4$, perform 3 iterations of Newton-Raphson's method with starting point $x_0 = 2$. Find the errors and percentage errors of x_0, x_1, x_2 and x_3. [6]

SOLUTION

Here,

$f(x) = x^3 - 4$

$f'(x) = 3x^2$

and $x_0 = 2$

By Newton Raphson method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore\ x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(2^3 - 4)}{3 \times 2^2} = 1.66667$$

$$\text{Error } (E_1) = \left|\frac{x_1 - x_0}{x_1}\right| = \left|\frac{1.66667 - 2}{1.66667}\right| = 0.1999$$

Percentage error = $E_1 \times 100 = 0.1999 \times 100 = 19.99\%$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.66667 - \frac{(1.66667)^3 - 4}{3 \times (1.66667)^2} = 1.59111$$

$$\text{Error } (E_2) = \left|\frac{x_2 - x_1}{x_2}\right| = \left|\frac{1.59111 - 1.66667}{1.59111}\right| = 0.0475$$

Percentage Error = $E_2 \times 100 = 0.0475 \times 100 = 4.75\%$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.59111 - \frac{(1.59111)^3 - 4}{3 \times (1.59111)^2} = 1.58741$$

$$\text{Error } (E_3) = \left|\frac{x_3 - x_2}{x_3}\right| = \left|\frac{1.58741 - 1.59111}{1.58741}\right| = 0.0023$$

Percentage Error $= 0.0023 \times 100$

$= 0.23\%$

58. 2075 Set C Q.No. 19

Use Newton-Raphson method (formula) to find the solutions of $f(x) = 1 - 12x + x^3$ correct upto four decimal places. [6]

SOLUTION

We draw the graph of the function in order to make the initial guess.

Here,

$f(x) = 1 - 12x + x^3$

$f'(x) = 3x^2 - 12$

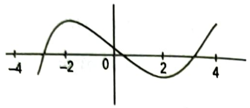
$f''(x) = 6x$

Now,

$f'(x) = 0 \Rightarrow 3x^2 - 12 = 0$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$



Also, $f(2) = -15$ and $f(-2) = 17$

Hence f has local maximum at – 2 and minimum at 2.

Also at $x = 0, y = 1$.

From the graph, it is clear that there are three solutions, one near to 0, another less than –2 and the remaining is greater than 0.

Solution near 0

Take $x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{(-12)} = \frac{1}{12} = 0.08333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.08333 - \frac{1 - 12(0.08333) + (0.08333)^3}{3(0.08333)^2 - 12} = 0.08338$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.08338 - \frac{1 - 12(0.08338) + (0.08338)^3}{3(0.08338)^2 - 12} = 0.08338$$

Hence the solution is 0.0833 upto four places of decimal.

Solution greater than 2

Take $x_0 = 3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{1 - 12 \times 3 + 3^3}{3 \times 3^2 - 12}$$

$$= 3.53333$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3.426797$$

$x_3 = 3.421669$

$x_4 = 3.421658$

∴ Required root correct to 4 place of decimal = 3.4216

Solution less than –2

Take $x_0 = -3$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= (-3) - \frac{1 - 12 \times (-3) + (-3)^3}{3 \times (-3)^2 - 12}$$

$$= -3.66666$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -3.515032$$

$x_3 = -3.505081$

$x_4 = -3.505039$

Thus, the required root upto 4 places of decimal in this case is –3.5050.

Hence, the required solutions are 0.0833, 3.4216 and –3.5050

UNIT 18

COMPUTATIONAL METHOD (CONTINUED)

A. GAUSS ELIMINATION METHOD

2 MARKS QUESTIONS

1. 2070 Supp. Q.No. 16 c

Test the consistency of the following system by the Gauss elimination method:

$x - y - 2z = -1$, $2x + y + z = 2$, $3x + 2y + 9z = 4$ [2]

SOLUTION

Given equations are:

$x - y - 2z = -1$ ...(i)

$2x + y + z = 2$ ...(ii)

$3x + 2y + 9z = 4$ ... (iii)

Multiplying equation (i) by 2, then subtracting from equation (ii), we have

$$\begin{array}{rrrl} 2x & +y & +z & = 2 \\ 2x & -2y & -4z & = -2 \\ - & + & + & + \\ \hline & 3y & +5z & = 4 \end{array} \quad \text{...(iv)}$$

Again, multiplying equation (i) by 3 and then subtracting from equation (iii), we have

$$\begin{array}{rrrl} 3x & +2y & +9z & = 4 \\ 3x & -3y & -6z & = -3 \\ - & + & + & + \\ \hline & 5y & +15z & = 7 \end{array} \quad \text{... (v)}$$

Multiplying equation (iv) by $\frac{5}{3}$ and subtracting from equation (v)

$$\begin{array}{rrl} 5y & +15z & = 7 \\ 5y & +\frac{25}{3}z & = \frac{20}{3} \\ - & - & - \\ \hline & \frac{20}{3}z & = \frac{1}{3} \end{array} \quad \text{... (vi)}$$

Now, we have the following three equations

$x - y - 2z = -1$ ... (i)

$3y + 5z = 4$ ...(iv)

$\frac{20}{3}z = \frac{1}{3}$ ... (vi)

From equation (vi), we can find a finite value of z and then finite values of x and y. So, the system of equation is consistent.

2. 2071 Supp. Q.No. 16c

By Gauss elimination method, solve

$2x + 3y = 4$, $3x + 2y = -4$ [2]

SOLUTION

Given equations are:

$2x + 3y = 4$ ... (i)

$3x + 2y = -4$ ... (ii)

Multiplying equation (i) by $\frac{3}{2}$ and then subtracting from (ii),

$$\begin{array}{rrl} 3x & +2y & = -4 \\ 3x & +\frac{9}{2}y & = 6 \\ - & - & - \\ \hline & -\frac{5}{2}y & = -10 \end{array} \quad \text{...(iii)}$$

Now, we have the following equations

$2x + 3y = 4$ ...(i)

$-\frac{5}{2}y = -10$ ... (iii)

From equation (iii), we have $y = 4$

Using $y = 4$ in equation (i), we have

$2x + 3 \times 4 = 4$

or, $x = -4$

∴ The required solution is $x = -4$, $y = 4$.

4 MARKS QUESTIONS

3. 2074 Supp Q.No. 17a

Using Gauss-elimination method, solve the following system of equation: [4]

$x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$.

➥ Please refer to 2070 Set C Q.N. 17a

4. 2075 Set A Q.No. 17a

Solve the following system of equations by Gauss-elimination method. [4]

$x + 3y - z = -2$, $3x + 2y - z = 3$, $-6x - 4y - 2z = 18$

➥ Please refer to Model Set II Q.No. 17a

5. 2069 (Set A) Q.No. 17a

Using Gauss elimination method, solve the following system of equations: [4]

$x - 2y + 3z = 2$, $2x - 3y + z = 1$, $3x - y + 2z = 9$

SOLUTION

Given equations are:

$x - 2y + 3z = 2$ ...(i)

$2x - 3y + z = 1$ ...(ii)

$3x - y + 2z = 9$ ... (iii)

Multiplying equation (i) by 2 and then subtracting from equation (ii)

$$\begin{array}{rrrrl} 2x & -3y & +z & =1 & \\ 2x & -4y & +6z & =4 & \\ - & + & - & - & \\ \hline & y & -5z & =-3 & \ldots(iv) \end{array}$$

Again, multiplying equation (i) by 3 then subtracting from equation (iii)

$$\begin{array}{rrrrl} 3x & -y & +2z & =9 & \\ 3x & -6y & +9z & =6 & \\ - & + & - & - & \\ \hline & 5y & -7z & =3 & \ldots(v) \end{array}$$

Multiplying equation (iv) by 5, then subtracting from (v)

$$\begin{array}{rrrl} 5y & -7z & =3 & \\ 5y & -25z & =-15 & \\ - & + & + & \\ \hline & 18z & =18 & \ldots(vi) \end{array}$$

Now, we have the following three equations

$$\begin{array}{rrrrl} x & -2y & +3z & =2 & \ldots(i) \\ & y & -5z & =-3 & \ldots(iv) \\ & & 18z & =18 & \ldots(vi) \end{array}$$

From equation (vi), we have z = 1

Using z = 1 in equation (iv), we have

$y - 5 \times 1 = -3$

$\therefore$ y = 2

Again, using y = 2 and z = 1 in equation (i), we have

$x - 2 \times 2 + 3 \times 1 = 2$

$\therefore$ $x = 3$

$\therefore$ The required solution is $x = 3$, y = 2, z = 1.

6. 2069 (Set B) Q.No. 17a

Using Gauss elimination method, solve the following system of equations:
$x_1 - 2x_2 + 3x_3 = 10$, $2x_1 + 3x_2 - 2x_3 = 1$, and $-x_1 - 2x_2 + 4x_3 = 13$. [4]

SOLUTION

Given equations are:

$x_1 - 2x_2 + 3x_3 = 10$ ...(i)

$2x_1 + 3x_2 - 2x_3 = 1$ ...(ii)

$-x_1 - 2x_2 + 4x_3 = 13$ ...(iii)

Multiplying equation (i) by 2 and then subtracting from equation (ii), we have

$$\begin{array}{rrrrl} 2x_1 & +3x_2 & -2x_3 & =1 & \\ 2x_1 & -4x_2 & +6x_3 & =20 & \\ - & + & - & - & \\ \hline & 7x_2 & -8x_3 & =-19 & \ldots(iv) \end{array}$$

Again, adding equation (i) and equation (iv) we have

$$\begin{array}{rrrrl} -x_1 & -2x_2 & +4x_3 & =13 & \\ x_1 & -2x_2 & +3x_3 & =10 & \\ \hline & -4x_2 & +7x_3 & =23 & \ldots(v) \end{array}$$

Multiplying equation (iv) by $\frac{4}{7}$ and adding with equation (v)

$$\begin{array}{rrrl} -4x_2 & +7x_3 & =23 & \\ 4x_2 & -\frac{32}{7}x_3 & =\frac{-76}{7} & \\ \hline & \frac{17}{7}x_3 & =\frac{85}{7} & \ldots(vi) \end{array}$$

Now, we have the following three equations

$$\begin{array}{rrrrl} x_1 & -2x_2 & +3x_3 & =10 & \ldots(i) \\ & 7x_2 & -8x_3 & =-19 & \ldots(iv) \\ & & \frac{17}{7}x_3 & =\frac{85}{7} & \ldots(vi) \end{array}$$

From equation (i), we have $x_3 = 5$

Using $x_3 = 5$ in equation (iv), we have

$7x_2 - 40 = -19$

$\therefore$ $x_2 = 3$

Using $x_2 = 3$, $x_3 = 5$ in equation (i), we have

$x_1 - 2 \times 3 + 3 \times 5 = 10$

$\therefore$ $x_1 = 1$

$\therefore$ The required solution is $x_1 = 1$, $x_2 = 3$, $x_3 = 5$.

7. 2070 Set C Q.No. 17 a

Solve, using Gauss elimination method, the following equations. [4]
$x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$.

SOLUTION

Given equations are:

$x + 3y - 2z = 5$ ... (i)

$3x + 5y + 6z = 7$ ... (ii)

$2x + 4y + 3z = 8$ ... (iii)

Multiplying equation (i) by 3 and then subtracting from (ii), we have

$$\begin{array}{rrrrl} 3x & +5y & +6z & =7 & \\ 3x & +9y & -6z & =15 & \\ - & - & + & - & \\ \hline & -4y & +12z & =-8 & \ldots(iv) \end{array}$$

Again, multiplying equation (i) by 2 and then subtracting from equation (iii), we have

$$\begin{array}{rrrrl} 2x & +4y & +3z & =8 & \\ 2x & +6y & -4z & =10 & \\ - & - & + & - & \\ \hline & -2y & +7z & =-2 & \ldots(v) \end{array}$$

Multiplying equation (iv) by 1/2 and then subtracting from (v)

$$\begin{array}{rrrl} -2y & +7z & =-2 & \\ -2y & +6z & =-4 & \\ + & - & + & \\ \hline & z & =2 & \ldots(vi) \end{array}$$

Now, we have the following three equations

$$\begin{array}{rrrrl} x & +3y & -2z & =5 & \ldots(i) \\ & -4y & +12z & =-8 & \ldots(iv) \\ & & z & =2 & \ldots(vi) \end{array}$$

From equation (vi), we have z = 2.

Using z = 2 in equation (iv), we have y = 8

Using y = 8 and z = 2 in equation (i), we have $x = -15$.

$\therefore$ The required solution is $x = -15$, y = 8, z = 2.

8. 2070 Set D Q.No. 17 a

Solve the following system of equation by Gaussian elimination method. [4]
$x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$.

➥ Please refer to 2070 Set C Q.No. 17 a

9. 2071 Set C Q.No. 17 a

Using Gauss-elimination method, solve the following system of equation.
$x + 3y - z = -2$, $3x + 2y - z = 3$, $-6x - 4y - 2z = 18$ [4]

➥ Please refer to Model Set II, Q.No. 17a

10. 2071 Set D Q.No. 17 a

Using Gauss-elimination method, solve the following system of equation: $x - 2y + 3z = 2$, $2x - 3y + z = 1$, $3x - y + 2z = 9$. [4]

➥ Please refer to 2069 (Set A) Q.No. 17a

11. 2072 Set C Q.No. 17a OR

Solve by Gauss elimination method:
$x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$ [4]

➥ Please refer to 2070 Set C Q.No. 17 a

12. 2072 Set D Q.No. 17a OR

Use Gauss elimination method to solve: [4]
$4x - y + z = 8$, $2x + 5y + 2z = 3$, $x + 2y + 4z = 11$.

SOLUTION

Given equations are:

$4x - y + z = 8$ ...(i)

$2x + 5y + 2z = 3$ ... (ii)

$x + 2y + 4z = 11$ ... (iii)

Multiplying equation (i) by 1/2 and then subtracting from equation (ii)

$$\begin{array}{rrrrl} 2x & +5y & +2z & =3 & \\ 2x & -\frac{1}{2}y & +\frac{1}{2}z & =4 & \\ - & + & - & - & \\ \hline & \frac{11}{2}y & +\frac{3}{2}z & =-1 & \ldots(iv) \end{array}$$

Again, multiplying equation (i) by $\frac{1}{4}$ and then subtracting from equation (iii)

$$\begin{array}{rrrrl} x & +2y & +4z & =11 & \\ 2x & -\frac{1}{4}y & +\frac{1}{4}z & =2 & \\ - & + & - & - & \\ \hline & \frac{9}{4}y & +\frac{15}{4}z & =9 & \ldots(v) \end{array}$$

Again, multiplying equation (iv) by $\frac{9}{22}$ and subtracting from (v)

$$\begin{array}{rrrl} \frac{9}{4}y & +\frac{15}{4}z & =9 & \\ \frac{9}{4}y & +\frac{27}{44}z & =-\frac{9}{22} & \\ - & - & + & \\ \hline & \frac{69}{22}z & =\frac{207}{22} & \ldots(vi) \end{array}$$

Now, we have the following three equations

$$\begin{array}{rrrrl} 4x & -y & +z & =8 & \ldots(i) \\ & \frac{11}{2}y & +\frac{3}{2}z & =-1 & \ldots(iv) \\ & & \frac{69}{22}z & =\frac{207}{22} & \ldots(vi) \end{array}$$

From equation (vi), we have z = 3.

Using z = 3 in equation (iv), we have y = –1

Using y = –1 and z = 3 in equation (i), we have $x = 1$

$\therefore$ The required solution is $x = 1$, y = –1, z = 3.

13. 2072 Set E Q.No. 17a

Using Gauss-elimination method, solve the following system of equations.
$2x_2 + 3x_3 = 7$, $3x_1 - 2x_2 + 2x_3 = 1$, $2x_1 + 3x_2 - 3x_3 = 5$. [4]

SOLUTION

The given equations are:

$2x_2 + 3x_3 = 7$

$3x_1 - 2x_2 + 2x_3 = 1$

$2x_1 + 3x_2 - 3x_3 = 5$

The coefficient of the first variable x_1 is zero in the first equation. So, interchanging the first two equations, we have

$3x_1 - 2x_2 + 2x_3 = 1$ ...(i)

$2x_2 + 3x_3 = 7$ ...(ii)

$2x_1 + 3x_2 - 3x_3 = 5$ ...(iii)

Multiplying equation (i) by $\frac{2}{3}$ and the subtracting from equation (iii)

$$\begin{array}{rrrrl} 2x_1 & +3x_2 & -3x_3 & =5 & \\ 2x_1 & -\frac{4}{3}x_2 & +\frac{4}{3}x_3 & =\frac{2}{3} & \\ - & + & - & - & \\ \hline & \frac{13}{3}x_2 & -\frac{13}{3}x_3 & =\frac{13}{3} & \\ \text{or,} & x_2 & -x_3 & =1 & \ldots(iv) \end{array}$$

Again, multiplying equation (ii) by $\frac{1}{2}$ and subtracting from equation (iv), we have

$$\begin{array}{rrrl} x_2 & -x_3 & =1 & \\ x_2 & +\frac{3}{2}x_3 & =\frac{7}{2} & \\ - & - & - & \\ \hline & -\frac{5}{2}x_3 & =-\frac{5}{2} & \ldots(v) \end{array}$$

Now, we have the following three equations

$$\begin{array}{rrrrl} 3x_1 & -2x_2 & +2x_3 & =1 & \ldots(i) \\ & x_2 & -x_3 & =1 & \ldots(ii) \\ & & -\frac{5}{2}x_3 & =-\frac{5}{2} & \ldots(v) \end{array}$$

From equation (v), $x_3 = 1$

Using $x_3 = 1$ in equation (iv), we have $x_2 = 2$

Again, using $x_2 = 2$ and $x_1 = 1$ in equation (i), we have

$x_1 = 1$

$\therefore$ The required solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 1$.

14. 2072 Supp Q.No. 17a

Solve the following system of equations using Gauss elimination method: [4]
$x + 3y - 2z = 5$; $3x + 5y + 6z = 7$; $2x + 4y + 3z = 8$

➥ Please refer to 2070 Set C Q.No. 17 a

15. 2073 Set C Q.No. 17b

Solve by Gauss elimination method. **[4]**

$3x_1 + x_2 + x_3 = 5$, $x_1 - 4x_2 + x_3 = -2$, $x_1 + x_2 - 3x_3 = -1$.

SOLUTION

Given equations are:

$3x_1 + x_2 + x_3 = 5$ ...(i)

$x_1 - 4x_2 + x_3 = -2$ ...(ii)

$x_1 + x_2 - 3x_3 = -1$ ...(iii)

Multiplying equation (i) by $\frac{1}{3}$ and subtracting from (ii)

$x_1 - 4x_2 + x_3 = -2$

$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = \frac{5}{3}$

− − − −

$-\frac{13}{3}x_2 - \frac{2}{3}x_3 = -\frac{11}{3}$ ...(iv)

Again, multiplying equation (i) by $\frac{1}{3}$ and subtracting from (iii),

$x_1 + x_2 - 3x_3 = -1$

$x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 = \frac{5}{3}$

− − − −

$\frac{2}{3}x_2 - \frac{10}{3}x_3 = -\frac{8}{3}$ ...(v)

Again, multiplying equation (iv) by $\frac{2}{13}$ and adding with equation (v).

$-\frac{2}{3}x_2 + \frac{4}{39}x_3 = -\frac{22}{39}$

$\frac{2}{3}x_2 - \frac{10}{3}x_3 = -\frac{8}{3}$

$-\frac{126}{39}x_3 = -\frac{126}{39}$ ...(vi)

$-\frac{2}{3}x_2 + \frac{4}{39}x_3 = -\frac{22}{39}$

$\frac{2}{3}x_2 - \frac{10}{3}x_3 = -\frac{8}{3}$

$-\frac{126}{39}x_3 = -\frac{126}{39}$ ...(vi)

Now, we have the following system of equations.

$3x_1 + x_2 + x_3 = 5$ ...(i)

$-\frac{13}{3}x_2 + \frac{2}{3}x_3 = -\frac{11}{3}$ ...(iv)

$-\frac{126}{39}x_3 = -\frac{126}{39}$ ...(vi)

From equation (vi), we have, $x_3 = 1$

Using $x_3 = 1$ in (iv)

$-\frac{13}{3}x_2 = -\frac{11}{3} - \frac{2}{3}$

$\therefore x_2 = 1$

Again, using $x_1 = 1$ and $x_2 = 1$ in (i), we get

$3x_1 + 1 + 1 = 5$

$x_1 = 1$

$\therefore x_1 = 1, x_2 = 1, x_3 = 1$

16. 2073 Set D Q.No. 17a

Using Gauss-elimination method, solve the following system of equations:

$2x - 3y + 3z = 27$, $4x + y - 2z = 0$, $-6x - 4y + 2z = 0$. **[4]**

SOLUTION

Given equations are

$2x - 3y + 3z = 27$ ...(i)

$4x + y - 2z = 0$ ...(ii)

$-6x - 4y + 2z = 0$ ...(iii)

Multiplying equation (i) by 2 and subtracting from equation (ii)

$4x + y - 2z = 0$

$4x - 6y + 6z = 54$

− + + −

$7y - 8z = -54$ ...(iv)

Again, multiplying equation (i) by 3 and adding with equation (iii)

$6x - 9y + 9z = 81$

$-6x - 4y + 2z = 0$

$-13y + 11z = 81$ ...(v)

Again, multiplying equation (iv) by $\frac{13}{7}$ and adding with equation (v)

$13y - \frac{104}{7}z = \frac{-702}{7}$

$-13y + 11z = 81$

$-\frac{27}{7}z = \frac{-135}{7}$ ...(vi)

Now, we have the following system of equations

$2x - 3y + 3z = 27$ ...(i)

$7y - 8z = -54$ ...(iv)

$-\frac{27}{7}z = -\frac{135}{7}$ ...(vi)

From equation (vi), $z = 5$

Using $z = 5$ in equation (iv),

$7y = -14$

$\therefore y = -2$

Again, using $z = 5$ and $y = -2$ in (i), we get,

$2x - 3 \times (-2) + 3 \times 5 = 27$

$\therefore x = 3$

Hence, the solution is $x = 3$, $y = -2$, $z = 5$.

17. 2074 Set A Q.No. 17b

Solve the following system of equations by Gauss elimination method,

$2x + 3y + 4z = 20$, $3x + 4y + 5z = 26$, $3x + 5y + 6z = 31$. **[4]**

SOLUTION

Given equations are

$2x + 3y + 4z = 20$ ... (i)

$3x + 4y + 5z = 26$ ... (ii)

$3x + 5y + 6z = 31$ ... (iii)

Multiplying equation (i) by $\frac{3}{2}$ and then subtracting from (ii), we get

$3x + 4y + 5z = 26$

$3x + \frac{9}{2}y + 6z = 30$

− − − −

$-\frac{1}{2} - z = -4$ ... (iv)

Again, multiplying equation (i) by $\frac{3}{2}$ & subtracting, from equation (iii)

$3x + 5y + 6z = 31$

$3x + \frac{9}{2}y + 6z = 30$

− − − −

$\frac{1}{2}y - 0.z = 1$ ... (v)

Adding equation (iv) & equation (v)

$\frac{1}{2}y - 0.z = 1$

$-\frac{1}{2}y - z = -4$

$-z = -3$ ... (vi)

Now, we have the following three equations

$2x + 3y + 4z = 20$ ... (i)

$-\frac{1}{2}y - z = -4$ ... (iv)

$-z = -3$ ... (vi)

From equation (vi), we have $z = 3$

Putting the value of z in (iv), we get

$-\frac{1}{2}y - 3 = -4$

or, $-\frac{1}{2}y = -4 + 3 = -1$

or, $y = 2$

Again putting the value of y & z in (i)

$2x + 3 \times 2 + 4 \times 3 = 20$

or, $2x = 20 - 18 = 2$

or, $x = 1$

$\therefore x = 1, y = 2, z = 3$

18. 2074 Set B Q.No. 17b

Solve by Gauss elimination method: **[4]**

$x + 3y - 2z = 5$, $3x + 5y + 6z = 7$, $2x + 4y + 3z = 8$.

➥ Please refer to 2070 Set C Q.No. 17a

6 MARKS QUESTIONS

19. 2075 Set B Q.No. 19

What are two steps of Gauss elimination method? Find the approximate solution of the following system of equations by Gauss elimination method: $3x - y + z = -2$, $x + 5y + 2z = 6$, $2x + 3y + z = 0$. **[6]**

SOLUTION

The two steps of Gauss Elimination method are

i. Forward Elimination of unknowns

ii. Backward Substitution.

Next part:

Given equations are

$3x - y + z = -2$ ... (i)

$x + 5y + 2z = 6$ ... (ii)

$2x + 3y + z = 0$ ... (iii)

Multiplying equation (i) by $\frac{1}{3}$ and subtracting from equation (ii)

$x + 5y + 2z = 6$

$x - \frac{1}{3}y + \frac{1}{3}z = -\frac{2}{3}$

− + − +

$\frac{16}{3}y + \frac{5}{3}z = \frac{20}{3}$ ... (iv)

Again, multiplying (i) by $\frac{2}{3}$ and subtracting from (iii),

$2x + 3y + z = 0$

$2x - \frac{2}{3}y + \frac{2}{3}z = -\frac{4}{3}$

− + − +

$\frac{11}{3}y + \frac{1}{3}z = \frac{4}{3}$ ... (v)

Multiplying equation (iv) by $\frac{11}{16}$ and subtracting from (v),

$\frac{11}{3}y + \frac{1}{3}z = \frac{4}{3}$

$\frac{11}{3}y + \frac{55}{48}z = \frac{55}{12}$

− − −

$-\frac{13}{16}z = \frac{-13}{4}$ ... (vi)

Now, we have the following three equations

$3x - y + z = -2$ ... (i)

$\frac{16}{3}y + \frac{5}{3}z = \frac{20}{3}$ ... (iv)

$-\frac{13}{16}z = -\frac{13}{4}$ ... (vi)

From (vi), we have $z = 4$

Using $z = 4$ in (iv),

$\frac{16}{3}y + \frac{5}{3} \times 4 = \frac{20}{3}$

or $y = 0$

Using $z = 4$ & $y = 0$ in (i), we get

$3x - 0 + 4 = -2$

or, $3x = -6$

$x = -2$

$\therefore x = -2, y = 0, z = 4$

B. GAUSS-SEIDEL METHOD

2 MARKS QUESTIONS

20. 2069 (Set A) Q.No. 16c

Examine whether the following system of equations are ill conditioned.
$2x_1 + x_2 = 25$; $2.001x_1 + x_2 = 25.01$ [2]

SOLUTION

The condition for the ill-conditioned system of equations is that the determinant of coefficient matrix is small enough or approximately equal to zero.

Here, determinant of coefficient matrix is

$$\begin{vmatrix} 2 & 1 \\ 2.001 & 1 \end{vmatrix}$$
$= 2 \times 1 - 1 \times 2.001 = 2 - 2.001$
$= -0.001$

which is small enough in magnitude. Hence, the given system of equations is ill-conditioned.

21. 2072 Set C Q.No. 16c

Write the conditions for the system of equations $a_{11}x + a_{12}y = b_1$, $a_{21}x + a_{22}y = b_2$, to be ill conditioned. [2]

SOLUTION

Given equations are:
$a_{11}x + a_{12}y = b_1$
$a_{21}x + a_{22}y = b_2$

If the two straight lines represented by above two equations are almost parallel, then slopes must nearly be equal

i.e. $\frac{a_{11}}{a_{12}} \approx \frac{a_{21}}{a_{22}}$

$\Rightarrow a_{11}a_{22} - a_{12}a_{21} \approx 0$

The left hand expression is the determinant of the coefficient matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Thus, the condition for the ill conditioned system of equations is that the determinant of the coefficient matrix is small enough or approximately equal to zero.

22. 2072 Set D Q.No. 16c

Test whether the system of equations $12x + 3y - 5z = 1$, $x + 5y + 3z = 28$ and $3x + 7y + 13z = 1$ is diagonally consistent? [2]

SOLUTION

Here,
$|12| > |3| + |-5| = 8$
$|15| > |1| + |3| = 4$
$|13| > |3| + |7| = 10$

Since the absolute value of the diagonal coefficient is greater than the sum of the absolute values of the other coefficients in that row, so the system is diagonally dominant.

23. 2072 Supp Q.No. 16c

Examine whether the following equations are diagonally dominant: [2]
$8x_1 - 2x_2 + 3x_3 = -1$
$-3x_1 + 9x_2 - x_3 = 2$
$2x_1 - x_2 - 7x_3 = 3$

SOLUTION

Here,
$|8| > |-2| + |3| = 5$
$|9| > |-3| + |-1| = 4$
$|-7| = 7 > |2| + |-1| = 3$

Hence, the given system is diagonally dominant.

24. 2073 Set C Q.No. 16c

Define well-conditioned and ill-conditioned of a system of equation. [2]

SOLUTION

Well-conditioned: If a small change in the coefficient of the variable shows only a small deviation in the solution. Capital then the system of equations is said to be well-conditioned.

Ill-conditioned: If a small change in the coefficient of the variable shows large deviation in the solution, then the system of equations is said to be ill-conditioned.

25. 2073 Supp Q.No. 16c

Interpret geometrically that a system of equations in two variables is ill-conditioned. [2]

SOLUTION

Consider the system of equations
$x - y = 1$
$x - 1.001y = 0$

Solving, $x = 1001$, $y = 1000$

Now, changing the coefficient of y in second equation to 0.999 then
$x - y = 1$
$x - 0.999y = 0$

Solving, $x = -999$, $y = -1000$

Here, small change in coefficient results the very large change in the solution. Such system is called ill conditioned system.

Geometrically, the ill - conditioned system represents the straight lines that are almost parallel. Even a slight shift on any line can make a great move of the intersection point in ill conditioned system.

26. 2074 Set A Q.No. 16c

Is the system $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ well-conditioned? Justify your answer. [2]

SOLUTION

Given $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

i.e $x + 2y = 4$ & $2x + 3y = 7$

Solving, we get $x = 2$ & $y = 1$

i.e. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Let us make a small change in the coefficient matrix of the system of equations as follows:

$$\begin{bmatrix} 1.0001 & 2.0001 \\ 2.0001 & 3.0001 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

i.e. $1.0001x + 2.0001y = 4$
$2.0001 + 3.0001y = 7$

Solving we get $x = 2.0003$ & $y = 0.9997$

Again, let us make a small change in the right hand side vector of the equation as follows:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4.0001 \\ 7.0001 \end{bmatrix}$$

i.e. $x + 2y = 4.0001$ & $2x + 3y = 7.0001$

Solving $x = 1.9999$ & $y = 1.0001$

Hence the system is well conditioned because small changes in coefficient matrix or right hand side vector gave the small change in the solution.

27. 2074 Set B Q.No. 16c

Examine whether the system of equations $x + 3y - 2z = 0$, $2x - 3y + z = 1$ and $4x - 3y + z = 3$ is diagonally dominant or not. [2]

SOLUTION

Given equations are: $x + 3y - 3z = 0$
$2x - 3y + z = 1$
$4x - 3y + z = 3$

Here, the absolute value of coefficient of diagonal element in first equation $= |1| = 1$

And the sum of the absolute value of coefficients of the non-diagonal elements in first equation $= |3| + |-3| = 3 + 3 = 6$

Since $1 < 6$, the system is not diagonally dominant.

28. 2075 Set A Q.No. 16c

Are the followings system of equation diagonally dominant? [2]
$12x_1 + 3x_2 - 5x_3 = 1$, $x_1 + 5x_2 + 3x_3 = 28$, $3x_1 + 7x_2 + 13x_3 = 1$.

➥ Please refer to 2072 Set D Q.No. 16c

4 MARKS QUESTIONS

29. 2069 (Set A) Q.No. 17a or

Solve the following equations using Guess-Seidel method:
$2x_1 - x_2 = 8$; $3x_1 + 7x_2 = -5$ [4]

➥ Please refer to Model Set II, Q.No. 17a or

30. 2069 (Set B) Q.No. 17a Or

Solve the following equations using Guess-Seidel method:
$3x_1 + x_2 = 5$, $x_1 - 3x_2 = 5$ [4]

SOLUTION

$x_1 = \frac{1}{3}(5 - x_2)$, $x_2 = \frac{1}{3}(x_1 - 5)$

Initially, $x_1 = 0$, $x_2 = 0$

1st iteration:
$x_1 = \frac{1}{3}(5 - 0) = 1.667$, $x_2 = \frac{1}{3}(1.667 - 5) = -1.111$

2nd iteration:
$x_1 = \frac{1}{3}(5 + 1.111) = 2.037$, $x_2 = \frac{1}{3}(2.037 - 5) = -0.988$

3rd iteration:
$x_1 = \frac{1}{3}(5 + 0.988) = 1.996$, $x_2 = \frac{1}{3}(1.996 - 5) = -1.001$

4th iteration:
$x_1 = \frac{1}{3}(5 + 1.001) = 2.000$, $x_2 = \frac{1}{3}(2.000 - 5) = -1.000$

5th iteration:
$x_1 = \frac{1}{3}(5 + 1) = 2$, $x_2 = \frac{1}{3}(2 - 5) = -1$

From 4th and 5th iterations, the value of x_1 and x_2 are equal.

$\therefore$ $x_1 = 2$, $x_2 = -1$.

31. 2070 Set C Q.No. 17 a or

Solve the following equation using Gauss Seidel method $3x_1 + x_2 = 5$; $x_1 + 2x_2 = 5$. [4]

SOLUTION

Given equations can be written as:
$x_1 = \frac{1}{3}(5 - x_2)$, $x_2 = \frac{1}{2}(5 - x_1)$

Initially, $x_1 = 0$, $x_2 = 0$

1st iteration:
$x_1 = \frac{1}{3}(5 - 0) = 1.667$, $x_2 = \frac{1}{2}(5 - 1.667) = 1.667$

2nd iteration:
$x_1 = \frac{1}{3}(5 - 1.667) = 1.111$, $x_2 = \frac{1}{2}(5 - 1.111) = 1.945$

3rd iteration:
$x_1 = \frac{1}{3}(5 - 1.945) = 1.018$, $x_2 = \frac{1}{2}(5 - 1.018) = 1.991$

4th iteration:
$x_1 = \frac{1}{3}(5 - 1.991) = 1.003$, $x_2 = \frac{1}{2}(5 - 1.003) = 1.998$

5th iteration:
$x_1 = \frac{1}{3}(5 - 1.998) = 1.001$, $x_2 = \frac{1}{2}(5 - 1.001) = 2.000$

6th iteration:
$x_1 = \frac{1}{3}(5 - 2) = 1$, $x_2 = \frac{1}{2}(5 - 1) = 2$

The values of x_1 and x_2 in 5th and 6th iteration are same.

$\therefore$ $x_1 = 1$, $x_2 = 2$

32. 2070 Set D Q.No. 17 a Or

Solve the following system of equations by Gauss Seidel method

$3x + y - z = 2$, $2x - 5y + z = 20$, $x - 3y - 8z = 3$ [4]

SOLUTION

Given equations can be written as:

$x = \frac{1}{3}(2 - y + z)$ ...(i)

$y = \frac{1}{5}(2x + z - 20)$ ... (ii)

$z = \frac{1}{8}(x - 3y - 3)$ ...(iii)

Initially, $x = 0, y = 0, z = 0$

1st iteration:

Put $x = 0, y = 0, z = 0$ in equation (i), we get

$x = \frac{1}{3}(2 - 0 + 0) = 0.667$

Again, put most recent values i.e. $x = 0.667$ and $z = 0$ in equation (ii),

$y = \frac{1}{5}(2 \times 0.667 + 0 - 20) = -3.733$

And, put $x = 0.667$ and $y = -3.733$ in equation (iii),

$z = \frac{1}{8}(0.667 + 3 \times 3.733 - 3) = 1.108$

2nd iteration

$x = \frac{1}{3}(2 + 3.733 + 1.108) = 2.280$

$y = \frac{1}{5}(2 \times 2.280 + 1.108 - 20) = -2.866$

$z = \frac{1}{8}(2.280 + 3 \times 2.866 - 3) = 0.985$

3rd iteration

$x = \frac{1}{3}(2 + 2.866 + 0.985) = 1.950$

$y = \frac{1}{5}(2 \times 1.950 + 0.985 - 20) = -3.023$

$z = \frac{1}{8}(1.950 + 3 \times 3.023 - 3) = 1.002$

4th iteration

$x = \frac{1}{3}(2 + 3.023 + 1.002) = 2.008$

$y = \frac{1}{5}(2 \times 2.008 + 1.002 - 20) = -2.996$

$z = \frac{1}{8}(2.008 + 3 \times 2.996 - 3) = 1.000$

5th iteration

$x = \frac{1}{3}(2 + 2.996 + 1) = 1.999$

$y = \frac{1}{5}(2 \times 1.999 + 1 - 20) = -3.001$

$z = \frac{1}{8}(1.999 + 3 \times 3.001 - 3) = 1.000$

6th iteration

$x = \frac{1}{3}(2 + 3.001 + 1) = 2.000$

$y = \frac{1}{5}(2 \times 2 + 1 - 20) = -3$

$z = \frac{1}{8}(2 + 3 \times 3 - 3) = 1$

7th iteration

$x = \frac{1}{3}(2 + 3 + 1) = 2$

$y = \frac{1}{5}(2 \times 2 + 1 - 20) = -3$

$z = \frac{1}{8}(2 + 3 \times 3 - 3) = 1$

From 6th and 7th iterations, x, y and z have equal values

So, $x = 2, y = -3, z = 1$

33. 2072 Set C Q.No. 17a

Using Gauss Seidel method, solve the equations $3x + 2y = -9$, $2x - 3y = -6$ [4]

SOLUTION

From the given equations we have,

$x = \frac{-9 - 2y}{3}$, $y = \frac{2x + 6}{3}$

Initially, $x = 0, y = 0$

Iteration I:

$x = \frac{-9 - 2 \times 0}{3} = -3$, $y = \frac{2 \times (-3) + 6}{3} = 0$

Iteration II:

$x = \frac{-9 - 2 \times 0}{3} = -3$, $y = \frac{2 \times (-3) + 6}{3} = 0$

From 1st and 2nd iterations, the values of x and y are equal, so $x = -3, y = 0$

34. 2072 Set D Q.No. 17a

Using Gauss Seidel method, solve:

$3x + 4y + 8z = 7$, $x + 20y + z = -18$, $25x + y - 5z = 19$ [4]

SOLUTION

Rewriting the given equations so as to make them diagonally dominant, we have

$25x + y - 5z = 19$

$x + 20y + z = -18$

$3x + 4y + 8z = 7$

From these equations, we have,

$x = \frac{1}{25}(19 - y + 5z)$, $y = \frac{1}{20}(-18 - x - z)$

$z = \frac{1}{8}(7 - 3x - 4y)$

Initially $x = 0, y = 0, z = 0$

1st iteration

$x = \frac{1}{25}(19 - 0 + 5 \times 0) = 0.76$

$y = \frac{1}{20}(-18 - 0.76 - 0) = -0.938$

$z = \frac{1}{8}\{(7 - 3 \times 0.76 - 4 \times (-0.938)\} = 1.059$

2nd iteration

$x = \frac{1}{25}(19 + 0.938 + 5 \times 1.059) = 1.009$

$y = \frac{1}{20}(-18 - 1.009 - 1.059) = -1.003$

$z = \frac{1}{8}(7 - 3 \times 1.009 + 4 \times 1.003) = 0.998$

3rd iteration

$x = \frac{1}{25}(19 + 1.003 + 5 \times 0.998) = 1.000$

$y = \frac{1}{20}(-18 - 1.000 - 0.998) = -1.000$

$z = \frac{1}{8}(7 - 3 \times 1.000 + 4 \times 1.000) = 1.000$

4th iteration

$x = \frac{1}{25}(19 + 1 + 5 \times 1) = 1$

$y = \frac{1}{20}(-18 - 1 - 1) = -1$

$z = \frac{1}{8}(7 - 3 \times 1 + 4 \times 1) = 1$

From 3rd and 4th iterations, the values of x, y and z are equal, so $x = 1, y = -1, z = 1$

35. 2072 Supp Q.No. 17a OR

Solve the following equation using Gauss-Seidel method:

$3x_1 + x_2 = 5$ [4]

$x_1 - 3x_2 = 5$

➥ Please refer to 2069 (Set B) Q.No. 17a Or

36. 2073 Set D Q.No. 17a OR

Using Gauss-Seidel method, solve the following system of equations: [4]

$4x_1 + x_2 + x_3 = 7$, $2x_1 - 5x_2 + 2x_3 = 1$, $x_1 - x_2 + 3x_3 = 6$.

SOLUTION

Given equation are

$4x_1 + x_2 + x_3 = 7$

$2x_1 - 5x_2 + 2x_3 = 1$

$x_1 - x_2 + 3x_3 = 6$

From these equations, we have

$x_1 = \frac{1}{4}(7 - x_2 - x_3)$, $x_2 = \frac{1}{5}(2x_1 + 2x_3 - 1)$

$x_3 = \frac{1}{3}(6 - x_1 + x_2)$

Initially, $x_1 = 0, x_2 = 0, x_3 = 0$

Iteration I

$x_1 = \frac{1}{4}(7 - 0 - 0) = 1.75$

$x_2 = \frac{1}{5}(2 \times 1.75 + 2 \times 0 - 1) = 0.5$

$x_3 = \frac{1}{3}(6 - 1.75 + 0.5) = 1.583$

Iteration II

$x_1 = \frac{1}{4}(7 - 0.5 - 1.583) = 1.229$

$x_2 = \frac{1}{5}(2 \times 1.229 + 2 \times 1.583 - 1) = 0.925$

$x_3 = \frac{1}{3}(6 - 1.229 + 0.925) = 1.899$

Iteration III

$x_1 = \frac{1}{4}(7 - 0.925 - 1.899) = 1.044$

$x_2 = \frac{1}{5}(2 \times 1.044 + 2 \times 1.899 - 1) = 0.977$

$x_3 = \frac{1}{3}(6 - 1.044 + 0.977) = 1.978$

Iteration IV

$x_1 = \frac{1}{4}(7 - 0.977 - 1.978) = 1.011$

$x_2 = \frac{1}{5}(2 \times 1.011 + 2 \times 1.978 - 1) = 0.996$

$x_3 = \frac{1}{3}(6 - 1.011 + 0.966) = 1.995$

Iteration V

$x_1 = \frac{1}{4}(7 - 0.996 - 1.995) = 1.002$

$x_2 = \frac{1}{5}(2 \times 1.002 + 2 \times 1.995 - 1) = 0.999$

$x_3 = \frac{1}{3}(6 - 1.002 + 0.999) = 1.999$

Iteration VI

$x_1 = \frac{1}{4}(7 - 0.999 - 1.999) = 1.001$

$x_2 = \frac{1}{5}(2 \times 1.001 + 2 \times 1.999 - 1) = 1.000$

$x_3 = \frac{1}{3}(6 - 1.001 + 1.000) = 2.000$

Iteration VII

$x_1 = \frac{1}{4}(7 - 1 - 2) = 1$

$x_2 = \frac{1}{5}(2 \times 1 + 2 \times 2 - 1) = 1$

$x_3 = \frac{1}{3}(6 - 1 + 1) = 2$

From 6th and 7th iterations, the value of x_1, x_2 and x_3 are equal, so $x_1 = 1, x_2 = 1, x_3 = 2$.

6 MARKS QUESTIONS

37. 2070 Supp. Q.No. 19

Given the system

$3x - 6y + 2z = 23$

$-4x + y - z = -8$

$x - 3y + 7z = 17$

Make it diagonally dominant and solve by Gauss-siedel method with error less than 0.005. [6]

SOLUTION

Writing the given equation so that the system will be diagonally dominant

$-4x + y - z = -8$

$3x - 6y + 2z = 23$

$x - 3y + 7z = 17$

From these equations, we have

$x = \frac{1}{4}(8 + y - z)$

$y = \frac{1}{6}(-23 + 3x + 2z)$

$z = \frac{1}{7}(17 - x + 3y)$

Initially, $x = 0, y = 0, z = 0$

1st iteration

$x = \frac{1}{4}(8 + 0 - 0) = 2$

$y = \frac{1}{6}(-23 + 3 \times 2 + 2 \times 0) = -2.833$

$z = \frac{1}{7}(17 - 2 - 3 \times 2.833) = 0.929$

2nd iteration

$x = \frac{1}{4}(8 - 2.833 - 0.929) = 1.06$

$y = \frac{1}{6}(-23 + 3 \times 1.06 + 2 \times 0.929) = -2.994$

$z = \frac{1}{7}(17 - 1.06 - 3 \times 2.994) = 0.994$

3rd iteration

$x = \frac{1}{4}(8 - 2.994 - 0.994) = 1.003$

$y = \frac{1}{6}(-23 + 3 \times 1.003 + 2 \times 0.994) = -3.001$

$z = \frac{1}{7}(17 - 1.003 - 3 \times 3.001) = 0.999$

4th iteration

$x = \frac{1}{4}(8 - 3.001 - 0.999) = 1$

$y = \frac{1}{6}(-23 + 3 \times 1 + 2 \times 0.999) = -3$

$z = \frac{1}{7}(17 - 1 - 3 \times 3) = 1$

5th iteration

$x = \frac{1}{4}(8 - 3 - 1) = 1$

$y = \frac{1}{6}(-23 + 3 \times 1 + 2 \times 1) = -3$

$z = \frac{1}{7}(17 - 1 - 3 \times 3) = 1$

Therefore, exact solution of given system is

$x = 1, y = -3, z = 1$

The solution with error less than 0.005 is obtained in 3rd iteration which is

$x = 1.003$, $y = -3.001$, $= 0.999$

38. 2070 Supp. Q.No. 19 OR

Given the system

$3x + 1.52y = 1$

$2x + 1.02y = 1$

Determine whether the system is ill-conditioned by changing the coefficient 1.02 to 1.03. [6]

SOLUTION

Given equations are:

$3x + 1.52y = 1$

$2x + 1.02y = 1$

The given equations in matrix form can be written as:

$$\begin{bmatrix} 3 & 1.52 \\ 2 & 1.02 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is in the form of $AX = B$

$\Rightarrow X = A^{-1}B$ ... (i)

For A^{-1}

$$|A| = \begin{vmatrix} 3 & 1.52 \\ 2 & 1.02 \end{vmatrix} = 3 \times 1.02 - 2 \times 1.52$$

$= 3.06 - 3.04 = 0.02 \neq 0$

So, A^{-1} exists.

$$\text{Adj. A} = \begin{bmatrix} 1.02 & -1.52 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj. A} = \frac{1}{0.02}\begin{bmatrix} 1.02 & -1.52 \\ -2 & 3 \end{bmatrix}$$

From (i)

$X = A^{-1}B$

$$= \frac{1}{0.02}\begin{bmatrix} 1.02 & -1.52 \\ -2 & +3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{0.02}\begin{bmatrix} 1.02 - 1.52 \\ -2 + 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -25 \\ 50 \end{bmatrix}$$

$\therefore$ $x = -25, y = 50$

Again, changing the coefficient 1.02 to 1.03, we have the following equations

$3x + 1.52y = 1$

$2x + 1.03y = 1$

The given equations in matrix form can be written as:

$$\begin{bmatrix} 3 & 1.52 \\ 2 & 1.03 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is in the form of $AX = B$

$\Rightarrow X = A^{-1}B$ ... (i)

For A^{-1}

$$|A| = \begin{vmatrix} 3 & 1.52 \\ 2 & 1.03 \end{vmatrix} = 3 \times 1.03 - 1.25 \times 2$$

$= 3.09 - 3.04 = 0.05 \neq 0$

So, A^{-1} exists.

$$\text{Adj. A} = \begin{bmatrix} 1.03 & -1.52 \\ -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj. A} = \frac{1}{0.05}\begin{bmatrix} 1.03 & -1.52 \\ -2 & 3 \end{bmatrix}$$

From (ii)

$X = A^{-1}B$

$$= \frac{1}{0.05}\begin{bmatrix} 1.03 & -1.52 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{0.05}\begin{bmatrix} 1.03 - 1.52 \\ -2 + 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9.8 \\ 20 \end{bmatrix}$$

$\therefore$ $x = -9.8, y = 20$

From above two sets of solutions, we see a large change in the solution when there is a small change in the coefficient. So, the system is ill conditioned.

39. 2071 Supp. Q.No. 19

Use the Gauss-Siedel method to solve the systems

$4x - y + z = 8$, $2x + 5y + 2z = 3$, $x + 2y + 4z = 11$. [6]

SOLUTION

From the given equations, we have

$x = \frac{1}{4}(8 + y - z)$, $y = \frac{1}{5}(3 - 2x - 2z)$

$z = \frac{1}{4}(11 - x - 2y)$

Initially, $x = 0, y = 0, z = 0$

1st iteration:

$x = \frac{1}{4}(8 + 0 - 0) = 2$

$y = \frac{1}{5}(3 - 2 \times 2 - 2 \times 0) = -0.2$

$z = \frac{1}{4}(11 - 2 + 2 \times 0.2) = 2.35$

2nd iteration:

$x = \frac{1}{4}(8 - 0.2 - 2.35) = 1.363$

$y = \frac{1}{5}(3 - 2 \times 1.363 - 2 \times 2.35) = -0.885$

$z = \frac{1}{4}(11 - 1.363 + 2 \times 0.885) = 2.852$

3rd iteration:

$x = \frac{1}{4}(8 - 0.885 - 2.852) = 1.066$

$y = \frac{1}{5}(3 - 2 \times 1.066 - 2 \times 2.852) = -0.967$

$z = \frac{1}{4}(11 - 1.066 + 2 \times 0.967) = 2.967$

4th iteration:

$x = \frac{1}{4}(8 - 0.967 - 2.967) = 1.017$

$y = \frac{1}{5}(3 - 2 \times 1.017 - 2 \times 2.967) = -0.994$

$z = \frac{1}{4}(11 - 1.017 + 2 \times 0.994) = 2.993$

5th iteration:

$x = \frac{1}{4}(8 - 0.994 - 2.993) = 1.003$

$y = \frac{1}{5}(3 - 2 \times 1.003 - 2 \times 2.993) = -0.998$

$z = \frac{1}{4}(11 - 1.003 + 2 \times 0.998) = 2.998$

6th iteration:

$x = \frac{1}{4}(8 - 0.998 - 2.998) = 1.001$

$y = \frac{1}{5}(3 - 2 \times 1.001 - 2 \times 2.998) = -1.000$

$z = \frac{1}{4}(11 - 1.001 + 2 \times 1.000) = 3.000$

7th iteration:

$x = \frac{1}{4}(8 - 1 - 3) = 1$

$y = \frac{1}{5}(3 - 2 \times 1 - 2 \times 3) = -1$

$z = \frac{1}{4}(11 - 1 + 2 \times 1) = 3$

From 6th and 7th iterations, the values of x, y and z are equal, so

$x = 1, y = -1$ and $z = 3$

40. 2071 Supp. Q.No. 19 OR

Given the system

$0.835x + 0.667y = 0.168$, $0.333x + 0.266y = 0.067$

Determine whether the system is ill conditioned by changing the coefficient 0.667 to 0.666. [6]

SOLUTION

Given equations are:

$0.835x + 0.667y = 0.168$

$0.333x + 0.266y = 0.067$

The given equations in matrix form can be written as:

$$\begin{bmatrix} 0.835 & 0.667 \\ 0.333 & 0.266 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.168 \\ 0.067 \end{bmatrix}$$

which is in the form of $AX = B$

$\Rightarrow X = A^{-1}B$ ... (i)

$$|A| = \begin{vmatrix} 0.835 & 0.667 \\ 0.333 & 0.266 \end{vmatrix}$$

$= 0.835 \times 0.266 - 0.667 \times 0.333 = -0.000001 \neq 0$

So, A^{-1} exists.

$$\text{Adj. A} = \begin{bmatrix} 0.266 & -0.667 \\ -0.333 & 0.835 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj. A}$$

$$= \frac{1}{-0.000001}\begin{bmatrix} 0.266 & -0.667 \\ -0.333 & 0.835 \end{bmatrix}$$

From (i)

$X = A^{-1}B$

$$= \frac{1}{-0.000001}\begin{bmatrix} 0.266 & -0.667 \\ -0.333 & 0.835 \end{bmatrix}\begin{bmatrix} 0.168 \\ 0.067 \end{bmatrix}$$

$$= \frac{1}{-0.000001}\begin{bmatrix} 0.266 \times 0.168 - 0.667 \times 0.067 \\ -0.333 \times 0.168 + 0.835 \times 0.067 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore$ $x = 1, y = -1$

Again, changing the coefficient 0.667 to 0.666, we have the following equations

$0.835x + 0.666y = 0.168$

$0.333x + 0.266y = 0.067$

The given equations in matrix form can be written as:

$$\begin{bmatrix} 0.835 & 0.666 \\ 0.333 & 0.266 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0.168 \\ 0.067 \end{bmatrix}$$

which is in the form of $AX = B$

$\Rightarrow X = A^{-1}B$ ... (ii)

$$\sim\left[\begin{array}{ccc:ccc}1&4&7&0&1&0\\0&-36&-72&0&-10&1\\0&0&-22&1&-3&0\end{array}\right]R_2\leftrightarrow R_3$$

$$\sim\left[\begin{array}{ccc:ccc}1&4&7&0&1&0\\0&1&2&0&\frac{5}{18}&-\frac{1}{36}\\0&0&-22&1&-3&0\end{array}\right]R_2\to\frac{1}{(-36)}R_2$$

$$\sim\left[\begin{array}{ccc:ccc}1&0&-1&0&-\frac{1}{9}&\frac{1}{9}\\0&1&2&0&\frac{5}{18}&-\frac{1}{36}\\0&0&-22&1&-3&0\end{array}\right]R_1\to R_1-4R_2$$

$$\sim\left[\begin{array}{ccc:ccc}1&0&-1&0&-\frac{1}{9}&\frac{1}{9}\\0&1&2&0&\frac{5}{18}&-\frac{1}{36}\\0&0&1&-\frac{1}{22}&\frac{3}{22}&0\end{array}\right]R_3\to-\left(\frac{1}{22}\right)R_3$$

Applying $R_1\to R_1+R_3$, $R_2\to R_2-2R_3$,

$$\sim\left[\begin{array}{ccc:ccc}1&0&0&-\frac{1}{22}&\frac{5}{198}&\frac{1}{9}\\0&1&0&\frac{1}{11}&\frac{1}{198}&-\frac{1}{36}\\0&0&1&-\frac{1}{22}&\frac{3}{22}&0\end{array}\right]$$

$$\therefore\ A^{-1}=\begin{bmatrix}-\frac{1}{22}&\frac{5}{198}&\frac{1}{9}\\\frac{1}{11}&\frac{1}{198}&-\frac{1}{36}\\-\frac{1}{22}&\frac{3}{22}&0\end{bmatrix}$$

From (i) $X=\begin{bmatrix}-\frac{1}{22}&\frac{5}{198}&\frac{1}{9}\\\frac{1}{11}&\frac{1}{198}&-\frac{1}{36}\\-\frac{1}{22}&\frac{3}{22}&0\end{bmatrix}\begin{bmatrix}28\\2\\20\end{bmatrix}$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-\frac{28}{22}+\frac{10}{198}+\frac{20}{9}\\\frac{28}{11}+\frac{2}{198}-\frac{20}{36}\\-\frac{28}{22}+\frac{6}{22}+0\end{bmatrix}=\begin{bmatrix}1\\2\\-1\end{bmatrix}$$

$\therefore\ x=1, y=2, z=-1.$

5. 2074 Supp Q.No. 17a OR

Solve the following system of equations using inverse matrix method: [4]

$3x+y+z=15$, $x+y+z=3$, $y-z=-1$.

➥ Please refer to 2071 Set C Q.N. 17a OR

6. 2075 Set C Q.No. 17a

Solve the following system of equations using Gauss-elimination or inverse matrix method. [4]

$x+y+z=6$, $3x-4y=-5$, $4z-3x+2y=13$

SOLUTION

Gauss Elimination Method

Given equations are

$x+y+z=6$ ... (i)

$3x-4y=-5$ ... (ii)

$-3x+2y+4z=13$ ... (iii)

Multiplying equation (i) by 3 and subtracting from equation (ii)

$$\begin{array}{rl}3x-4y&=-5\\ \underline{-}3x\underline{+}3y\underline{+}3z&=\underline{-}18\\\hline -7y-3z&=-23\end{array}\ \dots\text{(iv)}$$

Again, multiplying equation (1) by 3 and adding with (iii),

$$\begin{array}{rl}3x-2y+4z&=13\\ \underline{-}3x\underline{+}3y\underline{+}3z&=\underline{-}18\\\hline 5y+7z&=31\end{array}\ \dots\text{(v)}$$

Multiplying (iv) by $\frac{5}{7}$ and adding with equation (v)

$$\begin{array}{rl}5y+7z&=31\\-5y-\frac{15}{7}z&=-\frac{115}{7}\\\hline\frac{37}{7}z&=\frac{102}{7}\end{array}\ \dots\text{(vi)}$$

Now, we have the following three equations

$x+y+z=6$ ... (i)

$-7y-3z=-23$ ... (iv)

$\frac{34}{7}z=\frac{102}{7}$ ... (vi)

From the equation (vi), we have $z=3$

Using $z=3$ in (iv), we get

$-7y-3\times3=-23$

or, $-7y=-14$

$\therefore\ y=2$

Using $y=2$, $z=3$ in equation (i), we get

$x+2+3=6$

or, $x=1$

$\therefore\ x=1, y=2, z=3$

Matrix Inversion Method

Writing the given equations in matrix form:

$$\begin{pmatrix}1&1&1\\3&-4&0\\-3&2&4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-5\\13\end{pmatrix}$$

which is in the form of $AX=B$ ... (i)

where,

$A=\begin{pmatrix}1&1&1\\3&-4&0\\-3&2&4\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $B=\begin{pmatrix}6\\-5\\13\end{pmatrix}$

Now, we augment the matrix A with identify matrix I,

$$(A:I)=\left(\begin{array}{ccc:ccc}1&1&1&1&0&0\\3&-4&0&0&1&0\\-3&2&4&0&0&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&-7&-3&-3&1&0\\0&5&7&3&0&1\end{array}\right)\begin{array}{l}R_2\to R_2-3R_1\\R_3\to R_3+3R_1\end{array}$$

$$\sim\left(\begin{array}{ccc:ccc}1&1&1&1&0&0\\0&1&\frac{3}{7}&\frac{3}{7}&-\frac{1}{7}&0\\0&5&7&3&0&1\end{array}\right)R_2\to\left(-\frac{1}{7}\right)R_2$$

$$\sim\left(\begin{array}{ccc:ccc}1&0&\frac{4}{7}&\frac{4}{7}&\frac{1}{7}&0\\0&1&\frac{3}{7}&\frac{3}{7}&-\frac{1}{7}&0\\0&0&\frac{34}{7}&\frac{6}{7}&\frac{5}{7}&1\end{array}\right)\begin{array}{l}R_1\to R_1-R_2\\R_3\to R_3-5R_2\end{array}$$

$$\sim\left(\begin{array}{ccc:ccc}1&0&\frac{4}{7}&\frac{4}{7}&\frac{1}{7}&0\\0&1&\frac{3}{7}&\frac{3}{7}&-\frac{1}{7}&0\\0&0&1&\frac{3}{17}&\frac{5}{34}&\frac{7}{34}\end{array}\right)R_3\to\frac{7}{34}R_3$$

$$\sim\left(\begin{array}{ccc:ccc}1&0&0&\frac{8}{17}&\frac{1}{17}&-\frac{2}{17}\\0&1&0&\frac{6}{17}&-\frac{7}{34}&-\frac{3}{34}\\0&0&1&\frac{3}{17}&\frac{5}{34}&\frac{7}{34}\end{array}\right)\begin{array}{l}R_1\to R_1-\frac{4}{7}R_3\\R_2\to R_2-\frac{3}{7}R_2\end{array}$$

$$\therefore\ A^{-1}=\begin{pmatrix}\frac{8}{17}&\frac{1}{17}&-\frac{2}{17}\\\frac{6}{17}&-\frac{7}{34}&-\frac{3}{34}\\\frac{3}{17}&\frac{5}{34}&\frac{7}{34}\end{pmatrix}$$

From (i) $X=A^{-1}B$

$$X=\begin{pmatrix}\frac{8}{17}&\frac{1}{17}&-\frac{2}{17}\\\frac{6}{17}&-\frac{7}{34}&-\frac{3}{34}\\\frac{3}{17}&\frac{5}{34}&\frac{7}{34}\end{pmatrix}\begin{pmatrix}6\\-5\\13\end{pmatrix}$$

$$\text{or, } X=\begin{pmatrix}\frac{8}{17}\times6-\frac{1}{17}\times5-\frac{2}{17}\times13\\\frac{6}{17}\times6+\frac{7}{34}\times5-\frac{3}{34}\times13\\\frac{3}{17}\times6-\frac{5}{34}\times5+\frac{7}{34}\times13\end{pmatrix}$$

$$\text{or, }\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{48-5-26}{17}\\\frac{72+35-39}{34}\\\frac{36-25+91}{34}\end{pmatrix}$$

$$\text{or, }\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{17}{17}\\\frac{68}{34}\\\frac{102}{34}\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$\therefore\ x=1, y=2, z=3$

6 MARKS QUESTIONS

7. 2075 Set B Q.No. 19 OR

Solve the following system of equations by matrix inversion method:

$3x+5z=14$,

$2x+y-3z=3$,

$x+y+z=4$.

SOLUTION

Writing the given equations in matrix form

$$\begin{pmatrix}3&0&5\\2&1&-3\\1&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}14\\3\\4\end{pmatrix}$$

which is in the form of $AX=B$ ... (i)

where,

$A=\begin{pmatrix}3&0&5\\2&1&-3\\1&1&1\end{pmatrix}$, $X=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $B=\begin{pmatrix}14\\3\\4\end{pmatrix}$

We augment the matrix A with identify matrix I,

$$(A:I)=\left(\begin{array}{ccc:ccc}3&0&5&1&0&0\\2&1&-3&0&1&0\\1&1&1&0&0&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc:ccc}1&1&1&0&0&1\\2&1&-3&0&1&0\\3&0&5&1&0&0\end{array}\right)R_1\leftrightarrow R_3$$

$$\sim\left(\begin{array}{ccc:ccc}1&1&1&0&0&1\\0&-1&-5&0&1&-2\\0&-3&2&1&0&-3\end{array}\right)\begin{array}{l}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{array}$$

$$\sim\left(\begin{array}{ccc:ccc}1&1&1&0&0&1\\0&1&5&0&-1&2\\0&-3&2&1&0&-3\end{array}\right)R_2\to(-1)R_2$$

$$\sim\left(\begin{array}{ccc:ccc}1&0&-4&0&1&-1\\0&1&5&0&-1&2\\0&0&17&1&-3&3\end{array}\right)\begin{array}{l}R_3\to R_3+3R_2\\R_1\to R_1-R_2\end{array}$$

$$\sim\left(\begin{array}{ccc:ccc}1&0&-4&0&1&-1\\0&1&5&0&-1&2\\0&0&1&\frac{1}{17}&\frac{-3}{17}&\frac{3}{17}\end{array}\right)R_3\to\frac{1}{17}R_3$$

$$\sim\left(\begin{array}{ccc:ccc}1&0&0&\frac{4}{17}&\frac{5}{17}&\frac{-5}{17}\\0&1&0&\frac{-5}{17}&\frac{-2}{17}&\frac{19}{17}\\0&0&1&\frac{1}{17}&\frac{-3}{17}&\frac{3}{17}\end{array}\right)\begin{array}{l}R_1\to R_1+4R_3\\R_2\to R_2-5R_3\end{array}$$

$$\therefore\ A^{-1}=\begin{pmatrix}\frac{4}{17}&\frac{5}{17}&\frac{-5}{17}\\\frac{-5}{17}&\frac{-2}{17}&\frac{19}{17}\\\frac{1}{17}&\frac{-3}{17}&\frac{3}{17}\end{pmatrix}$$

From (i) AX = B

$\Rightarrow X = A^{-1}B$

or, $X = \begin{pmatrix} \frac{4}{17} & \frac{5}{17} & \frac{-5}{17} \\ \frac{-5}{17} & \frac{-2}{17} & \frac{19}{17} \\ \frac{1}{17} & \frac{-3}{17} & \frac{3}{17} \end{pmatrix} \begin{pmatrix} 14 \\ 3 \\ 4 \end{pmatrix}$

or, $X = \begin{pmatrix} \frac{4}{17}\times 14 + \frac{5}{17}\times 3 - \frac{5}{17}\times 4 \\ \frac{-5}{17}\times 14 - \frac{2}{17}\times 3 + \frac{19}{17}\times 4 \\ \frac{1}{17}\times 14 - \frac{3}{17}\times 3 + \frac{3}{17}\times 4 \end{pmatrix}$

or, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{51}{17} \\ \frac{0}{17} \\ \frac{17}{17} \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$\therefore$ x = 3, y = 0, z = 1

UNIT 19

NUMERICAL INTEGRATION

A. TRAPEZOIDAL RULES

2 MARKS QUESTIONS

1. 2069 (Set B) Q.No. 16c

Given $I = \int_0^4 x^3\,dx$, n = 4. Estimate the value of I using Trapezoidal rule. [2]

SOLUTION

Here, $y = f(x) = x^3$

$a = 0, b = 4, n = 4$

We have, $h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

The five points to be considered are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$. The values of the function at these points are tabulated below.

End point	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$	$x_4 = 4$
$y = f(x) = x^3$	0	1	8	27	64

Using trapezoidal rule, we have

$$\int_0^4 x^3 dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{1}{2}[0 + 2\times 1 + 2\times 8 + 2\times 27 + 64] = 68$$

2. 2070 Set C Q.No. 16 c

Using trapezoidal rule, evaluate: $\int_0^{\pi/2} \sqrt{\sin x}\,dx$, n = 2. [2]

SOLUTION

Here, $y = f(x) = \sqrt{\sin x}$

$a = 0, b = \pi/2, n = 2$

We have, $h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{2} = \frac{\pi}{4}$

The three points to be considered are:

$x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{2}$

The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = \frac{\pi}{4}$	$x_2 = \frac{\pi}{2}$
$y = f(x) = \sqrt{\sin x}$	0	0.8409	1

Hence, by using trapezoidal rule, we have

$$\int_0^{\pi/2} \sqrt{\sin x}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + y_2] = \frac{\frac{\pi}{4}}{2}[0 + 2\times 0.8409 + 1] = 1.0531$$

3. 2070 Set D Q.No. 16 c

Using trapezoidal rule, evaluate $\int_0^3 (3x^2 - 4x)dx$, n = 3. [2]

SOLUTION

Here, $f(x) = 3x^2 - 4x$

a = 0, b = 3, n = 3

We have, $h = \frac{b-a}{n} = \frac{3-0}{3} = 1$

The four points to be considered at $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$.

The values of the function at these points are tabulated as follows:

End point	$x_0 = 0$	$x_1 = 1$	$x_2 = 2$	$x_3 = 3$
$y = f(x) = 3x^2 - 4x$	0	-1	4	15

Using trapezoidal rule, we have

$$\int_0^3 (3x^2 - 4x)\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + y_3] = \frac{1}{2}[0 + 2 \times (-1) + 2 \times 4 + 15]$$

$$= \frac{1}{2}[-2 + 8 + 15] = 10.5$$

4. 2071 Set C Q.No. 1 6 c

Using the trapezoidal rule evaluate: $\int_0^1 \frac{dx}{1+x^2}$, n = 2 [2]

SOLUTION

Here, a = 0, b = 1, n = 2.

$\therefore$ $h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

Three points to be considered are $x_0 = 0, x_1 = 0.5, x_2 = 1$.

The values of function at these points are tabulated below.

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$
$y = f(x) = (1 + x^2)^{-1} = \frac{1}{1+x^2}$	1	0.8	0.5

Hence, by using Trapezoidal rule, we have

$$\int_0^1 (1 + x^2)^{-2}\,dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

$$= \frac{0.5}{2}[1 + 2 \times 0.8 + 0.5] = 0.775$$

5. 2071 Set D Q.No. 16 c

Using the trapezoidal rule, evaluate: $\int_0^2 (2x^2 - 1)dx$, n = 4. [2]

SOLUTION

First part:

Here, a = 0, b = 2, n = 4

$\therefore$ $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

The five points to be considered are $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5, x_4 = 2$

The values of function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y = 2x^2 - 1$	-1	-0.5	1	3.5	7

Hence, by using trapezoidal rule, we have

$$\int_0^2 (2x^2 - 1)\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{0.5}{2}[-1 + 2 \times (-0.5) + 2 \times 1 + 2 \times 3.5 + 7]$$

$$= 0.25[-1 - 1 + 2 + 7 + 7] = 3.5$$

6. 2074 Supp Q.No. 16c

Using trapezoidal rule, evaluate $\int_0^1 \frac{dx}{1+x^2}$, n = 2. [2]

➥ Please refer to 2071 Set C Q.N. 16c

7. 2075 Set C Q.No. 16c

Use the Trapezoidal Rule to approximate the integral $\int_1^2 \frac{1}{x}dx$. Find the error for the approximation. [2]

SOLUTION

Here,

a = 1, b = 2

$f(x) = \frac{1}{x}$

$\therefore$ $f(1) = 1, f(2) = \frac{1}{2}$

Using trapezoidal rule,

$$\int_1^2 \frac{1}{x}dx \approx (b-a)\left[\frac{f(a) + f(b)}{2}\right]$$

$$= (2-1)\left[\frac{1 + \frac{1}{2}}{2}\right]$$

$$= \frac{3}{4} = 0.75$$

Actual value $= \int_1^2 \frac{1}{x}dx$

$= [\ln x]_1^2$

$= \ln 2 - \ln 1$

$= 0.693147$

Error = Actual value – approximate value

$= 0.693147 - 0.75$

$= -0.056853$

4 MARKS QUESTIONS

8. 2069 (Set A) Q.No. 17b

Estimate the following integral using Trape-Zoidal rule. $\int_0^1 \frac{dx}{1+x}$, n = 4

Estimate the error with respect to the actual value. [4]

SOLUTION

Here, $y = f(x) = \frac{1}{1+x}$

a = 0, b = 1, n = 4

We have, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

The five points to be considered are $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$.

The values of the function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \frac{1}{1+x}$	1	0.8	0.66666	0.57143	0.5

Using trapezoidal rule, we have

$$\int_0^1 \frac{1}{1+x}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{0.25}{2}[1 + 2\times 0.8 + 2\times 0.66666 + 2\times 0.57143 + 0.5]$$

$$= (0.125)(5.57619) = 0.67701$$

Actual value $= \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2 = 0.69315$

Error = Actual value – Approximate value
= 0.69315 – 0.67701 = 0.01614

9. 2070 Supp. Q.No. 17 b

Compute two approximate values for $\int_1^2 x^{-2}\,dx$ using $h = \frac{1}{2}$ and $h = \frac{1}{4}$ by the composite trapezoid rule. [4]

SOLUTION

Here, $y = f(x) = x^{-2}$
$a = 1, b = 2$

Case I: When $h = \frac{1}{2}$

We have,

$h = \frac{b-a}{n}$

or, $\frac{1}{2} = \frac{2-1}{n}$

or, $n = 2$

The three points to be considered are $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$ respectively. The values of the function at these points are tabulated below.

End point:	$x_0 = 1$	$x_1 = 1.5$	$x_2 = 2$
$y = f(x) = x^{-2} = \frac{1}{x^2}$	1	0.4444	0.25

Hence, by using trapezoidal rule, we have

$$\int_1^2 x^{-2}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + y_2] = \frac{1/2}{2}[1 + 2\times 0.4444 + 0.25] = 0.5347$$

Case II: When $h = \frac{1}{4}$

We have, $h = \frac{b-a}{b}$

or, $\frac{1}{4} = \frac{2-1}{n}$

$\therefore$ $n = 4$

The five points to be considered are $x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75$ and $x_4 = 2$ respectively.

The values of the function at these points are tabulated below;

End point:	$x_0 = 1$	$x_1 = 1.25$	$x_2 = 1.5$	$x_3 = 1.75$	$x_4 = 2$
$y = f(x) = x^{-2} = \frac{1}{x^2}$	1	0.64	0.4444	0.3265	0.25

Using trapezoidal rule, we have

$$\int_1^2 x^{-2}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{1/4}{2}[1 + 2\times 0.64 + 2\times 0.4444 + 2\times 0.3265 + 0.25]$$

$$= \frac{1}{8}[1 + 1.28 + 0.8888 + 0.653 + 0.25] = 0.509$$

10. 2071 Supp. Q.No. 17b

Compute an approximate value of $\int_0^1 (1+x^2)^{-1}\,dx$ by the composite trapezoid rule with three points. Then comparing it with the actual value of the integral, find the error. [4]

SOLUTION

First part: Please see 2071 Set C Q.No. 1 6 c

Next part:

Actual value $= \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \tan^{-1}1 = \frac{\pi}{4} = 0.7854$

Error = Actual value – Approximate value
= 0.7854 – 0.775 = 0.0104

11. 2072 Set E Q.No. 17b

Evaluate using composite trapezoidal rule, the integral $\int_0^{\pi} \sin x\,dx$, n = 4. [4]

SOLUTION

Here, $a = 0, b = \pi, n = 4$

$\therefore$ $h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$

The five points to be considered are $x_0 = 0, x_1 = \frac{\pi}{4}, x_2 = \frac{\pi}{2}, x_3 = \frac{3\pi}{4}, x_4 = \pi$ respectively. The values of function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = \frac{\pi}{4}$	$x_2 = \frac{\pi}{2}$	$x_3 = \frac{3\pi}{4}$	$x_4 = \pi$
$y = \sin x$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

Using trapezoidal rule, we have

$$\int_0^{\pi} \sin x\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{\pi}{8}\left[0 + 2\cdot\frac{1}{\sqrt{2}} + 2\cdot 1 + 2\cdot\frac{1}{\sqrt{2}} + 0\right]$$

$$= \frac{\pi}{8}(2\sqrt{2} + 2) = 1.896$$

12. 2075 Set A Q.No. 17b OR

Evaluate, using composite trapezoidal rule the integral: $\int_0^1 \frac{dx}{1+x}$, n = 5. [4]

SOLUTION

Here, a = 0, b = 1, n = 5

We have, $h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$

The six points to be considered are $x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$ and $x_5 = 1$. The values of the integrand at each of the points are as follows:

x	0	0.2	0.4	0.6	0.8	1
$y = f(x) = \frac{1}{1+x}$	1	0.83333	0.71429	0.625	0.55556	0.5

Using Composite Trapezoidal rule, we have

$$\int_0^1 \frac{1}{1+x}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5]$$

$$= \frac{0.2}{2}[1 + 2 \times 0.83333 + 2 \times 0.71429 + 2 \times 0.625 + 2 \times 0.55556 + 0.5]$$

$$= \frac{0.2}{2} \times 6.95636$$

$$= 0.695636$$

13. 2075 Set B Q.No. 17b

Compute an approximate value of $\int_0^1 (1 + x^2)^{-1}\,dx$ by using the composite trapezoid rule with three points. Then comparing with the actual value of the integral, find the error. [4]

➥ **First part:** Please refer to 2071 Set C Q.No. 16c

Second part: Please refer to 2071 Supp Q.No. 17b

14. 2075 Set B Q.No. 17b OR

Approximate $\int_0^2 2^x\,dx$ using Simpson's $\frac{1}{3}$ rule with $h = \frac{1}{2}$.

SOLUTION

Here,

$a = 0, b = 2, h = \frac{1}{2}$

We have,

$h = \frac{b-a}{h}$

or, $n = \frac{b-a}{h} = \frac{2-0}{\frac{1}{2}} = 4$

Now, the end points of four subintervals are 0, 0.5, 1, 1.5 and 2.

The value of the integrand at these points are tabulated below:

x	0	0.5	1	1.5	2
$y = f(x) = 2^x$	1	1.4142	2	2.82842	4

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^2 2^x dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{\frac{1}{2}}{3}[1 + 4 \times 1.4142 + 2 \times 2 + 4 \times 2.82842 + 4]$$

$$= \frac{1}{6}(25.97008) = 4.3283$$

6 MARKS QUESTIONS

15. 2072 Set C Q.No. 19 OR

State and prove Trapezoidal rule of numerical approximation. [6]

➥ Please see Model Set I, Q.No. 19

16. 2072 Set D Q.No. 19

Approximate the value using trapezoidal rule for $\int_{-1}^{1} e^x\,dx$; n = 2. [6]

SOLUTION

Here, a = –1, b = 1, n = 2

We have, $h = \frac{b-a}{n} = \frac{1+1}{2} = 1$

The three points to be considered are $x_0 = -1, x_1 = 0, x_2 = 1$ respectively.

The values of function at these points are tabulated below

End point	$x_0 = -1$	$x_1 = 0$	$x_2 = 1$
$y = e^x$	0.3679	1	2.7183

Hence, by using trapezoidal rule, we have

$$\int_{-1}^{1} e^x\,dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)]$$

$$= \frac{1}{2}[0.3679 + 2 \times 1 + 2.7183] = 2.5431$$

17. 2073 Set C Q.No. 19

Define Trapezoidal rule. Evaluate using Trapezoidal rule: $\int_0^1 \frac{dx}{1+x}$ for n = 4. [6]

SOLUTION

First part: Please see Model Set I, Q.No. 19

Second part:

Here, $y = f(x) = \frac{1}{1+x}$

a = 0, b = 1, n = 4

We have, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

The five points to be considered are $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$.

The values of the function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \frac{1}{1+x}$	1	0.8	0.66666	0.57143	0.5

Using trapezoidal rule, we have

$$\int_0^1 \frac{1}{1+x}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{0.25}{2}[1 + 2 \times 0.8 + 2 \times 0.66666 + 2 \times 0.57143 + 0.5]$$

$$= (0.125)(5.57619) = 0.67701$$

18. 2073 Set D Q.No. 17b

Using the trapezoidal rule, compute $\int_0^2 (2x^2 - 1)\,dx$ with 4 intervals. Find the absolute error of approximation from its actual value. [4]

SOLUTION

First Part: Please refer to 2071 Set D Q.No. 16c

Second part:

Actual value $= \int_0^2 (2x^2 - 1)\,dx$

$= \left[\frac{2x^3}{3} - x\right]_0^2 = \frac{2 \times 2^3}{3} - 2 = 3.3333$

Absolute error $= |3.3333 - 3.5| = 0.1667$

19. 2074 Set A Q.No. 19

Using trapezoidal rule, evaluate $\int_0^1 \frac{dx}{1 + x}$, n = 4. Estimate the error of approximation from its actual value. [6]

➥ Please refer to 2069 Set A Q.No. 17b

20. 2074 Set B Q.No. 19

State Trapezoidal rule, hence evaluate $\int_0^2 \frac{dx}{1 + x^4}$ for n = 4 correct to 3 places of decimal. [6]

SOLUTION

1st part: Please refer to Model Set I Q.No. 19

2nd part:

Here, a = 0, b = 2, n = 4

$\therefore h = \frac{b - a}{n} = \frac{2 - 0}{4} = 0.5$

So, the five points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$. The value of the integrand at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
Value of $y = f(x) = \frac{1}{1 + x^4}$	1	0.94118	0.5	0.16495	0.05882

By Trapezoidal rule, we have

$$\int_0^2 \frac{dx}{1 + x^4} \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{0.5}{2}[1 + 2 \times 0.94118 + 2 \times 0.5 + 2 \times 0.16495 + 0.05882]$$

$$= \frac{0.5}{2}(4.27108)$$

= 1.06777 = 1.068 (upto 3 places of decimals)

B. SIMPSON'S RULE

2 MARKS QUESTIONS

1. 2072 Set E Q.No. 16c

Find the approximate value of $\int_0^{0.2} \sqrt{1 - 2x^2}\,dx$, n=2, using Simpson's $\frac{1}{3}$ rule. [2]

SOLUTION

Here, $y = f(x) = \sqrt{1 - 2x^2}$

a = 0, b = 0.2, n = 2

We have, $h = \frac{b - a}{n} = \frac{0.2 - 0}{2} = 0.1$

The three points to be considered at $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$ respectively. The values of the functions at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
$y = f(x) = \sqrt{1 - 2x^2}$	1	0.9899	0.9592

Using Simpson's $\frac{1}{3}$ rule,

we have $\int_0^1 \sqrt{1 - 2x^2}\,dx \approx \frac{h}{3}[y_0 + 4y_1 + y_2] = \frac{0.1}{3}[1 + 4 \times 0.9899 + 0.9592]$

= 0.1973

2. 2073 Set D Q.No. 16c

Using Simpson's $\frac{1}{3}$ rule, evaluate: $\int_0^{0.2} \sqrt[3]{1 - 2x^2}\,dx$, n = 2. [2]

SOLUTION

Here, a = 0, b = 0.2, n = 2

$\therefore h = \frac{b - a}{n} = \frac{0.2 - 0}{2} = 0.1$

The three points to be taken are 0, 0.1 and 0.2.

The values of the integrand at these points are tabulated below.

x	0	0.1	0.2
$y = \sqrt[3]{1 - 2x^2}$	1	0.9933	0.9726

Now, using Simpson's $\frac{1}{3}$ rule,

$$\int_0^{0.2} \sqrt[3]{1 - 2x^2} \approx \frac{h}{3}(y_0 + 4y_1 + y_2)$$

$$= \frac{0.1}{3}(1 + 4 \times 0.9933 + 0.9726)$$

= 0.1982

3. 2075 Set B Q.No. 16c

Apply Simpson's rule to approximate the value of $\int_1^4 e^x \ln x\,dx$ with n = 3. [2]

SOLUTION

Here,
a = 1, b = 4, n = 3

$\therefore$ $h = \frac{b-a}{n} = \frac{4-1}{3} = 1$

The 4 points to be considered are $x_0 = a = 1$, $x_1 = x_0 + h = 1 + 1 = 2$, $x_2 = x_1 + h = 2 + 1 = 3$ and $x_3 = b = 4$.

The values of the integrand at each of these points are tabulated below.

x	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$
$y = f(x) = e^x \ln x$	0	5.12170	22.06621	75.68911

Since n = 3 which is odd, so we have to apply Simpson's $\frac{3}{8}$ rule. Using Simpson's $\frac{3}{8}$ rule, we have

$$\int_1^4 e^x \ln x\, dx \approx \frac{3h}{8}[y_0 + 3xy_1 + 3y_2 + y_3]$$

$$= \frac{3\times 1}{8}[0 + 3\times 5.12170 + 3\times 22.06621 + 75.68911]$$

$$= 58.969815$$

4 MARKS QUESTIONS

4. 2069 (Set B) Q.No. 17b

Evaluate the following integral using Simpson's rule: $\int_0^{\pi} \sin x\, dx$, n = 6. [4]

SOLUTION

Here, $y = f(x) = \sin x$
a = 0, b = π, n = 6

We have, $h = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6}$

The seven points to be considered are $x_0 = 0$, $x_1 = \frac{\pi}{6}$,

$x_2 = \frac{\pi}{3}$, $x_3 = \frac{\pi}{2}$, $x_4 = \frac{2\pi}{3}$,

$x_4 = \frac{5\pi}{6}$ and $x_6 = \pi$. The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = \frac{\pi}{6}$	$x_2 = \frac{\pi}{3}$	$x_3 = \frac{\pi}{2}$	$x_4 = \frac{2\pi}{3}$	$x_5 = \frac{5\pi}{6}$	$x_6 = \pi$.
$y = f(x) = \sin x$	0	0.5	0.8660	1	0.8660	0.5	0

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^{\pi} \sin x\, dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6]$$

$$= \frac{\pi/6}{3}[0 + 4\times 0.5 + 2\times 0.8660 + 4\times 1 + 2\times 0.8660 + 4\times 0.5 + 0]$$

$$= \frac{\pi}{18}(11.464) = 2.0008$$

5. 2070 Set C Q.No. 17 b

Using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^1 \frac{dx}{1+x^2}$, n = 4. [4]

➥ Please see Model Set II, Q.No. 17b

6. 2070 Set D Q.No. 17 b

Using the Simpson's $\frac{1}{3}$ rule, evaluate $\int_0^1 \frac{dx}{1+x}$, n = 4. [4]

SOLUTION

Given, $y = f(x) = \frac{1}{1+x}$

Here, a = 0, b = 1, n = 4

We have, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

The five points to be considered are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \frac{1}{1+x}$	1	0.8	0.6666	0.5714	0.5

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \frac{1}{1+x}dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] = \frac{0.25}{3}[1 + 4\times 0.8 + 2\times 0.6666 + 4\times 0.5714 + 0.5]$$

$$= \frac{0.25}{3}(8.3188)$$

$$= 0.6932$$

7. 2071 Set C Q.No. 17 b

Using Simpson's 1/3 rule, evaluate: $\int_0^1 \sqrt{1+2x^2}\, dx$, h = 0.25 [4]

SOLUTION

Here, $f(x) = \sqrt{1+2x^2}$
a = 0, b = 1, h = 0.25

We have,

$h = \frac{b-a}{n}$

or, $0.25 = \frac{1-0}{n}$

$\therefore$ n = 4

The five points to be considered are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$.

The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \sqrt{1+2x^2}$	1	1.0607	1.2247	1.4577	1.7321

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \sqrt{1+2x^2}\, dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{0.25}{3}[1 + 4\times 1.0607 + 2\times 1.2247 + 4\times 1.4577 + 1.7321]$$

$$= 1.2712$$

8. 2071 Set D Q.No. 17 b

Estimate the following integral using Simpson's 1/3 rule: $\int_0^{\pi} \sin x\, dx$, n = 6 **[4]**

➥ Please refer to 2069 Set B Q.No. 17b

9. 2072 Set C Q.No. 17b

Using Simpson's $\frac{1}{3}$ rule, calculate $\int_1^5 x^4\, dx$ with n = 4 **[4]**

SOLUTION

Here, $y = f(x) = x^4$

$a = 1, b = 5, n = 4$

We have, $h = \frac{b-a}{n} = \frac{5-1}{4} = 1$

The five points to be considered are $x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 5$. The values of the function at these points are tabulated below.

End point	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$	$x_4 = 5$
$y = f(x) = x^4$	1	16	81	256	625

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_1^5 x^4 dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{1}{3}[1 + 4 \times 16 + 2 \times 81 + 4 \times 256 + 625] = 625.33$$

10. 2072 Supp Q.No. 17b

Evaluate the following integral using Simpson's rule: $\int_0^1 \frac{dx}{1+x^2}$, n = 4. **[4]**

➥ Please see Model Set II, Q.No. 17b

11. 2073 Supp Q.No. 17b

Evaluate the following using Simpson's rule: $\int_0^1 \frac{dx}{1+x^2}$, n= 4 **[4]**

➥ Please refer to Model Set II, Q.No. 17b

12. 2074 Supp Q.No. 17b

Using Simpson's $\frac{1}{3}$ rule, evaluate $\int_0^2 (2x-1)^2\, dx$, n = 4. **[4]**

SOLUTION

Here, a = 0, b = 2, n = 4

We have, $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

The five points to be considered are $x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5$ and $x_4 = 2$. The values of the integrand at each of the points are as follows:

x	0	0.5	1	1.5	2
$y = f(x) = (2x-1)^2$	1	0	1	4	9

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 (2x-1)^2 dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{0.5}{3}[1 + 4 \times 0 + 2 \times 1 + 4 \times 4 + 9]$$

$$= \frac{0.5}{3} \times 28 = 4.6667$$

13. 2075 Set A Q.No. 17b

Using Simpson's rule, evaluate: $\int_0^1 \frac{dx}{1+x}$, n = 4. **[4]**

➥ Please refer to 2070 Set D Q.No. 17b

14. 2075 Set C Q.No. 17b

Using Simpson's $\frac{1}{3}$ rule, evaluate $\int_0^2 (4x^2-4x+1)dx$, n=4. **[4]**

SOLUTION

Here,

a = 0, b = 2, n = 4

We have, length of each interval $(h) = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

Thus the end points of the 4 subintenals are 0, 0.5, 1, 1.5 and 2. The values of the integrand at each end points of the sub intervals are given below:

x	0	0.5	1	1.5	2
$y = 4x^2 - 4x + 1$	1	0	1	4	9

Now, using simpson's $\frac{1}{3}$ rule, we have

$$\int_0^2 (4x^2 - 4x + 1)dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{0.5}{3}[1 + 4 \times 0 + 2 \times 1 + 4 \times 4 + 9]$$

$$= \frac{0.5}{3} \times 28 = 4.6667$$

6 MARKS QUESTIONS

15. 2072 Set D Q.No. 19 OR

Evaluate $\int_0^1 \sqrt{1+x^3}\, dx$ using Simpson's $\frac{1}{3}$ rule with n = 4. **[6]**

SOLUTION

Given, $f(x) = \sqrt{1+x^3}$

a = 0, b = 1, n = 4

We have, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

The 5 points to be considered are $x_0 = 0, x_1 = 0.25, x_2 = 0.5, x_3 = 0.75$ and $x_4 = 1$. The values of the function at these points are tabulated below.

End point	$x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$x_4 = 1$
$y = f(x) = \sqrt{1+x^2}$	1	1.0078	1.0607	1.1924	1.4142

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \sqrt{1+x^3}\, dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{0.25}{3}[1 + 4 \times 1.0078 + 2 \times 1.0607 + 4 \times 1.1924 + 1.4142]$$

$$= 1.1114$$

16. 2073 Set C Q.No. 19 OR

Using Simpson's rule, evaluate $\int_0^2 \frac{dx}{1+x^4}$ for n= 4, correct to 3 places of decimal.

SOLUTION

Here, a = 0, b = 2, n = 4

$\therefore \; h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

The five points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$ and $x_4 = 2$.

The value of the integrand at these points are tabulated below.

x	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y = f(x) = \frac{1}{1+x^4}$	$y_0 = 1$	$y_1 = 0.9412$	$y_2 = 0.5$	$y_3 = 0.1649$	$y_4 = 0.0588$

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^2 \frac{dx}{1+x^4} \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{0.5}{3}[1 + 4 \times 0.9412 + 2 \times 0.5 + 4 \times 0.1649 + 0.0588]$$

$$= 1.081$$

17. 2074 Set B Q.No. 19 OR

Define Simpson's rule, hence evaluate $\int_0^1 \frac{dx}{1+x}$ for n = 4 correct to 4 places of decimal. [6]

SOLUTION

Simpson's Rule: If a function f is continuous on closed interval [a, b], then,

$$\int_b^a f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where [a, b] is divided into n equal subintervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$.... $[x_{n-1}, x_n]$, each of the length $\frac{b-a}{n}$.

Second Part:

Please refer to 2070 (Set D) Q. No. 17 b.

Haste Haste Din mere —
Raten Bhi Yeh. —
Hasin Ban Gaya. —
Aayush.